The Laplace Transform is frequently encountered in mathematics, physics, engineering and other fields. However, the spectral properties of the Laplace Transform tend to complicate its numerical treatment; therefore, the closely related “Truncated” Laplace Transforms are often used in applications. We have constructed efficient algorithms for the evaluation of the Singular Value Decomposition (SVD) of Truncated Laplace Transforms; in the current paper, we introduce algorithms for the evaluation of the right singular functions and singular values of Truncated Laplace Transforms. Algorithms for the computation of the left singular functions will be introduced separately in an upcoming paper. The resulting algorithms are applicable to all environments likely to be encountered in applications, including the evaluation of singular functions corresponding to extremely small singular values (e.g. $10^{-1000}$).

On the Analytical and Numerical Properties of the Truncated Laplace Transform I.

Roy R. Lederman†, Vladimir Rokhlin‡,
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1 Introduction

The Laplace Transform \( \mathcal{L} \) is a linear mapping \( L^2[0, \infty) \to L^2[0, \infty) \); for a function \( f \in L^2[0, \infty) \), it is defined by the formula:

\[
(\mathcal{L}(f))(\omega) = \int_0^\infty e^{-t\omega} f(t) dt.
\] (1)

As is well-known, \( \mathcal{L} \) has a continuous spectrum, and \( \mathcal{L}^{-1} \) is not continuous (see, for example, [1]). These and related properties tend to complicate the numerical treatment of \( \mathcal{L} \).

In addressing these problems, we find it useful to draw an analogy between the numerical treatment of the Laplace Transform, and the numerical treatment of the Fourier Transform \( \mathcal{F} \); for a function \( f \in L^1(\mathbb{R}) \), the latter is defined by the formula:

\[
(\mathcal{F}(f))(\omega) = \int_{-\infty}^{\infty} e^{-it\omega} f(t) dt,
\] (2)

where \( \omega \in \mathbb{R} \).

In various applications in mathematics and engineering, it is useful to define the “Truncated” Fourier Transform \( \mathcal{F}_c : L^2[-1,1] \to L^2[-1,1] \); for a given \( c > 0 \), \( \mathcal{F}_c \) of a function \( f \in L^2[-1,1] \) is defined by the formula:

\[
(\mathcal{F}_c(f))(\omega) = \int_{-1}^{1} e^{-ict\omega} f(t) dt.
\] (3)

The operator \( \mathcal{F}_c \) has been analyzed extensively; one of most notable observations, made by Slepian et al. around 1960, was that the integral operator \( \mathcal{F}_c \) commutes with a second order differential operator (see [2]). This property of \( \mathcal{F}_c \) was used in analytical and numerical investigations of the eigendecomposition of this operator, for example in [2, 3, 4, 5, 6, 7, 8].

For \( 0 < a < b < \infty \), the linear mapping \( \mathcal{L}_{a,b} : L^2[a,b] \to L^2[0, \infty) \) defined by the formula

\[
(\mathcal{L}_{a,b}(f))(\omega) = \int_a^b e^{-t\omega} f(t) dt,
\] (4)

will be referred to as the Truncated Laplace Transform of \( f \); obviously, \( \mathcal{L}_{a,b} \) is a compact operator (see, for example, [1]).

The Singular Value Decomposition (SVD) of \( \mathcal{L}_{a,b} \) has been analyzed, inter alia, in [1] and [9]; Bertero and Grünbaum observed that each of the symmetric operators \( (\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b} \) and \( \mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^* \) commutes with a differential operator (see [9]). Despite [9, 10, 11, 12, 13, 14, 15], much more is known about the numerical and analytical properties of \( \mathcal{F}_c \) than about the properties of \( \mathcal{L}_{a,b} \).

We have constructed algorithms for the efficient evaluation of the of the SVD of \( \mathcal{L}_{a,b} \). In this paper, we introduce algorithms for the efficient evaluation of the right singular functions and singular values of \( \mathcal{L}_{a,b} \). The remaining algorithms, including the algorithm for the numerical
evaluation of the left singular functions, will be discussed in upcoming papers along with additional analytical results.

The paper is organized as follows. Section 2 summarizes the various standard mathematical facts and simple derivations that are used later in the paper. Section 3 contains the derivation of various properties of the right singular functions of the Truncated Laplace Transform, which are used in the algorithms. Section 4 describes the algorithms for the evaluation of the right singular functions and singular values of the Truncated Laplace Transform. Section 5 contains numerical results obtained using the algorithms. Section 6 contains generalizations and conclusions.
2 Preliminaries

2.1 The Legendre Polynomials

In this subsection we summarize some of the properties of the standard Legendre Polynomials, and restate these properties for shifted and normalized forms of the Legendre Polynomials.

We define the *Shifted Legendre Polynomial* of degree \( k = 0, 1, \ldots \), which we will be denoting by \( P_k^* \), by the formula

\[
P_k^*(x) = P_k(2x - 1),
\]

where \( P_k \) is the Legendre Polynomial of degree \( k \); the standard definition of the Legendre Polynomials can be found, inter alia, in [16].

As is well-known, the Legendre Polynomials form an orthogonal basis in \( L^2[-1,1] \), but they are not normalized; it immediately follows that the Shifted Legendre Polynomials form an orthogonal basis in \( L^2[0,1] \) and that they are also not normalized. Therefore, we find it convenient to define the *Normalized Shifted Legendre Polynomial* of degree \( k = 0, 1, \ldots \), which we will be denoting by \( \overline{P}_k^* \), by the formula

\[
\overline{P}_k^*(x) = P_k^*(x)\sqrt{2k + 1};
\]

the Normalized Shifted Legendre Polynomials \( \overline{P}_0^*, \overline{P}_1^*, \ldots \) form an orthonormal basis in \( L^2[0,1] \).

The following well-known properties of the Legendre Polynomials can be found, inter alia, in [16], [17]:

\[
\int_{-1}^{1} (P_k(x))^2 \, dx = \frac{2}{2k + 1}
\]

\[
(k + 1)P_{k+1}(x) = (2k + 1)xP_k(x) - kP_{k-1}(x)
\]

\[
(1 - x^2) \frac{d}{dx} P_k(x) = -kxP_k(x) + kP_{k-1}(x)
\]

\[
\frac{d}{dx} \left((1 - x^2) \frac{d}{dx} P_k(x)\right) = -k(1 + k)P_k(x)
\]

\[
(2k + 1)P_k(x) = \frac{d}{dx} (P_{k+1}(x) - P_{k-1}(x))
\]
The following properties of the Shifted Legendre Polynomials are easily derived from the properties of the Legendre Polynomials by substituting (5) into (7-11).

\[ \int_{0}^{1} (P_{k}^{*}(x))^{2} dx = \frac{1}{2k + 1} \] (12)

\[ xP_{k}^{*}(x) = \frac{1}{2} \left( \frac{kP_{k-1}^{*}(x)}{1 + 2k} + P_{k}^{*}(x) + \frac{(1 + k)P_{k+1}^{*}(x)}{1 + 2k} \right) \] (13)

\[ x(1 - x) \frac{d}{dx} P_{k}^{*}(x) = \frac{k(1 + k)}{2(1 + 2k)} (P_{k-1}^{*}(x) - P_{k+1}^{*}(x)) \] (14)

\[ \frac{d}{dx} \left( x(1 - x) \frac{d}{dx} P_{k}^{*}(x) \right) = -k(1 + k)P_{k}^{*}(x) \] (15)

### 2.2 The Legendre Functions of the second kind

As is well-known, the Legendre Polynomial \( P_{k}(x) \) is not the only solution for the differential equation (10) in the interval \([-1, 1]\); the other solution is the Legendre Function of the second kind \( Q_{k}(x) \), defined by the formula

\[ Q_{k}(x) = \frac{1}{2} \int_{-1}^{1} (x - t)^{-1} P_{k}(t) dt, \] (16)

where \( P_{k} \) is the Legendre Polynomial.

Having defined the Shifted Legendre Polynomials, we find it convenient to similarly define the \textit{Shifted Legendre Function of the second kind} of degree \( k \), which we will be denoting by \( Q_{k}^{*} \), by the formula

\[ Q_{k}^{*}(x) = Q_{k}(2x - 1) \] (17)

The following identities can be found, for example, in [16], [17]:

\[ Q_{k}(z) = (-1)^{k+1} Q_{k}(-z), \] (18)

\[ Q_{k}(z) = \int_{0}^{\infty} \frac{d\varphi}{(z + \sqrt{z^2 - 1} \cosh(\varphi))^{k+1}} \] (19)

By (6), (16), (18) and (17),

\[ \int_{0}^{1} (x + y)^{-1} P_{k}^{*}(x) dx = 2(-1)^{k} Q_{k}(y + 1) \sqrt{2k + 1} \] (20)

for all \( y > 0 \).
2.3 Singular Value Decomposition (SVD) of integral operators

The Singular Value Decomposition (SVD) of integral operators and its key properties are summarized in the following theorem, which can be found, for example, in [18].

**Theorem 2.1.** Suppose that the function \( K : [c,d] \times [a,b] \to \mathbb{R} \) is square integrable, and \( T : L^2[a,b] \to L^2[c,d] \) is defined by the formula

\[
(T(f))(x) = \int_a^b K(x,t) f(t) dt.
\]

Then, there exist two orthonormal sequences of functions \( u_0, u_1, \ldots \), where \( u_n : [a,b] \to \mathbb{R} \) and \( v_0, v_1, \ldots \), where \( v_n : [c,d] \to \mathbb{R} \), and a sequence \( \alpha_0, \alpha_1, \ldots \in \mathbb{R} \), where \( \alpha_0 \geq \alpha_1 \geq \ldots \geq 0 \), such that

\[
(T(f))(x) = \sum_{n=0}^{\infty} \alpha_n \left( \int_a^b u_n(t) f(t) dt \right) v_n(x)
\]

for any \( f \in L^2[a,b] \). The sequence \( \alpha_0, \alpha_1, \ldots \) is uniquely determined by \( K \).

The functions \( u_0, u_1, \ldots \) are referred to as the **right singular functions**, the functions \( v_0, v_1, \ldots \) are referred to as the **left singular functions**, and the values \( \alpha_0, \alpha_1, \ldots \) are referred to as the **singular values** of the operator \( T \). Together, the right singular functions, the left singular functions and the singular values are referred to as the SVD of the operator \( T \).

It immediately follows from Theorem 2.1 that

\[
T(u_n) = \alpha_n v_n,
\]

\[
T^*(v_n) = \alpha_n u_n.
\]

**Observation 2.2.** The right singular functions \( u_0, u_1, \ldots \) of \( T \) are eigenfunctions of the operator \( T^* \circ T \) and the left singular functions \( v_0, v_1, \ldots \) are eigenfunctions of the operator \( T \circ T^* \); the singular values \( \alpha_0, \alpha_1, \ldots \) of \( T \) are the square roots of the eigenvalues of \( T^* \circ T \) and \( T \circ T^* \). In other words, for every \( n = 0, 1, \ldots \),

\[
((T^* \circ T)(u_n))(\tau) = \int_c^d K(x,\tau) \left( \int_a^b K(x,t) u_n(t) dt \right) dx = \alpha_n^2 u_n(\tau)
\]

and

\[
((T \circ T^*)(v_n))(\xi) = \int_a^b K(\xi,t) \left( \int_c^d K(x,t) v_n(x) dx \right) dt = \alpha_n^2 v_n(\xi)
\]
Remark 2.3. The function $K$ can be expressed using the singular functions as follows (see [18]),

$$K(x,t) = \sum_{n=0}^{\infty} v_n(x) \alpha_n u_n(t)$$

and it can be approximated by truncation of small singular values (also see [18]):

$$K(x,t) \simeq \sum_{n=0}^{p} v_n(x) \alpha_n u_n(t)$$
3 Analytical apparatus

3.1 Bounds on the Legendre functions of the second kind

For any \(x > 1\), the function \(Q_k(x)\) (defined in (16)) decays rapidly as \(k\) grows. More formally, for any \(\delta > 0\) and \(k = 0, 1, \ldots\), there is a uniform bound on \(|Q_k(x)|\), where \(x \geq 1 + \delta\); the bound decreases superalgebraically as \(k\) grows. The following lemma provides an explicit bound.

Lemma 3.1. Suppose that \(\delta > 0\); then, for all \(y \geq 0\),

\[
|Q_k(1 + \delta + y)| < \left( \log \left( 2 \frac{1 + \tilde{\delta}}{\delta} \right) + 1 \right) \left( \frac{1}{1 + \tilde{\delta}} \right)^{k+1},
\]

where

\[
\tilde{\delta} = \sqrt{(1 + \delta)^2 - 1}
\]

and \(Q_k\) is defined in (16).

Proof. By (16),

\[
|Q_k(1 + \delta + y)| = \int_0^\infty \frac{d\varphi}{(1 + \delta + y + \cosh(\varphi)\sqrt{(1 + \delta + y)^2 - 1})^{k+1}}.
\]

Since \((1 + \delta + y) \geq (1 + \delta)\),

\[
|Q_k(1 + \delta + y)| \leq \int_0^\infty \frac{d\varphi}{(1 + \delta + \cosh(\varphi)\sqrt{(1 + \delta)^2 - 1})^{k+1}}.
\]

Since \(\delta > 0\), clearly \(\tilde{\delta} > 0\) and therefore,

\[
|Q_k(1 + \delta + y)| < \int_0^\nu \frac{d\varphi}{\left( 1 + \tilde{\delta} \cosh(\varphi) \right)^{k+1}}.
\]

Introducing the notation

\[
\nu = \log \left( 2 \frac{1 + \tilde{\delta}}{\delta} \right),
\]

we break the integral in (33) into integrals on the two intervals \([0, \nu)\) and \([\nu, \infty)\):

\[
|Q_k(1 + \delta + y)| < \int_0^\nu \frac{d\varphi}{\left( 1 + \tilde{\delta} \cosh(\varphi) \right)^{k+1}} + \int_\nu^\infty \frac{d\varphi}{\left( 1 + \tilde{\delta} \cosh(\varphi) \right)^{k+1}}.
\]
Clearly,
\[
\frac{1}{(1 + \delta \cosh(\varphi))^{k+1}} \leq \frac{1}{(1 + \delta)^{k+1}} 
\]
and
\[
\frac{1}{(1 + \delta \cosh(\varphi))^{k+1}} \leq \frac{1}{(\delta \exp(\varphi)/2)^{k+1}},
\]
so that,
\[
|Q_k(1 + \delta + y)| < \nu \left( \frac{1}{(1 + \delta)^{k+1}} + \int_{\nu}^{\infty} \frac{d\varphi}{(\delta \exp(\varphi)/2)^{k+1}} \right).
\]
Substituting (34) into (38), we obtain
\[
|Q_k(1 + \delta + y)| < \frac{1}{(1 + \delta)^{k+1}} \left( \log \left( \frac{2 + \delta}{\delta} \right) + \frac{1}{k+1} \right),
\]
and from it, we obtain (29).

\[ \square \]

**Corollary 3.2.** By, (17) and (29),
\[
|Q^*_k(1 + \delta/2 + y)| < \left( \log \left( \frac{2 + \delta}{\delta} \right) + 1 \right) \left( \frac{1}{1 + \delta} \right)^{k+1}
\]
where
\[
\tilde{\delta} = \sqrt{(1 + \delta)^2 - 1},
\]
\[ \delta, y > 0 \text{ and } Q^*_k \text{ is defined in (17)}. \]

### 3.2 The Truncated Laplace Transform

**Definition 3.3.** For any pair of real numbers \(a, b\), such that \(0 < a < b < \infty\), the Truncated Laplace Transform \(L_{a,b}\) is the linear mapping \(L^2[a, b] \to L^2(0, \infty)\), defined by the formula
\[
(L_{a,b}(f))(\omega) = \int_a^b e^{-t\omega} f(t) dt,
\]
Obviously, the adjoint of \(L_{a,b}\) is
\[
((L_{a,b})^*(g))(t) = \int_0^\infty e^{-t\omega} g(\omega) d\omega.
\]
The operators \(L_{a,b}\) and \((L_{a,b})^*\) are compact, the range of \((L_{a,b})^*\) is dense in \(L^2[a, b]\) and the range of \(L_{a,b}\) is dense in \(L^2(0, \infty)\) (see, for example, [1]).
3.3 The SVD of the Truncated Laplace Transform

By Theorem 2.1, there exist an orthonormal sequence of right singular functions $u_0, u_1, \ldots \in L^2[a, b]$, an orthonormal sequence of left singular functions $v_0, v_1, \ldots \in L^2[0, \infty)$ and a sequence of real numbers $\alpha_0, \alpha_1, \ldots \in \mathbb{R}$ such that

$$(\mathcal{L}_{a,b}(f))(\omega) = \sum_{n=0}^{\infty} \alpha_n \left( \int_a^b u_n(t)f(t)dt \right) v_n(\omega),$$

and for all $n = 0, 1, \ldots$,

$$\mathcal{L}_{a,b}(u_n) = \alpha_n v_n,$$

$$(\mathcal{L}_{a,b})^*(v_n) = \alpha_n u_n,$$

and

$$\alpha_n \geq \alpha_{n+1} \geq 0.$$

Remark 3.4. The multiplicity of the singular values of $\mathcal{L}_{a,b}$ is one (see [9]); in other words, for all $n = 0, 1, \ldots$

$$\alpha_n > \alpha_{n+1}.$$

Remark 3.5. According to Observation 2.2, the right singular functions $u_0, u_1, \ldots$ of $\mathcal{L}_{a,b}$ are eigenfunctions of the integral operator $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b} : L^2[a, b] \to L^2[a, b]$ given by the formula

$$((\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b})(f)(t) = \int_0^{\infty} e^{-\omega t} \left( \int_a^b e^{-\omega s} f(s) ds \right) d\omega = \int_a^b \frac{1}{t+s} f(s) ds,$$

and the corresponding eigenvalues of $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ are $\alpha_0^2, \alpha_1^2, \ldots$, where $\alpha_n$ is the singular value of $\mathcal{L}_{a,b}$ associated with the right singular function $v_n$. In other words,

$$((\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b})(u_n)(t) = \int_a^b \frac{1}{t+s} u_n(s) ds = \alpha_n^2 u_n(t).$$

Similarly, the left singular functions $v_n$ of $\mathcal{L}_{a,b}$ are eigenfunctions of the integral operator $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^* : L^2[0, \infty) \to L^2[0, \infty)$ given by the formula

$$((\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*)(g))(\omega) =$$

$$\int_a^b e^{-\omega t} \left( \int_0^{\infty} e^{-\rho t} g(\rho) d\rho \right) dt = \int_0^{\infty} \frac{e^{-a(\omega+\rho)} - e^{-b(\omega+\rho)}}{\omega + \rho} g(\rho) d\rho,$$

and the corresponding eigenvalues $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$ are $\alpha_0^2, \alpha_1^2, \ldots$. In other words,

$$((\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*)(v_n))(\omega) = \int_0^{\infty} \frac{e^{-a(\omega+\rho)} - e^{-b(\omega+\rho)}}{\omega + \rho} v_n(\rho) d\rho = \alpha_n^2 v_n(\omega).$$
3.4 The differential operators $\tilde{D}_t$ and $\hat{D}_\omega$ associated with the singular functions of $L_{a,b}$

In this subsection we summarize several properties related to the differential operator $\tilde{D}_t$, defined by the formula

$$
\left(\tilde{D}_t(f)\right)(t) = \frac{d}{dt} \left( (t^2 - a^2)(b^2 - t^2) \frac{d}{dt} f(t) \right) - 2(t^2 - a^2)f(t),
$$

(53)

where $f \in C^2[a,b]$; and properties related to the differential operator $\hat{D}_\omega$, defined by the formula

$$
\left(\hat{D}_\omega(f)\right)(\omega) =
\frac{d^2}{d\omega^2} \left( \omega^2 \frac{d^2}{d\omega^2} f(\omega) \right) + (a^2 + b^2) \frac{d}{d\omega} \left( \omega^2 \frac{d}{d\omega} f(\omega) \right) + (-a^2b^2\omega^2 + 2a^2)f(\omega),
$$

(54)

where $f \in C^4[0,\infty) \cap L^2[0,\infty)$. For a derivation of these properties, see [9].

Theorem 3.6. The differential operator $\tilde{D}_t$, defined in (53), commutes with the integral operator $(L_{a,b})^* \circ L_{a,b}$, (specified in (49)) in $L^2[a,b]$. In other words,

$$
\tilde{D}_t \circ ((L_{a,b})^* \circ L_{a,b}) = ((L_{a,b})^* \circ L_{a,b}) \circ \tilde{D}_t
$$

(55)

Theorem 3.7. The differential operator $\hat{D}_\omega$, defined in (54), commutes with the integral operator $L_{a,b} \circ (L_{a,b})^*$, (specified in (51)) in $L^2[0,\infty)$. In other words,

$$
L_{a,b} \circ (L_{a,b})^* \circ \hat{D}_\omega = \hat{D}_\omega \circ L_{a,b} \circ (L_{a,b})^*
$$

(56)

Theorem 3.8. The right singular functions $u_0, u_1, \ldots$ (defined in (45)) of $L_{a,b}$ (defined in (42)) are also the eigenfunctions of $\tilde{D}_t$.

Theorem 3.9. The left singular functions $v_0, v_1, \ldots$ (defined in (46)) of $L_{a,b}$ (defined in (42)) are also the eigenfunctions of $\hat{D}_\omega$.

We denote the eigenvalues of the differential operator $\tilde{D}_t$ by $\tilde{\chi}_0, \tilde{\chi}_1, \ldots$, and the eigenvalues of the differential operator $\hat{D}_\omega$ by $\chi_0^*, \chi_1^*, \ldots$. By Theorem 3.8, the singular function $u_n$ is the solution to the differential equation

$$
\frac{d}{dt} \left( (t^2 - a^2)(b^2 - t^2) \frac{d}{dt} u_n(t) \right) - 2(t^2 - a^2)u_n(t) = \tilde{\chi}_n u_n(t),
$$

(57)

and by Theorem 3.9, the left singular function $v_n$ is the solution to the differential equation

$$
- \frac{d^2}{d\omega^2} \left( \omega^2 \frac{d^2}{d\omega^2} v_k(\omega) \right) + (a^2 + b^2) \frac{d}{d\omega} \left( \omega^2 \frac{d}{d\omega} v_k(\omega) \right) + (-a^2b^2\omega^2 + 2a^2)v_k(\omega) = \chi_k^* v_k(\omega).
$$

(58)
Remark 3.10. The singular values $\alpha_n$ (defined in (45)) of the integral operator $L_{a,b}$ are known to decay exponentially as $n$ grows; consequently, the direct numerical computation of the singular functions of $L_{a,b}$ beyond the first few singular functions is impossible.

The differential operators $\hat{D}_t$ and $\hat{D}_\omega$ are advantageous in the numerical treatment of the singular functions $u_n$ and $v_n$ because their eigenvalues increase with $n$, and because the differential operators can be treated using numerical tools developed for differential equations. Such tools are developed below in Sections 3.4, 3.5, 3.6, 3.7 and 3.8 and used to construct the SVD of the operator $L_{a,b}$ in Section 4.

3.5 The operator $T_{\gamma}$ and the function $\psi_n$

The right singular functions $u_n$ (see (45)) of $L_{a,b}$ (see (42)) are defined of the interval $[a,b]$; we find it convenient to shift this interval to the interval $[0,1]$.

We introduce the operator $T_{\gamma} : [0,1] \rightarrow [0,\infty)$, defined be the formula

$$\langle T_{\gamma}(f) \rangle (\tilde{\omega}) = \int_0^1 e^{-\tilde{\omega}(x+\frac{1}{\gamma})} f(x)dx.$$  \hspace{1cm} (59)

This operator is related to the operator $L_{a,b}$, where

$$\gamma = \frac{b}{a}$$ \hspace{1cm} (60)

by a the changes of variables

$$x = \frac{t - a}{b - a}, \quad t = a + (b - a)x,$$ \hspace{1cm} (61)

and

$$\tilde{\omega} = \omega(b-a), \quad \omega = \frac{\tilde{\omega}}{b-a},$$ \hspace{1cm} (62)

We denote the singular values of $T_{\gamma}$ by $\tilde{\alpha}_0,\tilde{\alpha}_1,...$, the right singular functions of $T_{\gamma}$ by $\psi_0,\psi_1,...$ and the left singular functions of $T_{\gamma}$ by $\tilde{v}_0,\tilde{v}_1,...$

Suppose now that $0 < a < b < \infty$. Then a simple calculation shows that for any $n = 0,1,2,...$

$$\psi_n(x) = \sqrt{b-a} \ u_n(a+(b-a)x),$$ \hspace{1cm} (63)

$$\tilde{v}_n(\tilde{\omega}) = \frac{1}{\sqrt{b-a}} v_n(\tilde{\omega}/(b-a)),$$ \hspace{1cm} (64)

(up to the ambiguity in sign) and

$$\tilde{\alpha}_n = \alpha_n,$$ \hspace{1cm} (65)
where \( \alpha_n, u_n \) and \( v_n \) are a singular value, right singular function and left singular function of \( L_{a,b} \), and where \( \tilde{\alpha}_n, \psi_n \) and \( \tilde{v}_n \) are a singular value, right singular function and left singular function of \( T_\gamma \) and \( \gamma = b/a \).

The operator \( T_\gamma^* \circ T_\gamma \) is defined by the formula

\[
((T_\gamma^* \circ T_\gamma)(f))(x) = \int_0^1 \frac{1}{x + y + \beta} f(y) dy,
\]

with \( f \in L^2[0,1] \), and \( \beta \) is defined by the formula:

\[
\beta = \frac{2}{\gamma - 1} = \frac{2a}{b - a}.
\]

By Observation 2.2, the right singular functions \( \psi_0, \psi_1, \ldots \) of \( T_\gamma \) are the eigenfunctions of the integral operator \( T_\gamma^* \circ T_\gamma \). Clearly, \( T_\gamma^* \circ T_\gamma \) has the same eigenvalues as \( (L_{a,b})^* \circ L_{a,b} \):

\[
((T_\gamma^* \circ T_\gamma)(\psi_n))(x) = \int_0^1 \frac{1}{x + y + \beta} \psi_n(y) dy = \alpha_n^2 \psi_n(x).
\]

Similarly, by (53) and (61), \( \psi_0, \psi_1, \ldots \) are the eigenfunctions of the differential operator \( D_x \), defined by the formula

\[
(D_x(f))(x) = \frac{d}{dx} \left( x(1 - x)(\beta + x)(\beta + 1 + x) \frac{d}{dx} f(x) \right) - 2x(x + \beta)f(x).
\]

In other words, \( \psi_n \) is the solution to the differential equation

\[
\frac{d}{dx} \left( x(1 - x)(\beta + x)(\beta + 1 + x) \frac{d}{dx} \psi_n(x) \right) - 2x(x + \beta)\psi_n(x) - \chi_n \psi_n(x) = 0,
\]

with \( \chi_0, \chi_1, \ldots \) the eigenvalues of \( D_x \).

**Remark 3.11.** A simple computation shows that the eigenvalues \( \chi_0, \chi_1, \ldots \) of the operator \( D_x \) are related to the eigenvalues \( \tilde{\chi}_0, \tilde{\chi}_1, \ldots \) of the operator \( \tilde{D}_t \), defined in (53) by the formula

\[
\tilde{\chi}_n = (b - a)^2 \chi_n.
\]

**Remark 3.12.** The operator \( T_\gamma \) is determined by the single parameter \( \gamma \); therefore, the singular value decomposition of the operator \( L_{a,\gamma a} \) is determined by \( \gamma \), in the following sense. The sequence of singular values \( \alpha_0, \alpha_1, \ldots \) of the truncated Laplace transform \( L_{a,\gamma a} \) depends only on \( \gamma \), and is independent of the value of \( a \). If \( \gamma = \frac{b}{a} = \frac{\tilde{b}}{\tilde{a}} \), then the the sequences of right and left singular functions of \( L_{a,b} \) are identical to those of the truncated Laplace transform \( L_{\tilde{a},\tilde{b}} \) up to trivial scaling.
3.6 Expansion of $\psi_n$ in the basis of Legendre Polynomials

Let $f$ be a smooth function in $L^2[0, 1]$, then $f$ can be expressed in the basis of Normalized Shifted Legendre Polynomials $P_k^*$ (defined in (6)); let $h = (h_0, h_1, \ldots)^\top$ be the vector where

$$h_k = \int_0^1 f(x) P_k^*(x) \, dx,$$

then clearly $h$ is the vector of coefficients in the expansion,

$$f(x) = \sum_{k=0}^{\infty} h_k P_k^*(x).$$

(73)

We introduce the notation $h_n = (h_{n0}, h_{n1}, ...)^\top$ for the vector of coefficients of the expansion of the function $\psi_n$ (defined in (63)) in the basis of Normalized Shifted Legendre Polynomials; where the element $h_{nk}$ is defined by the formula

$$h_{nk} = \int_0^1 \psi_n(x) P_k^*(x) \, dx,$$

so that

$$\psi_n(x) = \sum_{k=0}^{\infty} h_{nk} P_k^*(x).$$

(75)

3.7 Decay of the coefficients

Since the function $\psi_n$ (defined in (63)) is a smooth solution of a differential equation (specified in (70)), we expect the coefficients $h_{nk}$ in the expansion of $\psi_n$ to decay rapidly. In this subsection we provide an estimate for the actual decay.

Lemma 3.13. Suppose that $0 < \beta < \infty$. Then,

$$\int_0^1 \left( \int_0^1 \frac{1}{x + y + \beta} P_k^*(x) \, dx \right)^2 \, dy \leq \left( \frac{2\sqrt{2k+1}}{(1 + \tilde{\beta})^{k+1}} \left( \log \left( \frac{1 + \tilde{\beta}}{\beta} \right) + 1 \right) \right)^2,$$

(76)

where $P_k^*$ is defined in (6) and

$$\tilde{\beta} = \sqrt{(1 + (2\beta))^2 - 1} = 2\sqrt{\beta(1 + \beta)}.$$
Proof. Based on (20),
\[
\left| \int_0^1 (x + y + \beta)^{-1} P^*_k(x) dx \right| = 2Q^*_k(y + \beta + 1)\sqrt{2n + 1}, \tag{78}
\]
where \(Q^*_k\) is defined in (17). So, by Corollary 3.2,
\[
\left| \int_0^1 (x + y + \beta)^{-1} P^*_k(x) dx \right| < \frac{2\sqrt{2k + 1}}{1 + \beta} \left( \log \left( \frac{2 + \beta}{\beta} \right) + 1 \right). \tag{79}
\]
By squaring (79) and integrating over \(y\), we obtain (76). \(\square\)

Lemma 3.14. Suppose that \(h^n_k\) is the \(k + 1\)-th coefficient in the expansion of \(\psi_n\), specified in (74); then,
\[
|h^n_k| \leq c_n \sqrt{2k + 1} \left( \frac{1}{1 + \beta} \right)^{k+1}, \tag{80}
\]
where, \(c_n\) is defined by the formula
\[
c_n = 2\alpha_n^{-2} \left( \log \left( \frac{2 + \beta}{\beta} \right) + 1 \right) = 2\alpha_n^{-2} \left( \log \left( 2 + \sqrt{\frac{\gamma - 1}{2(\gamma + 1)}} \right) + 1 \right), \tag{81}
\]
and \(\tilde{\beta}\) is defined by the formula
\[
\tilde{\beta} = \sqrt{(1 + (2\beta)^2) - 1} = \sqrt{4\beta(1 + \beta)} = 2\sqrt{2} \frac{\sqrt{\gamma + 1}}{\gamma - 1}. \tag{82}
\]
The parameters \(\alpha_n, \beta\) and \(\gamma\) in (81) and (82) are defined in (45), (67) and (60), respectively.

Proof. We substitute (68) into (74) and change the order of integration:
\[
h^n_k = \alpha_n^{-2} \int_0^1 \int_0^1 \frac{1}{x + y + \beta} P^*_k(x) \psi_n(y) dx dy = \alpha_n^{-2} \int_0^1 \psi_n(y) \left( \int_0^1 \frac{1}{x + y + \beta} P^*_k(x) dx \right) dy. \tag{83}
\]
By the Cauchy-Schwarz inequality,
\[
|h^n_k| \leq \alpha_n^{-2} \sqrt{\int_0^1 (\psi_n(y))^2 dy} \sqrt{\int_0^1 \left( \int_0^1 \frac{1}{x + y + \beta} P^*_k(x) dx \right)^2 dy}. \tag{84}
\]
Now, by (76),
\[
|h^n_k| \leq \alpha_n^{-2} \sqrt{1} \left( \frac{2\sqrt{2k + 1}}{1 + \beta} \left( \log \left( \frac{2 + \beta}{\beta} \right) + 1 \right) \right). \tag{85}
\]
\(\square\)
3.8 A matrix representation of the differential operator $D_x$ in the basis of $P_k^*$

The purpose of this subsection is to express the differential operator $D_x$ in the basis of Normalized Shifted Legendre Polynomials $P_k^*$ as the matrix $M$ described in Lemma 3.15. Theorem 3.16 shows that the matrix $M$ is in fact a five-diagonal matrix; and Corollary 3.17 provides the relation between the eigenvectors of $M$ and the functions $\psi_n$ defined in (63).

**Lemma 3.15.** Let $f$ be a smooth function with the expansion $h = (h_0, h_1, ...)^\top$ specified in (73):

$$f(x) = \sum_{k=0}^{\infty} h_k P_k^*(x).$$  \hfill (86)

Suppose that $\varphi = D_x(f)$, with the expansion $\eta = (H_0, H_1, ...)^\top$ such that

$$\varphi(x) = \sum_{k=0}^{\infty} \eta_k P_k^*(x).$$  \hfill (87)

Then,

$$\eta = Mh,$$  \hfill (88)

where the matrix elements $M_{jk}$ of $M$ are specified via the formula

$$M_{jk} = \int_0^1 P_j^*(x) \left( D_x(P_k^*) \right)(x) dx,$$  \hfill (89)

with $0 \leq j, k < \infty$.

**Proof.** By the linearity of the differential operator $D_x$ (defined in (69)),

$$\varphi(x) = (D_x(f))(x) = \sum_{k=0}^{\infty} h_k \left( D_x(P_k^*) \right)(x).$$  \hfill (90)

Combining (86) and (90),

$$\sum_{k=0}^{\infty} \eta_k P_k^*(x) = \sum_{k=0}^{\infty} h_k \left( D_x(P_k^*) \right)(x).$$  \hfill (91)

Now, by multiplying both sides of (91) by $P_j^*$ and integrating, we have

$$\eta_j = \int_0^1 \left( \sum_{k=0}^{\infty} h_k \left( D_x(P_k^*) \right)(x) \right) P_j^*(x) dx,$$  \hfill (92)
and by linearity

\[ \eta_j = \sum_{k=0}^{\infty} h_k \left( \int_0^1 \mathcal{P}_j^r(x) \left( D_x(\mathcal{P}_k^r) \right)(x) dx \right). \] (93)

**Theorem 3.16.** For any \( k \geq 0 \),

\[
\left( D_x(\mathcal{P}_k^r) \right)(x) =
\begin{align*}
&= -\frac{(k-1)^2k^2}{4\sqrt{2k-3}(2k-1)\sqrt{2k+1}} \mathcal{P}_{k-2}^r(x) \\
&\quad - \frac{k^3(1+\beta)}{\sqrt{2k-1}\sqrt{2k+1}} \mathcal{P}_{k-1}^r(x) \\
&\quad - \left( -4-6\beta-2k(2+3\beta)+k^2(7+12\beta+2\beta^2)+(2k^3+k^4)(7+16\beta+8\beta^2) \right) \mathcal{P}_k^r(x) \\
&\quad - \frac{(k+1)^3(1+\beta)}{\sqrt{2k+1}\sqrt{2k+3}} \mathcal{P}_{k+1}^r(x) \\
&\quad - \frac{(k+1)^2(k+2)^2}{4\sqrt{2k+1}(2k+3)\sqrt{2k+5}} \mathcal{P}_{k+2}^r(x),
\end{align*}
\] (94)

where \( \mathcal{P}_k^r \) is the Normalized Shifted Legendre Polynomial defined in (6) and \( \beta = \frac{2a}{b-a} \) is defined in (67).

In other words, \( M \) is the five-diagonal matrix

\[
M_{k-2,k} = -\frac{(k-1)^2k^2}{4\sqrt{2k-3}(2k-1)\sqrt{2k+1}} ,
\]

\[
M_{k-1,k} = -\frac{k^3(1+\beta)}{\sqrt{2k-1}\sqrt{2k+1}} ,
\]

\[
M_{k,k} = \frac{-4-6\beta-2k(2+3\beta)+k^2(7+12\beta+2\beta^2)+(2k^3+k^4)(7+16\beta+8\beta^2)}{2(2k-1)(2k+3)} ,
\]

\[
M_{k+1,k} = -\frac{(k+1)^3(1+\beta)}{\sqrt{2k+1}\sqrt{2k+3}} ,
\]

\[
M_{k+2,k} = -\frac{(k+1)^2(k+2)^2}{4\sqrt{2k+1}(2k+3)\sqrt{2k+5}}.
\] (95)

**Proof.** By the definition of \( D_x \) in (69),

\[
\left( D_x(\mathcal{P}_k^r) \right)(x) =
\begin{align*}
&= \frac{d}{dx} \left( (\beta + x)(\beta + 1 + x)x(1-x) \frac{d}{dx} \mathcal{P}_k^r(x) \right) - 2x(x + \beta)\mathcal{P}_k^r(x).
\end{align*}
\] (96)
Using the chain rule,

\[
(D_x(P^*_k))(x) = \\
= \left(\frac{d}{dx}(\beta + x)(\beta + 1 + x)\right) \left(x(1 - x) \frac{d}{dx} P^*_k(x)\right) \\
+ (\beta + x)(\beta + 1 + x) \frac{d}{dx} \left(x(1 - x) \frac{d}{dx} P^*_k(x)\right) - 2x(x + \beta)P^*_k(x) = \\
= (1 + 2x + 2\beta) \left(x(1 - x) \frac{d}{dx} P^*_k(x)\right) \\
+ (x^2 + x(1 + 2\beta) + \beta + \beta^2) \frac{d}{dx} \left(x(1 - x) \frac{d}{dx} P^*_k(x)\right) \\
- 2x(x + \beta)P^*_k(x). \\
\tag{97}
\]

Using identities (13), (14) and (15),

\[
(D_x(P^*_k))(x) = \\
= \frac{(-1 + k)2k^2P^*_k(x)}{4(-1 + 2k)(1 + 2k)} \\
- \frac{k^3(1 + \beta)P^*_k(x)}{1 + 2k} \\
- \frac{(-4 + 7k^2 + 14k^2 + 7k^4 - 6\beta - 4k\beta + 12k^2\beta + 32k^3\beta + 16k^4\beta - 6k\beta^2 + 2k^2\beta^2 + 16k^3\beta^2 + 8k^4\beta^2)}{2(-1 + 2k)(3 + 2k)} P^*_k(x) \\
- \frac{(1 + k)^3(1 + \beta)P^*_k(x)}{1 + 2k} \\
- \frac{(1 + k)^2(2 + k)^2P^*_k(x)}{4(1 + 2k)(3 + 2k)}. \\
\tag{98}
\]

Finally, substituting (6) into (98) gives (94).

**Corollary 3.17.** Suppose that \( h^n = (h^n_0, h^n_1, \ldots)^\top \) is the vector of coefficients defined in (74), in the expansion of the function \( \psi_n(x) \) defined in (63); then, \( h^n \) is the \( n + 1 \)-th eigenvector of \( M \):

\[
Mh^n = \chi_n h^n, \\
\tag{99}
\]

where \( M \) is the five-diagonal matrix (95), \( \chi_n \) are the eigenvalues of the differential operator \( D_x \), and \( k = 0, 1, 2, \ldots \).

**Proof.** By (70), \( \psi_n(x) \) is an eigenfunction of \( D_x \), with the eigenvalue \( \chi_n \):

\[
(D_x(\psi_n))(x) = \chi_n \psi_n(x), \\
\tag{100}
\]

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so that

\[
\left( D_x \left( \sum_{k=0}^{\infty} h_k^* P_k \right) \right)(x) = (D_x(\psi_n))(x) = \chi_n \sum_{k=0}^{\infty} h_k^* P_k(x). \tag{101}
\]

Therefore, by Lemma 3.15, we obtain (99). \hfill \Box

3.9 A relation between \( u_n, u_m \), and the ratio \( \alpha_n/\alpha_m \)

Lemma 3.18. For any \( n, m = 0, 1, \ldots \),

\[
\frac{\alpha_m^2}{\alpha_n^2} = \frac{\int_a^b u_n'(t) u_m(t) dt}{\int_a^b u_n(t) u_m'(t) dt},
\]

if the integrals are not 0; where \( u_n \) and \( u_m \) are right singular functions (defined in (45)) and \( \alpha_n \) and \( \alpha_m \) are the singular values (defined in (43)) of \( L_{a,b} \) (defined in (42)).

Similarly,

\[
\frac{\alpha_m^2}{\alpha_n^2} = \frac{\int_a^b \psi_n'(x) \psi_m(x) dx}{\int_a^b \psi_n(x) \psi_m'(x) dx},
\]

if the integrals are not 0; where \( \psi_n \) and \( \psi_m \) are defined in (63).

Proof. We recall from (46) that

\[
u_n(t) = \frac{1}{\alpha_n} (L^*(v_n))(t) = \frac{1}{\alpha_n} \int_0^\infty e^{-\omega t} v_n(\omega) d\omega.
\]

Therefore, the derivative of \( u_n(t) \) is

\[
u_n'(t) = \frac{1}{\alpha_n} \int_0^\infty (-\omega) e^{-\omega t} v_n(\omega) d\omega.
\]

We multiply both sides of the expression by \( u_m(t) \), integrate both sides, and change the order of integration:

\[
\int_a^b u_n'(t) u_m(t) dt = \frac{1}{\alpha_n} \int_a^b \left( \int_0^\infty (-\omega) e^{-\omega t} v_n(\omega) d\omega \right) u_m(t) dt.
\]

By rearranging the result, we obtain

\[
\int_a^b u_n'(t) u_m(t) dt = \frac{\alpha_m}{\alpha_n} \int_0^\infty (-\omega) v_n(\omega) v_m(\omega) d\omega.
\]

\( m \) and \( n \) are clearly interchangeable, so that

\[
\int_0^\infty (-\omega) v_n(\omega) v_m(\omega) d\omega = \frac{\alpha_m}{\alpha_n} \int_a^b u_n'(t) u_m(t) dt.
\]

By substituting (108) into (107), we obtain (102). Substituting (63) into (102) we obtain (103). \hfill \Box
4 Algorithms

4.1 Computing the right singular function \( u_n \)
In this section we introduce an algorithm for the numerical evaluation of \( u_n(t) \), the \( n+1 \)-th right singular function (defined in (45)) of \( \mathcal{L}_{a,b} \) (the operator defined in (42)).

**Step 1:** Compute \( h^n \), the \( n+1 \)-th eigenvector of the matrix \( M \), defined in (95).

**Step 2:** Compute the function \( \psi_n(x) \) from \( h^n \), using the expansion specified in (75).

**Step 3:** Obtain \( u_n(t) \) from \( \psi_n(x) \) using (63).

**Remark 4.1.** For computations in precision \( \epsilon \), the vector \( h^n = (h^n_0, h^n_1, \ldots) \top \) is truncated at \( K \), such that \( |h^n_k| \ll \epsilon \) for all \( k > K \). By lemma 3.13 the coefficients \( h^n_k \) decay rapidly as \( k \) grows; the actual position of the last significant coefficient, larger in magnitude than \( \epsilon \), is given in Figure 6 and Table 4 in Section 5 for several combinations of \( \gamma \) and \( n \).

4.2 Computing the singular value \( \alpha_n \)
In this subsection we present an algorithm for computing the \( N+1 \) first singular values \( \alpha_0, \alpha_1, \ldots, \alpha_N \) (defined in (45)) of \( \mathcal{L}_{a,b} \) (the operator defined in (42)).

**Step 1:** Compute the first singular value \( \alpha_0 \), for example, by the formula
\[
\alpha_0 = \sqrt{\frac{\int_a^b \frac{1}{s+t} u_0(s)ds}{u_0(t)}},
\]
derived from (50), where \( t \in [a,b] \) and \( u_0(t) \) is computed using the algorithm from the previous section.

**Step 2:** For every \( n > 0 \), compute \( \alpha_n \) from \( \alpha_{n-1} \) using the relation in Lemma 3.18 and the functions \( u_n(t) \) and \( u_{n-1}(t) \) computed using the algorithm in the previous section.
5 Numerical results

In this section we present results of several numerical experiments. The algorithms for computing the right singular functions $u_n$ and singular values $\alpha_n$ of $L_{a,b}$ (the operator defined in (42)) were implemented in FORTRAN 77, using double precision arithmetic, and compiled using GFORTRAN.

In Figures 1, 2 and 3 we present examples of right singular functions of the operator $L_{a,b}$, where $a = 1$ and $b = 1.1$, $b = 10$ and $b = 100000$ respectively.

In Figure 4 and Table 1 we present the singular values $\alpha_n$ of the operator $L_{a,b}$, for several ratios $\gamma = b/a$; $\alpha_n$ depends only on $\gamma$ and $n$ (see remark 3.12). In table 2 we present several singular values smaller than $10^{-1000}$; the Fujitsu compiler with quadruple precision was used in this experiment.

In Figure 5 and Table 3 we present the eigenvalues of the matrix $M$ defined in (95).

In Figure 6 and Table 4 we present for several combinations of $\gamma$ and $n$ the position of the last significant coefficient $h_{nk}$ in the expansion defined in (75), that is larger in magnitude than $\epsilon = 10^{-16}$. In numerical computations, the vectors are truncated around that point (see Remark 4.1).

In figure 5 we present the CPU time required for the computation of the expansion of the 101-st right singular function $u_{100}$ of $L_{1,\gamma}$, for varying $\gamma$; The experiment was performed on a ThinkPad X230 laptop with Intel Core i7-3520 CPU and 16GB RAM.
Figure 1: Right Singular functions of $\mathcal{L}_{1,1,1}$. 
Figure 2: Right Singular functions of $\mathcal{L}_{1,10}$. 
Figure 3: Right Singular functions of $\mathcal{L}_{1,100000}$. 
Figure 4: Singular values $\alpha_n$ of $\mathcal{L}_{a,b}$, with $\gamma = b/a$. 
Figure 5: Magnitude of the eigenvalues of the matrix $M$ defined in (95), with $\gamma = \frac{2+\beta}{\beta}$.
Figure 6: The position of the last significant coefficient larger in magnitude than $10^{-16}$. 
Figure 7: CPU time required for computing the expansion of the 101-st right singular function of $L_{1,\gamma}$, as a function of $\gamma$. The experiment was performed on a ThinkPad X230 laptop with an Intel Core i7-3520 CPU and 16GB RAM.
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Table 2: Examples of singular values $\alpha_n$ smaller than $10^{-1000}$

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Table 3: Eigenvalues of $M$ defined in (95)

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6 Conclusions and generalizations

In this paper we have introduced effective algorithms for the evaluation of the right singular functions and singular values of the Truncated Laplace Transform $\mathcal{L}_{a,b}$.

As is evident from Remark 2.3 and the more detailed discussion in [18], the right singular functions of $\mathcal{L}_{a,b}$ are an efficient basis for representing decaying exponentials on the interval $[a, b]$.

An algorithm for the computation of the left singular functions of $\mathcal{L}_{a,b}$, which is the remaining component in the computation of the SVD, will be presented in a future paper. Additional asymptotic properties of the Truncated Laplace Transform and of the associated differential operators will also be discussed in a future paper.
References


