APPLIED MATHEMATICS AND STATISTICS LABORATORY

STANFORD UNIVERSITY

CALIFORNIA

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GEORGE J. RESNIKOFF

TECHNICAL REPORT NO. 8

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1. INTRODUCTION

In industrial acceptance sampling, the classical procedure, so-called inspection by "attributes", is to classify each item of a sample drawn from a lot of manufactured items as defective or non-defective, and accept or reject the lot according to whether the proportion defective in the sample is small or large.

When the classification into defective and non-defective is made on the basis of the measurement of a variable quality characteristic, inspection plans based on these measurements are called "variables" plans. Suppose an item is considered defective if the measurement exceeds a given value U, and suppose that the distribution of the measurements follows the normal law. A sample is drawn from a lot, the quality characteristic of each item in the sample is measured, and the mean $\bar{x}$ and standard deviation s of the sample measurements are computed. If the quantity $\bar{x} + ks$ is less than or equal to U the lot is accepted; otherwise it is rejected. The constant k is chosen so as to effect certain desired characteristics of the plan. Graphically, this is equivalent to rejecting if the point $(\bar{x}, s)$ lies to the right of the line $\bar{x} + ks = U$. The criterion, accept if $\bar{x} + ks \leq U$, is called a one-sided plan. Since inspection by variables makes greater use of the information concerning the lot than does inspection by attributes, whenever the testing of the individual items involves measurement, variables plans require smaller sample sizes to furnish the same degree of protection than do attributes plans, except for the case of samples of size two.

Two-sided inspection plans in acceptance sampling by variables are applied to the cases wherein manufactured items are considered defective if a measurement of the characteristic which defines quality lies either above some upper limit U, or below some lower limit L. For example, the constants of resistance and capacitance of electronics components such as resistors and condensers may be specified with permissible tolerances on either side of the specified value, thus indirectly defining upper and lower limits, or the maximum and minimum dimensions of a machine part may be specified directly. As in one-sided plans using measurements on a variable quality characteristic, a relatively small sample is drawn from a lot, and the quality characteristic for each item is measured. The mean $\bar{x}$, and the standard deviation s, of the sample measurements, are computed. Graphically, if the point $(\bar{x}, s)$ falls within a certain region the lot is accepted; otherwise, it is rejected.

In applying an acceptance criterion, the acceptance of some lots containing defective items is unavoidable, unless 100 per cent inspection is used. The probability L_p of accepting a lot with proportion defective p is a function of p. The graph of this function is called the operating characteristic curve (OC curve) of the test procedure. If a lot with no defective items is submitted for inspection $L_p = 1$, that is, the lot is certain to be accepted. If a lot in which every item is defective, it is certain to be rejected, and $L_p = 0$. For any other values of p of submitted lots L_p lies between 0 and 1.

For fixed values, the pair of numbers (N,k) defines the test procedure uniquely, in the sense that the plan has a single OC curve. This curve differs with each different plan, but for any one plan the location of the points of the curve depends only on the lot proportion defective. It is clearly desirable for users of a sampling plan to know its OC curve since the operating characteristic provides a basis for choosing an existing inspection procedure, as well as describing the long-run results once the plan is put to use.

All existing two-sided plans have the disadvantage that the probability of accepting a submitted lot with given proportion defective p does not depend on p alone, but is also a function of the actual lot mean μ, or equivalently, on the actual division of the proportion defective into components lying above and below the specification limits U and L. For this reason, it is not possible to compute a single OC curve giving the operating characteristics of a two-sided plan. Such plans do not yield a constant probability of acceptance for given proportion defective, but rather a spectrum of probabilities. Computing OC curves for all possible divisions of p results in a band of curves.

The two-sided test procedure described in this report is a graphical method. The decision as to whether to accept or reject a lot as a result of sample data is made by plotting a point on a graph. If the point lies within a closed region of the graph the lot is accepted. If the point lies outside of the region the lot is rejected. A typical region is shown in Figure 1. It may be seen that the upper boundary consists of three segments, two straight lines, and a curved portion. The straight lines correspond to one-sided tests $\bar{x} + ks \leq U$, $\bar{x} - ks \geq L$. Thus there is established, in a natural way, a unique correspondence between a one-sided test and a related two-sided test. Just as the pair of numbers (N,k)

TYPICAL TWO-SIDED ACCEPTANCE REGION

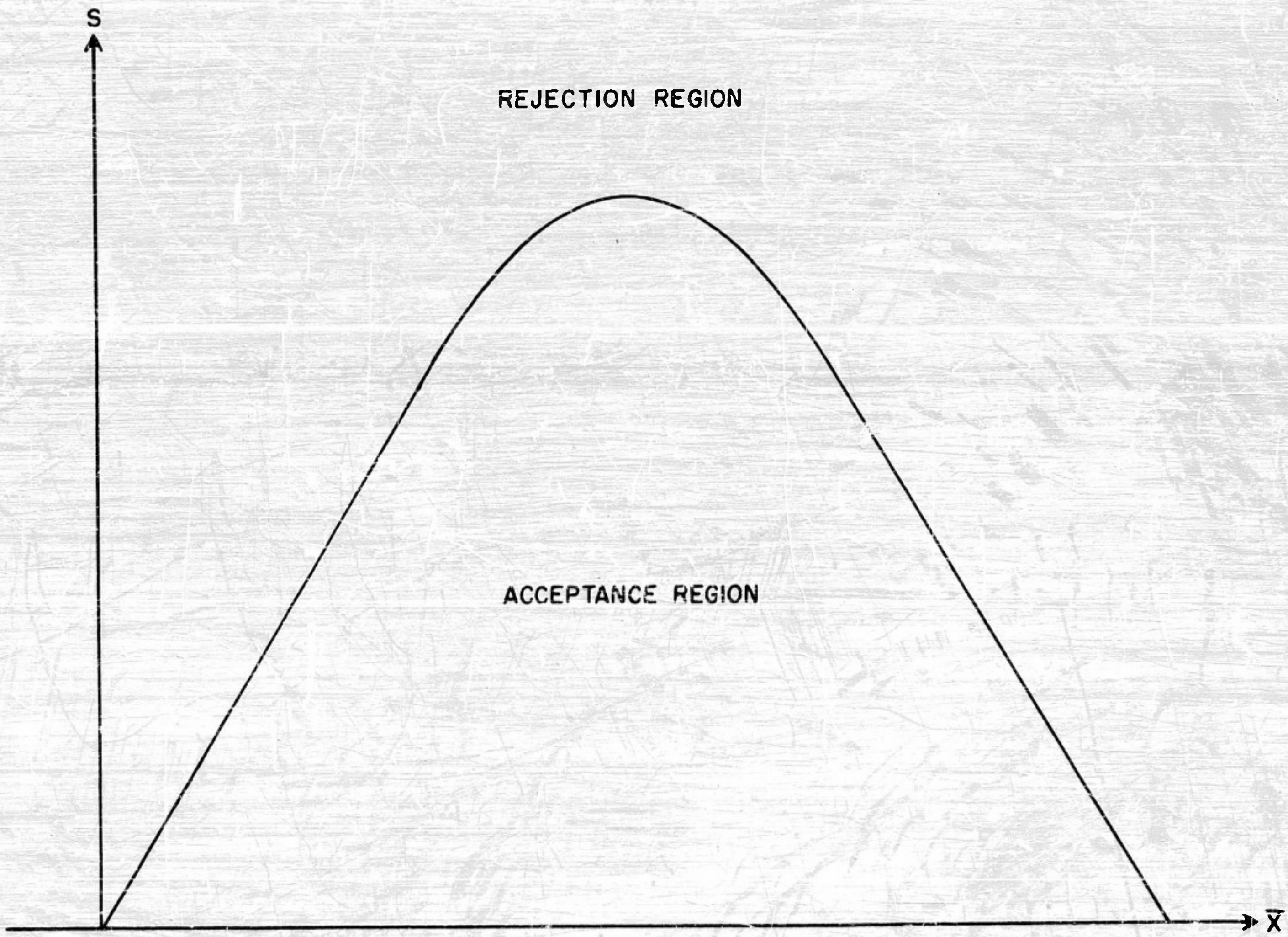

FIGURE 1

characterize a one-sided test, the same numbers (N,k), together with the graph of its boundary, may be used to define a two-sided test, provided the following conditions hold:

(i) The band of OC curves due to all possible divisions of the proportion defective is so narrow as to be, for practical purposes, a single curve.

(ii) The OC curve of the related one-sided test closely approximates the OC band of the two-sided test.

For a two-sided test procedure to be useful only the first condition is needed, since any one of the OC curves could be computed and used as an approximation to the others, for example, the OC curve for the case of equal division of the proportion defective into components above U and below L. However, the labor involved in obtaining such OC curves is considerably greater than the labor required to compute the OC curve for a one-sided plan. Moreover there exist collections of one-sided plans published together with their OC curves. Hence, if a two-sided procedure also satisfies the second condition it is only necessary to furnish the boundary points of its graph and the numbers (N,k) of its associated one-sided plan. The OC curve of the latter can be used as a very close approximation to the OC "curve" of the two-sided plan.

This is the point of view adopted in <u>Sampling Inspection by Variables</u>, by Bowker and Goode [1]. Charts of OC curves for 160 combinations of N and k are given. Corresponding to each pair (N,k) a set of points for the central portion of a two-sided region is tabled. To construct the region it is only necessary to draw the lines $\bar{x} + ks = U$, $\bar{x} - ks = L$, plot the points given in the table, and draw a smooth curve through these points. The OC curves may then be used for either the two-sided plans or the one-sided plans or for both.

Section 3 of this report contains data which indicates that the two-sided acceptance procedure presented here does have the properties (i) and (ii), stated above. A wide range of sample sizes $N = 3$, 25, 50, and 100 have been investigated. For each of these values of N, two regions corresponding to different values of k have been studied. A 5-point OC curve is found for each of the 8 regions, for equal division of the proportion defective. Several additional points in the OC band for each region are computed to illustrate the closeness of these points to the corresponding points on the associated one-sided OC curve. It may be seen from these tables that for practical purposes the probability of acceptance for a lot submitted to such an inspection procedure is virtually independent of the division of the proportion defective. Comparison of the points of the OC bands of the two-sided regions with the corresponding points of the one-sided plans bears out the contention that the latter's OC curve may be used as a very good approximation, since, for practical purposes, the differences are negligible.

Additional considerations which indicate that this sampling procedure leads to a satisfactory test are the following: (i) if the lot is actually one-sided, that is, all of the defectives are due to measurements of the quality characteristic exceeding only one of the specification limits, then the probability L_p of acceptance of the lot is nearly identical with the L_p of the one-sided test; (ii) if we take the lower limit L equal to $-\infty$, that is, we are in the case where we have only one specification limit, the estimate of the lot proportion defective p on which the test is based leads to the one-sided test $\bar{x} + ks \leq U$, (iii) the estimate of p which is used as the basis for the test procedure has minimum sampling variability among unbiased estimates of p.

Section 2 contains the analytical derivation of the test procedure and of the estimate of p on which it is based. This derivation is based on the references [2] and [3] in the bibliography.

2. A TWO-SIDED ACCEPTANCE REGION BASED ON AN OPTIMUM ESTIMATE OF THE PROPORTION DEFECTIVE

2.1 An Optimum Estimate of the Proportion of a Normal Population Lying Outside a Fixed Interval

It has been shown, as a consequence of a theorem of D. Blackwell [4], that if $\tilde{p}$ is any unbiased statistic for estimating p, that fraction of a normal population lying outside a fixed interval (L,U), then the conditional expected value of $\tilde{p}$, given the sufficient statistic $(\bar{x}, S^2)$ is also an unbiased estimate of p, having variance less than or equal to the variance of the original estimate $\tilde{p}$.

By a theorem of E. L. Lehmann and H. Scheffe [5], $\hat{p}$ is unique, i.e., no matter what unbiased estimate of p one starts with, the conditional expected value of this estimate given $(\bar{x}, S^2)$ is the same. Hence, it follows that $\hat{p}$ has a minimum variance among all unbiased estimates of p. An example of an unbiased estimate of p is that proportion of the observations in a sample which lie outside the interval (L,U). Another example is the statistic $\tilde{p}$ defined in the following paragraph. Because of the uniqueness, both of these estimates lead to the same value, $\hat{p}$.

Let y be any one of a sample of $N > 2$ observations $(x_1, x_2, \ldots, x_N)$, say x_1, and let $\tilde{p}$ be the unbiased estimate of p defined by

$$\begin{aligned} \tilde{p}(y, x_2, x_3, \ldots, x_N) &= 0 \quad \text{if} \quad L \leq y \leq U \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

Let

$$\bar{x} = \sum_{i=1}^{N} \frac{x_i}{N} \qquad S^2 = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N} \quad .$$

Let $g(y, \bar{x}, S^2)$ be the joint density of y, $\bar{x}$, and S^2, and let $h(\bar{x}, S^2)$ be the joint density of $\bar{x}$ and S^2. Then

$$\hat{p} = E(\tilde{p} \mid \bar{x}, S^2) = Pr(p = 1 \mid \bar{x}, S^2) = 1 - Pr(L \leq y \leq U \mid \bar{x}, S^2)$$

$$\hat{p} = 1 - \int_L^U f(y \mid \bar{x}, S^2)\,dy = \int_{-\infty}^{L} \frac{g(y, \bar{x}, S^2)}{h(\bar{x}, S^2)}\,dy + \int_U^{\infty} \frac{g(y, \bar{x}, S^2)}{h(\bar{x}, S^2)}\,dy \quad .$$

It is well known that $h(\bar{x}, S^2)$ is given by:

$$h(\bar{x}, S^2) = \frac{N^{N/2}}{\sigma\sqrt{2\pi}\, 2^{\frac{N-1}{2}}\, \Gamma\left(\frac{N-1}{2}\right) \sigma^{N-1}}\, e^{-\frac{1}{2\sigma^2}\left(N(\bar{x} - \mu)^2 + NS^2\right)} (S^2)^{\frac{N-3}{2}} \quad .$$

To find $f(y \mid \bar{x}, S^2) = \frac{g(y, \bar{x}, S^2)}{h(\bar{x}, S^2)}$ it is necessary to determine $g(y, \bar{x}, S^2)$ and divide by $h(\bar{x}, S^2)$. To do this, consider the joint density of the sample. This may be expressed as the joint density of y and $\bar{x}'$, S'^2 where

$$\bar{x}' = \sum_{i=2}^{N} \frac{x_i}{N-1} \qquad S'^2 = \sum_{i=2}^{N} \frac{(x_i - \bar{x})^2}{N-1}$$

This expression is the following

$$\left(\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\right)\cdot\left(\frac{\sqrt{N-1}}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{N-1}{2\sigma^2}(\bar{x}'-\mu)^2}\right)$$

$$\left(\frac{(N-1)^{\frac{N-2}{2}}}{(2\sigma^2)^{\frac{N-2}{2}}\,\Gamma(\frac{N-2}{2})}\, e^{-\frac{N-1}{2\sigma^2}S'^2}\,(S'^2)^{\frac{N-4}{2}}\right)$$

with the range of the variables being

$$-\infty < y < \infty$$

$$-\infty < \bar{x}' < \infty$$

$$0 < S'^2 < \infty \quad .$$

Now we make the transformation

$$\bar{x}' = \frac{N\bar{x} - y}{N-1}$$

$$S'^2 = \frac{N}{N-1}\, S^2 - \frac{N}{(N-1)^2}\,(\bar{x} - y)^2$$

$$y = y \quad .$$

Under this transformation the density becomes

$$K\left\{S^2\,\frac{N}{N-1} - (\bar{x}-y)^2\,\frac{N}{(N-1)^2}\right\}^{\frac{N-4}{2}} e^{-\frac{1}{2\sigma^2}\left[(y-\mu)^2 + (N-1)\left(\frac{N\bar{x}-y}{N-1} - \mu\right)^2 + (N-1)\left[S^2\,\frac{N}{N-1} - \frac{N}{(N-1)^2}(\bar{x}-y)^2\right]\right]}$$

where

$$K = \frac{N^2 (N-1)^{\frac{N-1}{2}}}{\Gamma(\frac{N-2}{2})\,\sigma^{N}(N-1)^2(2\pi)(2)^{\frac{N-2}{2}}}$$

and the ranges of the variables are:

$$-\infty < y < \infty$$

$$-\infty < \bar{x} < \infty$$

$$0 < S^2 < \infty$$

Now we divide this expression $g(y,\bar{x},S^2)$ by $h(\bar{x},S^2)$ and obtain $f(y|\bar{x},S^2)$.

$$f(y|\bar{x},S^2) = \frac{\Gamma(\frac{N-1}{2})(N-1)^{-\frac{1}{2}}}{\sqrt{\pi}\;\Gamma(\frac{N-2}{2})}\;\frac{1}{S}\left\{1-\left(\frac{\bar{x}-y}{S\sqrt{N-1}}\right)^2\right\}^{\frac{N-4}{2}} .$$

If we put

$$s^2 = \frac{N}{N-1}\,S^2 = \sum_{i=1}^{N}\frac{(x_i-\bar{x})^2}{N-1}$$

and

$$z = \frac{1}{2}+\frac{1}{2}\,\frac{\bar{x}-y}{S\sqrt{N-1}} ,$$

then since $-1 \leq \frac{\bar{x}-y}{S\sqrt{N-1}} \leq 1$ we have $0 \leq z \leq 1$; hence we obtain the density of the random variable z as a Beta density, with parameters $\frac{N}{2}-1$, $\frac{N}{2}-1$

$$g(z) = \frac{\Gamma(N-2)}{\Gamma(\frac{N-2}{2})\,\Gamma(\frac{N-2}{2})}\,z^{\frac{N}{2}-2}(1-z)^{\frac{N}{2}-2} \qquad 0 \leq z \leq 1 .$$

Hereafter we shall denote $\int g(z)dz$ as $\int dB(\frac{N}{2}-1,\frac{N}{2}-1)$ or $\int dB$, when there is no possibility of confusion or ambiguity.

Since $\hat{p} = Pr(y > U|x,s^2) + Pr(y < L|\bar{x},s^2)$, then

$$\hat{p} = Pr\left(z < \frac{1}{2}-\frac{1}{2}\,\frac{(U-\bar{x})\sqrt{N}}{s(N-1)}\right) + Pr\left(z > \frac{1}{2}-\frac{1}{2}\,\frac{(\bar{x}-L)\sqrt{N}}{s(N-1)}\right)$$

$$= \int_0^{\max[0,\frac{1}{2}-\frac{1}{2}\frac{(U-\bar{x})\sqrt{N}}{s(N-1)}]} dB + \int_0^{\max[0,\frac{1}{2}-\frac{1}{2}\frac{(\bar{x}-L)\sqrt{N}}{s(N-1)}]} dB .$$

$$k = \frac{t_0(N-1, \sqrt{N}\,K_p, L_p)}{\sqrt{N}} \quad .$$

Hence the one-sided test criterion $\bar{x} + ks \leq U$ is equivalent to the criterion $\hat{p} \leq p^*$ where p^* is such that

$$B_{p^*} = \frac{1}{2} - \frac{\sqrt{N}\,k}{2(N-1)} \quad .$$

2.3 The Two-sided Test Based on $\hat{p}$

It can be seen, from the range of the Beta density that if $\dfrac{U-\bar{x}}{s\,\frac{N-1}{\sqrt{N}}} > 1$, i.e., if $\bar{x} < U - s\,\frac{N-1}{\sqrt{N}}$ then there is no contribution to p from values of $\bar{x}$ lying above U. Similarly, if $\bar{x} > L + s\,\frac{N-1}{\sqrt{N}}$, there is no contribution from below L. If, however, $U - s\,\frac{N-1}{\sqrt{N}} < \bar{x} < L + s\,\frac{N-1}{\sqrt{N}}$ values of $\bar{x}$ beyond both limits contribute to $\hat{p}$. The boundary of the region will thus consist of three sections, each section corresponding to a range of $\bar{x}$.

In the first two cases one of the tests $\bar{x} + ks \leq U$ or $\bar{x} - ks \geq L$ applies. In terms of p^* these tests are

$$\bar{x} + (1 - 2B_{p^*})s\,\frac{N-1}{\sqrt{N}} \leq U \qquad \bar{x} - (1 - 2B_{p^*})s\,\frac{N-1}{\sqrt{N}} \geq L \quad .$$

We thus have, for the two end sections of the region, the lines of the one-sided test:

$$s = \frac{U-\bar{x}}{k} \qquad\qquad s = \frac{\bar{x}-L}{k} \quad .$$

If $U - s\,\frac{N-1}{\sqrt{N}} < \bar{x} < L + s\,\frac{N-1}{\sqrt{N}}$ both sides contribute. In this case, let:

$$p' = \int_0^{\left[\frac{1}{2} - \frac{1}{2}\frac{(U-\bar{x})\sqrt{N}}{s(N-1)}\right]} dB \quad ; \quad p'' = \int_0^{\left[\frac{1}{2} - \frac{1}{2}\frac{(\bar{x}-L)\sqrt{N}}{s(N-1)}\right]} dB \quad .$$

Since we are at the boundary of the acceptance region if $p = p^*$, we are on the boundary for all values of p' and p'' such that $p' + p'' = p^*$.

Let

$$B_{p'} = \frac{1}{2} - \frac{1}{2}\,\frac{(U-\bar{x})\sqrt{N}}{s(N-1)}$$

$$B_{p''} = \frac{1}{2} - \frac{1}{2}\,\frac{(\bar{x}-L)\sqrt{N}}{s(N-1)}$$

We may solve these equations for $\bar{x}$ and s and obtain:

$$s = \frac{(U-L)\sqrt{N}}{2(1-B_{p'}-B_{p''})(N-1)} \qquad \bar{x} = \frac{U(1-2B_{p''}) - L(2B_{p'}-1)}{2(1-B_{p'}-B_{p''})} .$$

All such points $(\bar{x}, s)$ which are solutions of these equations, for all the partitions of p^* into p', p'', are points of the central portion of the boundary.

The complete test, accept if $\hat{p} \leq p^*$, is equivalent to a sample point $(\bar{x}, s)$ lying in a region bounded by the $\bar{x}$-axis and the three sections.

(i) The line $\bar{x} = L + (1-2B_{p^*})s\,\frac{N-1}{\sqrt{N}}$ for $L < \bar{x} < L + \delta$

(ii) The line $\bar{x} = U - (1-2B_{p^*})s\,\frac{N-1}{\sqrt{N}}$ for $U - \delta < \bar{x} < U$

(iii) The curve determined by p' and p'' for $L + \delta < \bar{x} < U - \delta$ where $U - \delta$ is the intersection of $\bar{x} = L + s\,\frac{N-1}{\sqrt{N}}$ and $\bar{x} = U - (1-2B_{p^*})s\,\frac{N-1}{\sqrt{N}}$. It is easily verified that $\delta = \frac{(U-1)(1-2B_{p^*})}{2(1-B_{p^*})}$. In terms of the original one-sided test criterion k, the straight lines forming part of the boundary are

$$s = \frac{U-\bar{x}}{k} \quad \text{and} \quad s = \frac{\bar{x}-L}{k} .$$

The value $k = \frac{N-1}{\sqrt{N}}$ corresponds to the test criterion accept if $\hat{p} \leq p^* = 0$. For this value the region is simply the triangular region bounded by the $\bar{x}$-axis and the lines $s = \frac{(U-\bar{x})\sqrt{N}}{N-1}$, $s = \frac{(\bar{x}-L)\sqrt{N}}{N-1}$. No two-sided region can

can be constructed for $k > \frac{N-1}{\sqrt{N}}$, where the region is based on the optimum estimate $\hat{p}$. This restriction is of no consequence except for very small N. For N = 25, for example, the maximum k is 4.8 which corresponds to a one-sided test criterion which is so strict as to be of no practical consequence. Existing collections of one-sided sampling plans which have been examined use a maximum value of $k = 3^+$ for sample size N = 25.

Except for the cases N = 3 and N = 4 it is not possible to give a simple analytic representation of the curved part of the boundary. A graphical representation of a typical region is given in Figure .

2.4 Detailed Step-by-Step Construction of the Stanford Region

(i) Choose a one-sided sampling plan defined by N and k.

(ii) Compute $B_{p^*} = \frac{1}{2} - \frac{1}{2}\frac{\sqrt{N}\,k}{(N-1)}$.

(iii) By interpolation in a table of the Incomplete Beta Function, $B(\frac{N}{2}-1, \frac{N}{2}-1)$, determine p^*.

(iv) Partition p^* into p', p'', such that $p' + p'' = p^*$.

(v) By inverse interpolation in the Beta table find $B_{p'}$ and $B_{p''}$.

(vi) Solve for $\bar{x}$ and s in the equations

$$s = \frac{(U-L)\sqrt{N}}{2(1-B_{p'} - B_{p''})(N-1)} \qquad \bar{x} = \frac{U(1-2B_{p''}) - L(2B_{p'} - 1)}{2(1-B_{p'} - B_{p''})}.$$

(vii) Plot the $(\bar{x}, s)$ points, and draw the lines $\frac{U-\bar{x}}{k}$, $\frac{\bar{x}-L}{k}$. Sufficient points should be plotted to obtain a smooth curve in the interval $L + \delta \leq \bar{x} \leq U - \delta$.

2.5 A Simplification in Applying the Plan

An obvious simplification is to construct the two-sided regions with U = 1, L = -1. One computes the sample point $(a\bar{x} + b, as)$ instead of $(\bar{x}, s)$ and observes whether the point falls in this region defined on the interval [-1,1]. The constants a and b are determined by the equations

$$aU + b = 1$$

$$aL + b = -1$$

where U and L are the actual limits which define quality.

In this way a two-sided region can be associated with every one-sided plan independently of the limits U and L.

If several lots with the same specification limits, U and L, are to be inspected it would probably be more convenient to construct the graph of the region so that $\bar{x}$ and s are plotted directly. The procedure would then be as follows.

(1) Compute a and b from the above equations.

(2) Subtract from each point for the x -axis the constant b, and divide by a.

(3) Divide each point for the upper boundary of the curve by the constant a.

The lot is then accepted if the point $(\bar{x},s)$ falls in the region.

2.6 Stanford Regions for $N = 3$, $N = 4$

For the special cases where the sample size is 3 or 4, the analytic expression for the boundary of the region has a rather simple form. Moreover, it can be determined without reference to the Table of the Incomplete Beta Function, and can be expressed simply as a function of $\bar{x}$,s, and k. In the derivations given below, U is taken to be unity, and $L = -U$. No loss in generality for the region is caused by this since one can always make the linear transformation given in Section 2.5.

For $N = 4$, the region is a trapezoi , the curved portion reducing to a line $s =$ constant. This may be seen by considering the Incomplete Beta Function, with parameters $\frac{N}{2} - 1 = 1$. This is

$$\int_0^x dt = x \quad .$$

Therefore $B_{p^*} = p^*$, and taking $U = 1$, $L = -1$

$$p = \max(0, \frac{1}{2} - \frac{1-\bar{x}}{3s}) + \max(0, \frac{1}{2} - \frac{\bar{x}-1}{3s}) \quad .$$

If $\hat{p} = p^*$, and the test is one-sided, one of the terms is zero and the other equals p^*. Say the second term is zero, then

$$\frac{1}{2} - \frac{1-\bar{x}}{3s} = p^*$$

$$s = \frac{2(1-\bar{x})}{3(1-2p^*)} = \frac{1-\bar{x}}{k} \quad .$$

From this it can easily be seen that $p^* = \frac{3-2k}{6}$. Since

$$\delta = \frac{1-2B_{p^*}}{1-B_{p^*}} = \frac{1-2p^*}{1-p^*} \text{ ; then } \quad \delta = \frac{4k}{3+2k} \quad .$$

At $\bar{x} = 1 - \delta$, $s = \frac{4}{3+2k}$.

This region is sketched in Figure 2.

For $N = 3$, the region is derived in the $(\bar{x}, s^2)$ -plane for simplicity. Again U is taken to be unity and $L = -U$.

In this case the Incomplete Beta Function has the form

$$K \int_0^x \frac{dt}{\sqrt{t(1-t)}} = \frac{\cos^{-1}(1-2x)}{\pi}$$

$$p = \max(0, \frac{1}{\pi} \cos^{-1} \frac{\sqrt{3}}{2} \frac{1-\bar{x}}{s}) + \max(0, \frac{1}{\pi} \cos^{-1} \frac{\sqrt{3}}{2} \frac{\bar{x}+1}{s}) .$$

TWO-SIDED ACCEPTANCE REGION FOR N = 4

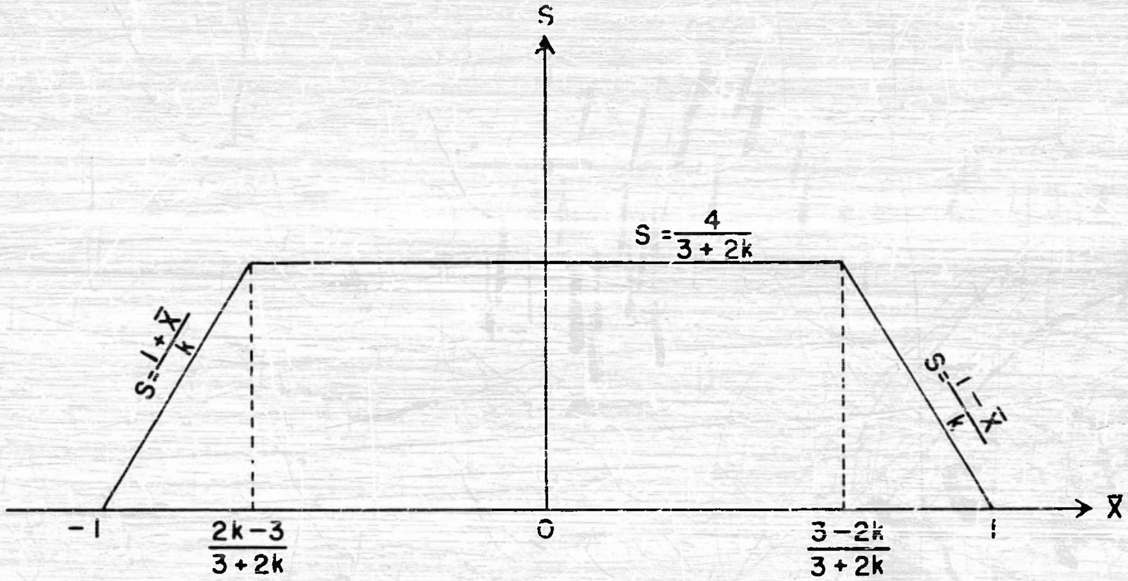

FIGURE 2

Let $\frac{\sqrt{3}}{2}\frac{1-\bar{x}}{s} = \theta_1$, $\frac{\sqrt{3}}{2}\frac{\bar{x}+1}{s} = \theta_2$, then $\cos\theta_1 + \cos\theta_2 = \frac{\sqrt{3}}{s}$. If $p = p^*$, then $\theta_1 + \theta_2 = \pi p^*$.

If the test is one-sided, $\cos \pi p^* = \frac{\sqrt{3}}{2}\frac{1-\bar{x}}{s} = \frac{\sqrt{3}}{2}k$; hence $k = \frac{2}{\sqrt{3}} \cos \pi p^*$.

If we make use of the two equations

$$\begin{cases} \cos\theta_1 + \cos\theta_2 = \frac{\sqrt{3}}{s} \\ \theta_1 + \theta_2 = p^* \end{cases},$$

we obtain for the curved portion of the region

$$1-\bar{x} = 1 + \sin\frac{\pi p}{2}\sqrt{\frac{4}{3}s^2 - \frac{2}{1+\cos\pi p}}$$

If for $\cos\pi p^*$ we substitute $\frac{\sqrt{3}}{2}k$ and simplify, we obtain for the entire region

$$s^2 = \left(\frac{3}{2+\sqrt{3}k} + \frac{3\bar{x}^2}{2-\sqrt{3}k}\right) \qquad \text{for } -1+\delta < \bar{x} < 1-\delta$$

$$s^2 = \left(\frac{1+\bar{x}}{k}\right)^2 \qquad \text{for } -1 < \bar{x} < -1+\delta$$

$$s^2 = \left(\frac{1-\bar{x}}{k}\right)^2 \qquad \text{for } 1-\delta < \bar{x} < 1 .$$

The intersection of the first two equations gives the value of $1-\delta$, and is easily determined if a numerical value of k is given.

A sketch of the region for $N = 3$, $k = \sqrt{1/3}$ is given in Figure 3.

3. NUMERICAL RESULTS

3.1 Tables

A 5-point operating characteristic curve for each of eight two-sided regions is presented. These curves are for equal division of the proportion defective. In each case they are compared with the OC curves of the associated

TWO-SIDED ACCEPTANCE REGION FOR $N=3$ AND $k=\sqrt{1/3}$

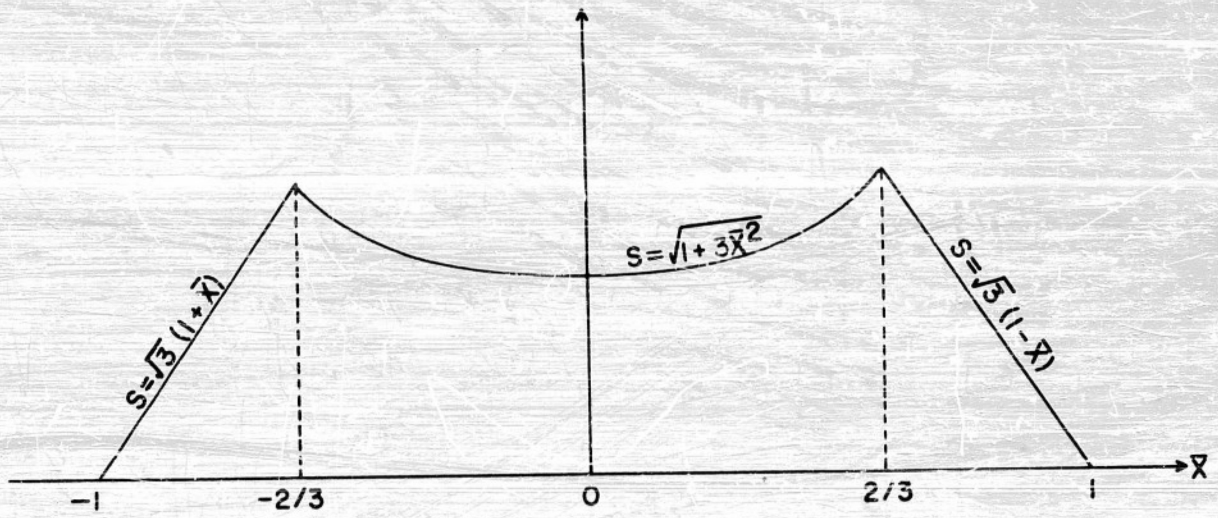

FIGURE 3

one-sided test (N,k). Comparison of the two indicates that, at least for the case of equal division of the proportion defective, the OC curve of a one-sided test may be used as a very good approximation to that of a two-sided test.

In addition, for each two-sided region, and for each fixed p, the probabilities of acceptance, L_p, for 3 possible divisions of p into (p', p''), are given. The results indicate that none of the curves of the two-sided region's band of OC curves differs widely, for practical purposes, from the OC curve of the one-sided test.

The range of N, from 3 to 100, and the use of two diverse values of k in each case, bear out the contention that the property indicated in the two preceding paragraphs is independent of the choice of the test, provided that $k \leq \frac{N-1}{\sqrt{N}}$.

It may be noted that with each pair (p', p'') there are given two numbers μ and σ. Since without loss of generality we may take the lot specification limits to be $(-1,1)$, these values μ and σ are the lot mean and lot standard deviation which, for fixed p, correspond to the division of p into p', the proportion defective due to the lower limit, -1, being violated, and p'', the proportion defective due to the upper limit, $+1$, being violated.

3.2 Detailed Construction of a Two-sided Acceptance Region

To illustrate the construction of a two-sided test, suppose that a lot of manufactured articles is considered acceptable if the per cent defective is 1.52 or less and unacceptable if the per cent defective is 5.92 or more, and suppose the maximum producer's and consumer's risks are each 10%.

(i) The first step is to find a one-sided plan (N,k) whose OC curve passes through the points $(p_1, L_{p_1}) = (.0152, .90)$ and $(p_2, L_{p_2}) = (.0592, .10)$. The procedure for obtaining N and k is given in detail in [6], hence only the result, $N = 50$, and $k = 1.8714$ is given here.

(ii) Next one computes $B_{p^*} = \frac{1}{2} - \frac{1}{2}\frac{\sqrt{N}\,k}{N-1} = \overset{.3650}{\cancel{1.8714}}$.

(iii) Calculate p* from the equation $\int_0^{B_{p^*}} t^{p-1}(1-t)^{q-1}dt = p^*$ where $p = q = \frac{N}{2} - 1 = 24$. To obtain $\overset{p^*}{\cancel{B_{p^*}}}$ one may use Tables of the Incomplete Beta Function [7]. In the notation used there $B_{p^*} = x$ and $p^* = I_x(p,q)$.

For example, if B_{p^*} were the tabled argument .40, one would find $x = .40$ in the first column on the page headed $q = 24$, and read $I_{.40}(24,24) = .0818765$ in the column headed $q = 24$. In our case $B_{p^*} = \overset{.3650}{\cancel{1.8714}}$ is not tabled and it is necessary to interpolate. The result obtained is $p^* = .0289344$.

(iv) Next, the value of p* obtained in Step(iii) is partitioned into several pairs (p_1^*, p_2^*) such that $p_1^* + p_2^* = p^*$. Judgment must be exercised in choosing p_1^* and p_2^*, for several reasons. First of all, there must be enough pairs so that a smooth curve may be drawn through the points $(\bar{x}, s)$ obtained from the pairs (p_1^*, p_2^*). Secondly, they must be chosen so that the resulting points $(\bar{x}, s)$ are spread enough to enable one to draw the entire curve. It will probably be best to make a few partitions first and then go on to the remaining steps of the construction, returning to Step (iv) as needed. It is also advantageous to choose p_1^* from as many of the tabled values in column x as is possible, in order to reduce the labor of inverse interpolation in Step (v).

(v) Now $B_{p_1^*}$ is such that $\int_0^{B_{p_1^*}} t^{p-1}(1-t)^{q-1}dt = p_1^*$. In Step (ii) B_{p^*} was known and it was required to find p*. In the present case p_1^* and p_2^*

are known and it is necessary to find $B_{p_1^*}$ and $B_{p_2^*}$. This is done by using the table inversely with the column headed $p = 24$ as the argument and the column headed x as the function. For example, if p_1^* is the tabled value .0603716 then $B_{p_1^*} = .40$. For those values of p_1^* and p_2^* which are not tabled it is necessary to interpolate in the column x.

It is necessary to exercise caution in both backward and forward interpolation in these tables to obtain accuracy. A discussion of interpolation methods and accuracy is given on page xxxv of the Incomplete Beta Tables.

The pairs (p_1^*, p_2^*) used in this example are given in Columns 2 and 3 of Table 9 of this report. The interpolated values of $B_{p_1^*}$ and $B_{p_2^*}$ are given in Columns 1 and 4 of Table 9.

(vi) Next, one solves for $\bar{x}$ and s in the equations

$$\bar{x} = \frac{U(1 - 2B_{p_2^*}) - L(2B_{p_1^*} - 1)}{2(1 - B_{p_1^*} - B_{p_2^*})}$$

$$s = \frac{(U - L)\sqrt{N}}{2(1 - B_{p_1^*} - B_{p_2^*})} .$$

The computed points corresponding to the pairs $(B_{p_1^*}, B_{p_2^*})$ are given in Columns 5 and 6 of Table 9.

(vii) The region may now be plotted. One draws the straight lines $\frac{U - \bar{x}}{k}$, $\frac{\bar{x} - L}{k}$, and plots all the points $(\bar{x}, s)$. A smooth curve is drawn through these points. This curve together with the straight lines form a closed region, which is the desired acceptance region.

3.3 TABLES

Table 1. OC curves for $N = 3$, $k = 1/\sqrt{3}$

One-sided		Two-sided					
L_p	.9481	L_p	.9515	.9511	.9481	.9481	
p	.06	p'	.03	.02	.0009	.0001	
		p''	.03	.04	.0591	.0599	
		μ	0	.0797	.3333	.4102	
		σ	.5317	.5257	.4267	.3792	
L_p	.7001	L_p	.7054	.7031	.7009	.7000	
p	.20	p'	.10	.05	.0288	.0001	
		p''	.10	.15	.1712	.1999	
		μ	0	.2269	.3333	.6308	
		σ	.7803	.7459	.7022	.4385	
L_p	.5112	L_p	.5132	.5123	.5098	.5110	
p	.30	p'	.15	.10	.05	.0001	
		p''	.15	.20	.25	.2999	
		μ	0	.2072	.4184	.7527	
		σ	.9648	.9420	.8623	.4713	
L_p	.2113	L_p	.2078	.2076	.2073	.2065	.2104
p	.50	p'	.25	.1779	.15	.05	.0001
		p''	.25	.3221	.35	.45	.4999
		μ	0	.3333	.4580	.8581	.9999
		σ	1.4826	1.4438	1.4067	1.1296	.6291
L_p	.0503	L_p	.0476	.0476	.0475		
p	.70	p'	.35	.30	.20		
		p''	.35	.40	.50		
		μ	0	.3485	1.		
		σ	2.5952	2.5715	2.3764		

Table 2. OC curves for $N = 3$, $k = 2/\sqrt{3}$

p	One-sided	Two-sided*
.0150	.9225	.9189
.0250	.8804	.8750
.0800	.6954	.6852
.1500	.5205	.5095
.2500	.3389	.3300
.3100	.2572	.2502
.4200	.1468	.1430

* This OC curve is for equal division of the proportion defective; thus, in each case $\mu = 0$.

Table 3. OC curves for $N = 25$, $k = 1.2$

One-sided		Two-sided			
L_p	.95	L_p	.9476	.9477	.9486
p		p'	.02605	.0100	.0021
		p''	.02605	.0421	.0500
		μ	0	.1479	.2702
		σ	.5149	.4934	.4437
L_p	.90	L_p	.8963	.8963	.8976
p	.0634	p'	.0317	.0200	.0034
		p''	.0317	.0434	.0600
		μ	0	.0906	.2703
		σ	.5387	.5310	.4693
L_p	.70	L_p	.6954	.6974	.6987
p	.0926	p'	.0463	.0300	.0026
		p''	.0463	.0626	.0900
		μ	0	.1018	.3515
		σ	.5946	.5858	.4837
L_p	.50	L_p	.4968	.4959	.5025
p	.1178	p'	.0589	.0400	.0078
		p''	.0589	.0778	.1100
		μ	0	.1043	.3269
		σ	.6394	.6308	.5487
L_p	.30	L_p	.2985	.2979	.3004
p	.1473	p'	.07365	.0500	.0073
		p''	.07365	.0973	.1400
		μ	0	.1182	.3866
		σ	.6901	.6798	.5678
L_p	.10	L_p	.1002	.1011	.1011
p	.1975	p'	.09875	.0600	.0075
		p''	.09875	.1375	.1900
		μ	0	.1750	.4696
		σ	.7760	.7557	.6042

Table 4. OC curves for N = 25, k = 2.5

One-sided		Two-sided			
L_p	.95	L_p	.9497	.9497	.9496
p	.0008	p'	.0004	.0003	.0001
		p''	.0004	.0005	.0007
		μ	0	.0230	.0813
		σ	.2978	.2969	.2876
L_p	.90	L_p	.9000	.9000	.8999
p	.0013	p'	.00065	.0004	.0001
		p''	.00065	.0009	.0012
		μ	0	.0330	.0930
		σ	.3115	.3098	.2988
L_p	.70	L_p	.7028	.7029	.7027
p	.0036	p'	.0018	.0012	.0006
		p''	.0018	.0024	.0030
		μ	0	.0357	.0801
		σ	.3438	.3419	.3348
L_p	.50	L_p	.5058	.5058	.5055
p	.0068	p'	.0034	.0028	.0010
		p''	.0034	.0040	.0058
		μ	0	.0217	.1006
		σ	.3695	.3689	.3563
L_p	.30	L_p	.3076	.3078	.3063
p	.0121	p'	.00605	.0031	.0011
		p''	.00605	.0090	.0110
		μ	0	.0725	.1436
		σ	.3986	.3921	.3739
L_p	.10	L_p	.1061	.1062	.1056
p	.0255	p'	.01275	.0075	.0035
		p''	.01275	.0180	.0220
		μ	0	.0743	.1453
		σ	.4476	.4415	.4244

Table 5. OC curves for N = 50, k = 1.8714

One-sided		Two-sided			
L_p	.9743	L_p	.9721	.9721	[illegible]
p	.01	p'	.005	.003	[illegible]
		p''	.005	.007	[illegible]
		μ	0	.0558	[illegible]
		σ	.3882	.3842	[illegible]
L_p	.90	L_p	.8959	.8959	[illegible]
p	.0152	p'	.0076	.0052	[illegible]
		p''	.0076	.0100	[illegible]
		μ	0	.0483	.3742
		σ	.4119	.4091	.2891
L_p	.70	L_p	.6989	.6991	.7007
p	.0236	p'	.0118	.0036	.000001
		p''	.0118	.0200	.023599
		μ	0	.1337	.4109
		σ	.4418	.4218	.2968
L_p	.50	L_p	.5014	.5013	.4990
p	.0315	p'	.01575	.0050	.000001
		p''	.01575	.0265	.031499
		μ	0	.1421	.4377
		σ	.4650	.4434	.3025
L_p	.10	L_p	.1066	.1065	.1041
p	.0592	p'	.0296	.0150	.000001
		p''	.0296	.0442	.059199
		μ	0	.1203	.5054
		σ	.5300	.5163	.3167

Table 6. OC curves for N = 50, k = 2.5

One-sided		Two-sided			
L_p	.95	L_p	.9483	.9484	.9490
p	.0015	p'	.00075	.0005	.0001
		p''	.00075	.0010	.0014
		μ	0	.0315	.1092
		σ	.3150	.3134	.2980
L_p	.90	L_p	.8982	.8984	.8991
p	.0021	p'	.00105	.0007	.0001
		p''	.00105	.0014	.0020
		μ	0	.0320	.1211
		σ	.3254	.3239	.3054
L_p	.70	L_p	.7011	.7012	.7013
p	.0042	p'	.0021	.0016	.0002
		p''	.0021	.0026	.0040
		μ	0	.0271	.1456
		σ	.3492	.3482	.3222
L_p	.50	L_p	[illegible]	[illegible]	[illegible]
p	[illegible]	p'	[illegible]	[illegible]25	[illegible]
		p''	.00325	.0040	.0060
		μ	0	.0285	.1345
		σ	.3674	.3663	.3445
L_p	.30	L_p	.3081	.3085	.3064
p	.0098	p'	.0049	.0038	.0008
		p''	.0049	.0060	.0090
		μ	0	.0301	.1423
		σ	.3873	.3861	.3626
L_p	.10	L_p	.1070	.1071	[illegible]
p	.0170	p'	.0085	.0050	[illegible]
		p''	.0085	.0120	[illegible]
		μ	0	.0654	[illegible]
		σ	.4191	.4140	[illegible]

Table 7. OC curves for $N = 100$, $k = 1.5567$

p	One-sided	Two-sided*
.0360	.95	.9465
.0406	.90	.8955
.0515	.70	.6966
.0603	.50	.5004
.0702	.30	.3042
.0866	.10	.1049

Table 8. OC curves for $N = 100$, $k = 3.0061$

p	One-sided	Two-sided*
.0004	.95	.9478
.0005	.90	.8973
.0009	.70	.6996
.0014	.50	.5034
.0020	.30	.3066
.0035	.10	.1061

* This OC curve is for equal division of the proportion defective; thus, in each case $\mu = 0$.

Table 9

1	2	3	4	5	6
$B_{p_1^*}$	p_1^*	p_2^*	$B_{p_2^*}$	$\bar{x}$	s
.17	.0000001	.0289343	.364972	.4193	.3103
.18	.0000003	.0289341	.364972	.4065	.3171
.19	.0000008	.0289336	.364971	.3931	.3243
.20	.0000021	.0289323	.364970	.3792	.3317
.21	.0000051	.0289293	.364967	.3646	.3395
.22	.0000118	.0289226	.364960	.3492	.3477
.23	.0000259	.0289085	.364945	.3332	.3563
.24	.0000543	.0288801	.364916	.3162	.3653
.25	.0001090	.0288254	.364859	.2982	.3747
.26	.0002098	.0287246	.364754	.2792	.3846
.27	.0003884	.0285460	.364567	.2588	.3949
.28	.0005212	.0282410	.364246	.2368	.4056
.29	.0011967	.0277377	.363709	.2129	.4167
.30	.0020003	.0269341	.362836	.1864	.4280
.31	.0032446	.0256898	.361441	.1566	.4392
.32	.0051153	.0238191	.359238	.1223	.4499
.33	.0078502	.0210842	.355749	.0819	.4592
.34	.0117428	.0171916	.350077	.0325	.4656
.345436	.0144672	.0144672	.345436	0	.4668

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