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Response of a Non-Linear Device to Noise

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(None)

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Massachusetts Inst. of Technology, Radiation Lab., Cambridge

Office of Scientific Research and Development, NDRC, Div 14

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(None)

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The problem is discussed of a non-linear device connected in series with an admittance, and with a random noise voltage impressed across the combination. The current-voltage function of the non-linear device and the admittance were assumed, and approximate statistical information was obtained about the voltage across the non-linear device. Explicit formulas depending only on the current-voltage relation of the non-linear device and on the admittance are given for the moments of all orders, and for the frequency spectrum of the voltage across the device. The method of solution consisted of solving for the voltage across part of the circuit in terms of the entire voltage, and then taking averages of the random voltages in accordance with known formulas.

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Electronics (3) Noise, Random - Analysis (66684);

Electronic Theory (12) Radar - Noise elimination (77310)

R-3-12

V-16S

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Radiation Laboratory

Response of a Non-Linear Device to Noise

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Radiation Laboratory

Response of a Non-Linear Device to Noise

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Abstract

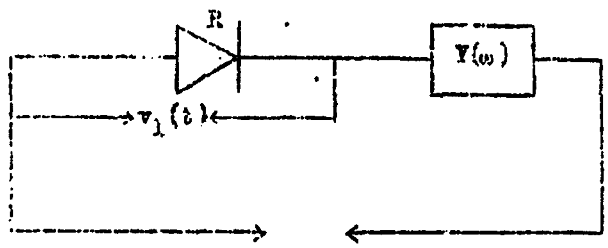
A non-linear device R is connected in series with an admittance $Y(\omega)$ and a random noise voltage $v(t)$ (e.g., arising from thermal agitation) impressed across the combination. Assume that one knows (a) the current-voltage function of R, and (b) the admittance $Y(\omega)$. Then we can obtain approximate statistical information about the voltage $v_1(t)$ across th non-linear device: explicit formulas depending only on (a) and (b) can be given for the moments of all orders of $v_1(t)$, and similarly for its frequency spectrum.

Title page
8 numbered pages

RESPONSE OF A NON-LINEAR DEVICE TO NOISE

Norbert Wiener

1. A non-linear device R is connected in series with an admittance $Y(\omega)$ and a random noise voltage $v(t)$ (e.g., thermal agitation) impressed across the combination



random voltage $v(t)$

Fig. 1

Assume that one knows

(a) the current-voltage function of R

and

(b) the admittance $Y(\omega)$.

Then one can obtain approximate statistical information about the voltage $v_1(t)$ across the non-linear device: explicit formulas depending only on (a) and (b) can be given for the moments of all orders of $v_1(t)$, and similarly for its frequency spectrum.

2. There are two ideas involved. The first is to express $v_1(t)$ in terms of $v(t)$, assuming (a) and (b) known. The second idea is then to make use of the random nature of $v(t)$ to get the statistical information

about $v_1(t)$.

The first idea employs an operator-series expansion for $v_1(t)$, and the second employs known averaging processes on Brownian functions.

3. In order to make matters definite we shall assume throughout the sequel that if the voltage across R is $v_1(t)$ the current through it is $v_1(t) + \epsilon(v_1(t))^2$.

4. Let us denote by $A(t)$ the indicial admittance corresponding to the frequency admittance $Y(\omega)$. Then it is well known that the current through $Y(\omega)$ is

$$\int_{-\infty}^{+\infty} A'(t-\tau)\ (v(\tau) - v_1(\tau))\ d\tau.$$

Since current through R = current through $Y(\omega)$,

$$v_1(t) + \epsilon(v_1(t))^2 = \int_{-\infty}^{+\infty} A'(t-\tau)\ (v(\tau) - v_1(\tau))\ d\tau.$$

Collecting terms in $v_1(t)$:

$$(1) \quad v_1(t) + \epsilon(v_1(t))^2 + \int_{-\infty}^{+\infty} A'(t-\tau)\ v_1(\tau)\ d\tau = \int_{-\infty}^{+\infty} A'(t-\tau)\ v(\tau)\ d\tau.$$

5. Now we come to the first basic step of the paper, that of solving for $v_1(t)$ in terms of $v(t)$. To do this we assume* that

$$(2) \quad v_1(t) = \int_{-\infty}^{+\infty} Q_1(t-\tau)\ v(\tau)\ d\tau + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q_2(t-\tau_1, t-\tau_2)\ v(\tau_1)\ v(\tau_2)\ d\tau_1\ d\tau_2$$

$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q_3(t-\tau_1, t-\tau_2, t-\tau_3)\ v(\tau_1)\ v(\tau_2)\ v(\tau_3)\ d\tau_1\ d\tau_2\ d\tau_3 + \cdots$$

---

* In general one will also include a constant term Q_0.

it is clear that we shall have secured the desired solution for $v_1(t)$ as soon as we have found Q_1, Q_2, Q_3, ...

6. Now substitute expression (2) for $v_1(t)$ in Eq. (1). We shall then equate linear part with linear part, quadratic part with quadratic part, etc

7. First for the first-degree terms in Eq. (1), we have

$$(3) \qquad \int_{-\infty}^{+\infty} Q_1(t-\tau)\, v(\tau)\, d\tau + \int_{-\infty}^{+\infty} A'(t-\sigma)\, d\sigma \int_{-\infty}^{+\infty} Q_1(\sigma-\tau)\, v(\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} A'(t-\tau)\, v(\tau)\, d\tau.$$

It is clear that (3) will be satisfied if

$$(4) \qquad Q_1(t-\tau) + \int_{-\infty}^{+\infty} A'(t-\sigma)\, d\sigma\, Q_1(\sigma-\tau) = A'(t-\tau).$$

Suppose Q_1 is the Fourier transform of q_1; then recalling that $Y(\omega)$ is the Fourier transform of $A'(t)$, i.e.,

$$(5) \qquad Q_1(t) = \int_{-\infty}^{+\infty} q_1(\omega)\, e^{i\omega t}\, d\omega$$

$$(6) \qquad A'(t) = \int_{-\infty}^{+\infty} Y(\omega)\, e^{i\omega t}\, d\omega,$$

Eq. (4) leads to

$$q_1(\omega)(1 + 2\pi\, Y(\omega)) = Y(\omega).$$

Hence

(7) $$q_1(\omega) = Y(\omega)/(1 + 2\pi\, Y(\omega));$$

thereby determining $Q_1(t)$.

8. Equating the second-degree terms in Eq. (1) to each other we have

(8) $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q_2(t-\tau_1, t-\tau_2)\, v(\tau_1)\, v(\tau_2)\, d\tau_1\, d\tau_2 + \epsilon\left(\int_{-\infty}^{+\infty} Q_1(t-\tau)\, v(\tau)\, d\tau\right)^2$$

$$+ \int_{-\infty}^{+\infty} A'(t-\sigma)\, d\sigma \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q_2(\sigma-\tau_1, \sigma-\tau_2)\, v(\tau_1)\, v(\tau_2)\, d\tau_1\, d\tau_2 = 0.$$

Eq. (8) will hold if

(9) $$Q_2(\tau_1, \tau_2) + \epsilon\, Q(\tau_1, \tau_2) = \int_{-\infty}^{+\infty} A'(\sigma-\tau)\, Q_2(\tau+\tau_1, \tau+\tau_2)\, d\tau.$$

Suppose

(10) $$Q_2(\tau_1, \tau_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(\omega_1\tau_1+\omega_2\tau_2)}\, q_2(\omega_1, \omega_2)\, d\omega_1\, d\omega_2,$$

i.e., $Q_2(\tau_1, \tau_2)$ is the double Fourier transform of $q_2(\omega_1, \omega_2)$. Recalling Eqs. (5) and (6), Eq. (10) leads to

$$q_2(\omega_1, \omega_2) + \epsilon\, q_1(\omega_1)\, q_1(\omega_2) + 2\pi\, Y(\omega_1+\omega_2)\, q_2(\omega_1, \omega_2) = 0.$$

Hence

(11) $$q_2(\omega_1, \omega_2) = -\frac{\epsilon\, q_1(\omega_1)\, q_1(\omega_2)}{2\pi\, Y(\omega_1+\omega_2) + 1} = -\frac{\epsilon\, Y(\omega_1)\, Y(\omega_2)}{(1+2\pi Y(\omega_1))(1+2\pi Y(\omega_2))(1+2\pi Y(\omega_1+\omega_2))}.$$

Note that q_2 contains the factor ϵ, indicating the effect of the non-linearity of the device E.

9. In the same way one computes $q_3(\omega_1,\omega_2,\omega_3)$:

$$(12)\quad q_3(\omega_1,\omega_2,\omega_3) = -\frac{2\pi\, q_1(\omega_1)\, q_2(\omega_2+\omega_3)}{1+2\pi Y(\omega_1+\omega_2+\omega_3)}$$

$$= \frac{2\epsilon^2\, Y(\omega_1)}{1+2\pi Y(\omega_1)}\; \frac{Y(\omega_2)\,Y(\omega_3)}{(1+2\pi Y(\omega_2))(1+2\pi Y(\omega_3))(1+2\pi Y(\omega_2+\omega_3))}\; \frac{1}{1+2\pi Y(\omega_1+\omega_2+\omega_3)},$$

where the q_n's are defined, in analogy with Eqs. (5) and (10), as multiple Fourier transforms of the Q_n's:

$$(13)\qquad Q_n(\tau_1,\ldots,\tau_n) = \int\cdots\int e^{i\sum_1^n \omega_k\tau_k}\, q_n(\omega_1,\ldots,\omega_n)\, d\omega_1 \cdots d\omega_n .$$

Note that q_3 contains ϵ^2. Similarly q_4 will contain ϵ^3, and so on. Thus taking higher powers of ϵ into account is equivalent to going out farther in the series of q's; this is the characteristic feature of perturbation methods.

10. At this point we interpose a formula which will be needed soon. It expresses an average of Q_n, n even, in terms of an average of q_n:

$$(14)\qquad \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} Q_n(\tau_1,\tau_1,\ldots,\tau_{\frac{n}{2}},\tau_{\frac{n}{2}})\, d\tau_1 \cdots d\tau_{\frac{n}{2}}$$

$$= (2\pi)^{\frac{n}{2}} \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} q_n(\omega_1,-\omega_1,\ldots,\omega_{\frac{n}{2}},-\omega_{\frac{n}{2}})\, d\omega_1 \cdots d\omega_{\frac{n}{2}}.$$

11. Eqs. (7), (11), and (12) tell us the first three terms of (2). In other words, we have accomplished (to three terms) our first basic task, that of expressing $v_1(t)$, the voltage across the non-linear device R in terms of the admittance $Y(\omega)$ and the voltage $v(t)$ across the entire circuit.

12. So far we have said nothing about $v(t)$, but we are now ready to make use of the fact that $v(t)$ is a random voltage. This will constitute the second step of the paper, and will be accomplished by taking **averages** of the random voltages in accordance with known formulas. In these formulas the average is taken with respect to the parameter α which in going from 0 to 1 runs through all Brownian motions; $x(t,\alpha)$ is a properly normed Brownian motion, whose differential is a random voltage and $K(t_1, \ldots, t_n)$ is a symmetric function of n variables:

(15) $$\int_0^1 d\alpha \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{n \text{ times}} K(t_1, \ldots, t_n)\, dx(t_1,\alpha) \cdots dx(t_n,\alpha) = 0 \quad \text{if } n \text{ is odd.}$$

(16) $$\int_0^1 d\alpha \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{n \text{ times}} K(t_1, \ldots, t_n)\, dx(t_1,\alpha) \cdots dx(t_n,\alpha)$$

$$= (n-1)(n-3)\cdots 1 \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{\frac{n}{2} \text{ times}} K(t_1, t_1, t_2, t_2, \ldots, t_{\frac{n}{2}}, t_{\frac{n}{2}})\, dt_1 \cdots dt_{\frac{n}{2}}$$

if n is even.

13. Referring to Eq. (2), we inquire as to the average of $v_1(t)$. We apply Eqs. (15) and (16) to the Q's, then express the result in terms of the q's by (14), and finally (Eqs. (7), (11), and (12)) expressing the q's in terms of the admittance $Y(\omega)$ we find that the first non-vanishing term of the average of $v_1(t)$ is

(17) $$-2\pi^2 \int_{-\infty}^{+\infty} \frac{Y(\omega)\, Y(-\omega)}{(1+2\pi Y(\omega))(1+2\pi Y(-\omega))(1+2\pi Y(\omega))}\, d\omega.$$

14. Similarly the average of $(v_1(t))^2$ is

$$(13) \quad 2\pi \int_{-\infty}^{+\infty} q_1(\omega)\, q_1(-\omega)\, d\omega$$

$$+ 4\pi^2 \left\{ 2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_2(\omega_1,\omega_2)\, q_2(-\omega_1,-\omega_2)\, d\omega_1\, d\omega_2 + \int_{-\infty}^{+\infty} q_2(\omega,-\omega)\, d\omega \right\}^2$$

$$+ 12\pi^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_1(\omega_1)\, q_3(\omega_1,\omega_2,-\omega_2)\, d\omega_1\, d\omega_2$$

$$+ 8\pi^3 \left\{ 6 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_3(\omega_1,\omega_2,\omega_3)\, q_3(-\omega_1,-\omega_2,-\omega_3)\, d\omega_1\, d\omega_2\, d\omega_3 \right.$$

$$\left. + 9 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_3(\omega_1,\omega_2,-\omega_2)\, q_3(-\omega_1,\omega_3,-\omega_3)\, d\omega_1\, d\omega_2\, d\omega_3 \right\} + \cdots .$$

15. In the same way the higher moments of $v_1(t)$ may be computed. So also may the moments of $v(t) - v_1(t)$, the voltage across $Y(\omega)$, be computed.

16. An average of much importance is that of $v_1(t)v_1(t+\sigma)$; this average is called the auto-correlation coefficient, and its Fourier transform gives the frequency distribution of the square of the voltage. The auto-correlation coefficient is the average of

$$\left\{ \int_{-\infty}^{+\infty} Q_1(t-\tau)\, v(\tau)\, d\tau + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Q_2(t-\tau_1, t-\tau_2)\, v(\tau_1)\, v(\tau_2)\, d\tau_1\, d\tau_2 + \cdots \right\}$$

$$\times \left\{ \int_{-\infty}^{+\infty} Q_1(t+\sigma-\tau)\, v(\tau)\, d\tau + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Q_2(t+\sigma-\tau_1, t+\sigma-\tau_2)\, v(\tau_1)\, v(\tau_2)\, d\tau_1\, d\tau_2 + \cdots \right\},$$

which is equal to

$$(19)\quad 2\pi\int_{-\infty}^{+\infty} q_1(\omega)\, q_1(-\omega)\, e^{-i\omega\sigma}\, d\sigma + 4\pi^2\int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_2\, q_2(\omega_1,-\omega_1)\, q_2(\omega_2,-\omega_2)$$

$$+ 8\pi^2\int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_2\, q_2(\omega_1,\omega_2)\, q_2(-\omega_1,-\omega_2)\, e^{-i\sigma(\omega_1+\omega_2)} + \cdots .$$

Since the second term does not contain σ, it is a constant, representing a DC component. The third term is equal to

$$8\pi^2\int_{-\infty}^{+\infty} e^{-i\sigma\omega}\int_{-\infty}^{+\infty} q_2(\omega,\omega-\omega_1)\, q_2(-\omega_1,\omega_1-\omega)\, d\omega_1 .$$

Thus to the frequency spectrum

$$2\pi\, q_1(\omega)\, q_1(-\omega)$$

present with no rectification (i.e., $\epsilon = 0$), there has been added

$$8\pi^2\int_{-\infty}^{+\infty} q_2(\omega_1,\omega-\omega_1)\, q_2(-\omega_1,\omega_1-\omega)\, d\omega_1 .$$

17. <u>Critique</u>. The method above, of first solving for the voltage across part of the circuit in terms of the entire voltage, and then getting statistical averages, is clearly quite general. The particular application of this method given here has two weaknesses, however:

(i) the current-voltage relation of R is over-simplified,

(ii) in every practical case a filter of finite band-width precedes the rectifier-admittance combination of Fig. 1. The problem in which (i) and (ii) have been taken into account can, and should be, set up and solved.

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