THIS REPORT HAS BEEN DELIMITED AND CLEARED FOR PUBLIC RELEASE UNDER DOD DIRECTIVE 5200.20 AND NO RESTRICTIONS ARE IMPOSED UPON ITS USE AND DISCLOSURE.

DISTRIBUTION STATEMENT A

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.
PLEASE RETURN THIS COPY TO:
ARMED SERVICES TECHNICAL INFORMATION AGENCY
DOCUMENT SERVICE CENTER
Knott Building, Dayton, Ohio

Because of our limited supply you are requested to return this copy as soon as it has served your purposes so that it may be made available to others for reference use. Your cooperation will be appreciated.

"NOTICE: When Government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the U.S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto."

UNCLASSIFIED
The U.S. GOVERNMENT IS ABSOLVED FROM ANY LITIGATION WHICH MAY ENSUE FROM THE CONTRACTORS INFRINGING ON THE FOREIGN PATENT RIGHTS WHICH MAY BE INVOLVED.

WRIGHT FIELD, DAYTON, OHIO
Response of a Non-Linear Device to Noise

Wiener, N.
Massachusetts Inst. of Technology, Radiation Lab., Cambridge
Office of Scientific Research and Development, NDRC, Div 14

April '42  Rest.  U.S.  Eng.  9 diagr

The problem is discussed of a non-linear device connected in series with an admittance, and with a random noise voltage impressed across the combination. The current-voltage function of the non-linear device and the admittance were assumed, and approximate statistical information was obtained about the voltage across the non-linear device. Explicit formulas depending only on the current-voltage relation of the non-linear device and on the admittance are given for the moments of all orders, and for the frequency spectrum of the voltage across the device. The method of solution consisted of solving for the voltage across part of the circuit in terms of the entire voltage, and then taking averages of the random voltages in accordance with known formulas.

Copies of this report obtainable from Air Documents Division; Attn: MCIDD
Electronics (3)  Noise, Random - Analysis (66684)
Electronic Theory (12)  Radar - Noise elimination (77310)

R-3-12
RESTRICTED

UNCLASSIFIED
RADIATION LABORATORY

LIMITED DISTRIBUTION SERVICE TO AFRICA

RESTRICTED

RESTRICTED

RESTRICTED

_provenance__

RADIATION LABORATORY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Best Available Copy
Response of a Non-Linear Device to Noise

Abstract

A non-linear device \( R \) is connected in series with an admittance \( Y(\omega) \) and a random noise voltage \( v(t) \) (e.g., arising from thermal agitation) impressed across the combination. Assume that one knows (a) the current-voltage function of \( R \), and (b) the admittance \( Y(\omega) \). Then we can obtain approximate statistical information about the voltage \( v_1(t) \) across the non-linear device: explicit formulas depending only on (a) and (b) can be given for the moments of all orders of \( v_1(t) \), and similarly for its frequency spectrum.
RESPONSE OF A NON-LINEAR DEVICE TO NOISE

Norbert Wiener

1. A non-linear device $R$ is connected in series with an admittance $Y(\omega)$ and a random noise voltage $v(t)$ (e.g., thermal agitation) impressed across the combination.

![Diagram of a non-linear device](image)

random voltage $v(t)$

Fig. 1

Assume that one knows

(a) the current-voltage function of $R$

and

(b) the admittance $Y(\omega)$.

Then one can obtain approximate statistical information about the voltage $v_1(t)$ across the non-linear device: explicit formulas depending only on (a) and (b) can be given for the moments of all orders of $v_1(t)$, and similarly for its frequency spectrum.

2. There are two ideas involved. The first is to express $v_1(t)$ in terms of $v(t)$, assuming (a) and (b) known. The second idea is then to make use of the random nature of $v(t)$ to get the statistical information.
about $v_1(t)$.

The first idea employs an operator-series expansion for $v_1(t)$; and the second employs known averaging processes on Brownian functions.

3. In order to make matters definite we shall assume throughout the sequel that if the voltage across $R$ is $v_1(t)$ the current through it is $\dot{v}_1(t) + c(v_1(t))^2$.

4. Let us denote by $A(t)$ the indicial admittance corresponding to the frequency admittance $Y(w)$. Then it is well known that the current through $Y(w)$ is

$$\int_{-\infty}^{\infty} A'(t-\tau) \left( v(\tau) - v_1(\tau) \right) d\tau.$$  

Since current through $R = current through Y(w)$,

$$v_1(t) + c(v_1(t))^2 = \int_{-\infty}^{\infty} A'(t-\tau) \left( v(\tau) - v_1(\tau) \right) d\tau.$$  

Collecting terms in $v_1(t)$:

1. $v_1(t) + c(v_1(t))^2 + \int_{-\infty}^{\infty} A'(t-\tau) v_1(\tau) d\tau = \int_{-\infty}^{\infty} A'(t-\tau) v(\tau) d\tau.$

5. Now we come to the first basic step of the paper, that of solving for $v_1(t)$ in terms of $v(t)$. To do this we assume that

$$v_1(t) = \int_{-\infty}^{\infty} Q_3(t,\tau) v(\tau) d\tau + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_3(t-\tau_1, t-\tau_2, t-\tau_3) v(\tau_1) v(\tau_2) v(\tau_3) d\tau_1 d\tau_2 d\tau_3 + \ldots$$

*In general one will also include a constant term $Q_0$.
it is clear that we shall have secured the desired solution for \( r_1(t) \) as soon as we have found \( q_1, q_2, q_3, \ldots \).

6. Now substitute expression (2) for \( r_1(t) \) in Eq. (1). We shall then equate linear part with linear part, quadratic part with quadratic part, etc.

7. First for the first-degree terms in Eq. (1), we have

\[
\int_{-\infty}^{\infty} q_1(t - \tau) \, v(t) \, d\tau = \int_{-\infty}^{\infty} A'(t - \tau) \, d\tau \int_{-\infty}^{\infty} q_1(\tau - \tau) \, v(t) \, d\tau = \int_{-\infty}^{\infty} A'(t - \tau) \, v(t) \, d\tau.
\]

It is clear that (3) will be satisfied if

\[
q_1(t - \tau) + \int_{-\infty}^{\infty} A'(t - \tau) \, d\tau \, q_1(\tau - \tau) = A'(t - \tau).
\]

Suppose \( Q \) is the Fourier transform of \( q_1 \); then recalling that \( Y(w) \) is the Fourier transform of \( A'(t) \), i.e.,

\[
Q_1(t) = \int_{-\infty}^{\infty} q_1(w) \, e^{iwt} \, dw.
\]

\[
A'(t) = \int_{-\infty}^{\infty} Y(w) \, e^{iwt} \, dw.
\]

Eq. (4) leads to

\[
q_1(w)(1 + 2\pi Y(w)) = Y(w).
\]

Hence

\[
q_1(t) = \int_{-\infty}^{\infty} q_1(w) \, e^{iwt} \, dw.
\]
(7) 
\[ q_1(\omega) = \frac{Y(\omega)}{1 + 2\pi Y(\omega)} \]

thereby determining \( q_1 \).

8. Equating the second-degree terms in Eq. (1) to each other we have

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_2(t_1 - t_2) \nu(\tau_1) \nu(\tau_2) \, dt_1 \, dt_2 + \epsilon \left( \int_{-\infty}^{+\infty} q_1(t) \nu(t) \, dt \right)^2 + \int_{-\infty}^{+\infty} A'(t-C) \, dt \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_2(t_1 - t_1, t - t_2) \nu(\tau_1) \nu(\tau_2) \, dt_1 \, dt_2 = 0. \]

Eq. (8) will hold if

\[ q_2(t_1, t_2) = \epsilon q(t_1, t_2) = \int_{-\infty}^{+\infty} A'(t-C) q_2(t+\tau_1, \tau_2) \, dt. \]

Support

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_2(t_1, t_2) \, dt_1 \, dt_2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} 1(t_1 + t_2) \, dt_1 \, dt_2. \]

\[ \omega \times q_2(t_1, t_2) \]

is the double Fourier transform of \( q_2(\omega_1, \omega_2) \). Recalling Eqs. (5) and (6), Eq. (10) leads to

\[ q_2(\omega_1, \omega_2) + \epsilon q_1(\omega_1) q_1(\omega_2) + 2\pi Y(\omega_1, \omega_2) q_2(\omega_1, \omega_2) = 0. \]

Hence

\[ q_2(\omega_1, \omega_2) = \frac{-\epsilon q_1(\omega_1) q_1(\omega_2)}{2\pi Y(\omega_1, \omega_2)} = \frac{-\epsilon Y(\omega_1) Y(\omega_2)}{(1 + 2\pi Y(\omega_1))(1 + 2\pi Y(\omega_2))}. \]

Note that \( q_2 \) contains the factor \( \epsilon \), indicating the effect of the nonlinearity of the device \( R \).
9. In the same way one computes \( q_3(w_1, w_2, w_3) \):

\[
(12) \quad q_3(w_1, w_2, w_3) = \frac{2\pi^2 q_2(w_1, w_2, w_3)}{\lambda^2 Y(w_1, w_2, w_3)}.
\]

\[
2\pi^2 \frac{Y(w_1) Y(w_3)}{1 + 2\pi^2 Y(w_1) Y(w_3)} \left( \frac{1}{1 + 2\pi^2 Y(w_1) Y(w_3)} \right) \left( \frac{1}{1 + 2\pi^2 Y(w_2) Y(w_3)} \right) \left( \frac{1}{1 + 2\pi^2 Y(w_1, w_2, w_3)} \right)
\]

where the \( q_n \)'s are defined, in analogy with Eqs. (5) and (10), as multiple Fourier transforms of the \( Q_n \)'s:

\[
(13) \quad q_n(T_1, \ldots, T_n) = \int \ldots \int e^{i2\pi T_k T} q_n(w_1, \ldots, w_n) \, dw_1 \ldots \, dw_n.
\]

Note that \( q_1 \) contains \( c^2 \). Similarly \( q_2 \) will contain \( c^3 \), and so on. Thus taking higher powers of \( c \) into account is equivalent to going out farther in the series of \( q \)'s; this is the characteristic feature of perturbation methods.

10. At this point we interpose a formula which will be needed soon. It expresses an average of \( Q_n \), a even, in terms of an average of \( q_n \):

\[
(14) \quad \int \ldots \int q_n(T_1, \ldots, T_n, T_2) \, dT_1 \ldots dT_2
\]

\[
+ \frac{2\pi^2}{\lambda^2 Y(\infty, \infty, \infty)} \left( \frac{1}{1 + 2\pi^2 Y(\infty, \infty, \infty)} \right) \left( \frac{1}{1 + 2\pi^2 Y(\infty, \infty, \infty)} \right) \left( \frac{1}{1 + 2\pi^2 Y(\infty, \infty, \infty)} \right)
\]

11. Eqs. (7), (11), and (12) tell us the first three terms of (8). In other words, we have accomplished (to three terms) our first basic task, that of expressing \( v_1(t) \), the voltage across the non-linear device \( R \) in terms of the admittance \( Y(w) \) and the voltage \( v(t) \) across the entire circuit.
12. So far we have said nothing about \( v(t) \), but we are now ready to make use of the fact that \( v(t) \) is a random voltage. This will constitute the second step of the paper, and will be accomplished by taking averages of the random voltages in accordance with known formulas. In these formulas the average is taken with respect to the parameter \( \alpha \) which in going from 0 to 1 runs through all Brownian motions; \( x(t, \alpha) \) is a properly normalized Brownian motion, whose differential is a random voltage and \( K(t_1, \ldots, t_n) \) is a symmetric function of \( n \) variables:

\[
\int_0^1 \, d\alpha \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} K(t_1, \ldots, t_n) \, dx(t_1, \alpha) \cdots dx(t_n, \alpha) = 0 \quad \text{if } n \text{ is odd,}
\]

\[
\int_0^1 \, d\alpha \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} K(t_1, \ldots, t_n) \, dx(t_1, \alpha) \cdots dx(t_n, \alpha) = \frac{(n-1)(n-3) \cdots 1}{2^{n-1}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} K(t_1, t_1, t_2, t_2, \ldots, t_n, t_n) \, dt_1 \cdots dt_n
\]

if \( n \) is even.

13. Referring to Eq. (2), we inquire as to the average of \( v_1(t) \). We apply Eqs. (15) and (16) to the \( q \)'s, then express the result in terms of the \( q \)'s by (14), and finally (Eqs. (7), (11), and (12)) expressing the \( q \)'s in terms of the admittance \( Y(w) \) we find that the first non-vanishing term of the average of \( v_1(t) \) is

\[
\int_{-\infty}^{+\infty} \frac{Y(w) Y(-w)}{(1+2\pi Y(w))(1+2\pi Y(-w))(1+2\pi Y(w))} \, dw.
\]
14. Similarly the average of \( \langle v_1^2 (t) \rangle \) is

\[
2\pi \int_{-\infty}^{+\infty} q_1(\omega) \, q_1(-\omega) \, d\omega
\]

\[
+ 4m^2 \left\{ 3 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_2(w_1, w_2) \, q_2(-w_1, -w_2) \, dw_1 \, dw_2 + \int_{-\infty}^{+\infty} q_2(\omega, -\omega) \, d\omega \right\}^2
\]

\[
+ 12m^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_1^4(w_1, w_2, w_3) \, dw_1 \, dw_2 \, dw_3
\]

\[
+ 3m^2 \left\{ 6 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_3(w_1, w_2, w_3) \, q_3(-w_1, -w_2, -w_3) \, dw_1 \, dw_2 \, dw_3
\]

15. In the same way the higher moments of \( v_1(t) \) may be computed.

\[
\text{So also the moments of } v(t) = v_1(t), \text{ the voltage across } Y(w), \text{ be computed.}
\]

16. An average of much importance is that of \( v_1(t) v_1(t+\sigma) \); this average is called the auto-correlation coefficient, and its Fourier transform gives the frequency distribution of the square of the voltage.

The auto-correlation coefficient is the average of

\[
\left\{ \int_{-\infty}^{+\infty} q_1(t-\tau) \, v(\tau) \, d\tau + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_2(t-\tau_1, t-\tau_2) \, v(\tau_1) \, v(\tau_2) \, d\tau_1 \, d\tau_2 + \cdots \right\}
\]

\[
\times \left\{ \int_{-\infty}^{+\infty} q_1(t+\sigma-\tau) \, v(\tau) \, d\tau + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_2(t+\sigma-\tau_1, t+\sigma-\tau_2) \, v(\tau_1) \, v(\tau_2) \, d\tau_1 \, d\tau_2 + \cdots \right\}
\]

which is equal to
(18) \[ 2n \int_{-\infty}^{+\infty} q_1(\omega) q_2(-\omega) e^{-j\omega\sigma} d\omega + 4\pi^2 \int_{-\infty}^{+\infty} dw_1 \int_{-\infty}^{+\infty} dw_2 q_2(w_1, -w_1) q_2(w_2, -w_2) \]

\[ + 8\pi^2 \int_{-\infty}^{+\infty} dw_1 \int_{-\infty}^{+\infty} dw_2 q_2(w_1, -w_1) q_2(-w_1, -w_2) e^{-j\sigma w_1 w_2} \]

Since the second term does not contain \( \sigma \), it is a constant, representing a DC component. The third term is equal to

\[ 8\pi^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q_2(w_1, -w_1) q_2(-w_1, w_1) dw_1. \]

Thus to the frequency spectrum

\[ 2n q_1(\omega) q_2(-\omega) \]

present with no rectification (i.e., \( \phi = 0 \)), there has been added

\[ 8\pi^2 \int_{-\infty}^{+\infty} q_2(w_1, -w_1) q_2(-w_1, w_1) dw_1. \]

17. Critique. The method above, of first solving for the voltage across part of the circuit in terms of the entire voltage, and then getting statistical averages, is clearly quite general. The particular application of this method given here has two weaknesses, however:

(1) the current-voltage relation of \( R \) is over-simplified,

(2) in every practical case a filter of finite bandwidth precedes the rectifier-admittance combination of Fig. 1. The problem in which (1) and (2) have been taken into account can, and should be, set up and solved.
The problem is discussed of a non-linear device connected in series with an admittance, and with a random noise voltage impressed across the combination. The current-voltage function of the non-linear device and the admittance were assumed, and approximate statistical information was obtained about the voltage across the non-linear device. Explicit formulas depending only on the current-voltage relation of the non-linear device and on the admittance are given for the moments of all orders, and for the frequency spectrum of the voltage across the device. The method of solution consisted of solving for the voltage across part of the circuit in terms of the entire voltage, and then taking averages of the random voltages in accordance with known formulas.