Broadcast Using Certified Propagation Algorithm in Presence of Byzantine Faults\textsuperscript{1}

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Abstract

We explore the correctness of the Certified Propagation Algorithm (CPA) [6, 1, 8, 5] in solving broadcast with locally bounded Byzantine faults. CPA allows the nodes to use only local information regarding the network topology. We provide a tight necessary and sufficient condition on the network topology for the correctness of CPA.

Keywords: Distributed computing, Byzantine broadcast, CPA, Tight condition

1. Introduction

In this work, we explore fault-tolerant broadcast with locally bounded Byzantine faults in synchronous point-to-point networks. We assume a $f$-locally bounded model, in which at most $f$ Byzantine faults occur in the neighborhood of every fault-free node [6]. In particular, we are interested in the necessary and sufficient condition on the underlying communication network topology for the correctness of the Certified Propagation Algorithm.

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We explore the correctness of the Certified Propagation Algorithm (CPA) [6, 1, 8, 5] in solving broadcast with locally bounded Byzantine faults. CPA allows the nodes to use only local information regarding the network topology. We provide a tight necessary and sufficient condition on the network topology for the correctness of CPA.
(CPA) – the CPA algorithm has been analyzed in prior work [6, 1, 8, 5, 7].

**Problem Formulation.** Consider an arbitrary directed network of \( n \) nodes. One node in the network, called the source \((s)\), is given an initial input, which the source node needs to transmit to all the other nodes. The source \( s \) is assumed to be fault-free. We say that CPA is correct, if it satisfies the following properties, where \( x_s \) denotes the input at source node \( s \):

- **Termination:** every fault-free node \( i \) eventually decides on an output value \( y_i \).

- **Validity:** for every fault-free node \( i \), its output value \( y_i \) equals the source’s input, i.e., \( y_i = x_s \).

We study the condition on the network topology for the correctness of CPA.

**Related Work.** Several researchers have addressed CPA problem. [6] studied the problem in an infinite grid. [1] developed a sufficient condition in the context of arbitrary network topologies, but the sufficient condition proposed is not tight. [8] provided necessary and sufficient conditions, but the two conditions are not identical (not tight). [5] provided another condition that can approximate (within a factor of 2) the largest \( f \) for which CPA is correct in a given graph. Independently, [7] presented the tight condition in undirected graphs. Similar condition under other contexts are also discovered by other researchers [9, 3]. Please refer to [11] for more discussions.

**System Model.** The synchronous communication network consisting of \( n \) nodes including source node \( s \) is modeled as a simple directed graph \( G(\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} \) is the set of \( n \) nodes, and \( \mathcal{E} \) is the set of directed edges between the nodes.
in $\mathcal{V}$. Node $i$ can transmit messages to another node $j$ if and only if the directed edge $(i, j)$ is in $\mathcal{E}$. Each node can transmit messages to itself as well; however, for convenience, we exclude self-loops from set $\mathcal{E}$. That is, $(i, i) \not\in \mathcal{E}$ for $i \in \mathcal{V}$. All the links (i.e., communication channels) are assumed to be point-to-point, reliable, FIFO (first-in first-out) and deliver each transmitted message exactly once. With a slight abuse of terminology, we will use the terms edge and link interchangeably.

For each node $i$, let $N^-_i$ be the set of nodes from which $i$ has incoming edges, i.e., $N^-_i = \{ j \mid (j, i) \in \mathcal{E} \}$. Similarly, define $N^+_i$ as the set of nodes to which node $i$ has outgoing edges, i.e., $N^+_i = \{ j \mid (i, j) \in \mathcal{E} \}$. Nodes in $N^-_i$ and $N^+_i$ are, respectively, said to be incoming and outgoing neighbors of node $i$. Since we exclude self-loops from $\mathcal{E}$, $i \not\in N^-_i$ and $i \not\in N^+_i$. However, we note again that each node can indeed transmit messages to itself.

We consider the $f$-local fault model, with at most $f$ incoming neighbors of any fault-free node becoming faulty. [6, 1, 8, 5, 7] also explored this fault model. Yet, to the best of our knowledge, the tight necessary and sufficient conditions for the correctness of CPA in directed networks under $f$-local fault model have not been developed previously.

2. Feasibility of CPA under $f$-local fault model

Certified Propagation Algorithm (CPA). We first describe the Certified Propagation Algorithm (CPA) from [6] formally. Note that the faulty nodes may deviate from this specification arbitrarily. Possible misbehavior includes sending incorrect and mismatching messages to different outgoing neighbors.

Source node $s$ commits to its input $x_s$ at the start of the algorithm, i.e.,
sets its output equal to \( x_s \). The source node is said to have committed to \( x_s \) in round 0. The algorithm for each round \( r \) (\( r > 0 \)), is as follows:

1. Each node that commits in round \( r - 1 \) to some value \( x \), transmits message \( x \) to all its outgoing neighbors, and then terminates.
2. If any node receives message \( x \) directly from source \( s \), it commits to output \( x \).
3. Through round \( r \), if a node has received messages containing value \( x \) from at least \( f + 1 \) distinct incoming neighbors, then it commits to output \( x \).

**The Necessary Condition.** For CPA to be correct, the network graph \( G(\mathcal{V}, \mathcal{E}) \) must satisfy the necessary condition proved in this section. We borrow two relations \( \Rightarrow \) and \( \not\Rightarrow \) from our previous paper [12].

**Definition 1.** For non-empty disjoint sets of nodes \( A \) and \( B \),

- \( A \Rightarrow B \) iff there exists a node \( v \in B \) that has at least \( f + 1 \) distinct incoming neighbors in \( A \), i.e., \( |N_v^- \cap A| > f \).
- \( A \not\Rightarrow B \) iff \( A \Rightarrow B \) is not true.

**Definition 2.** Set \( F \subseteq \mathcal{V} \) is said to be a feasible \( f \)-local fault set, if for each node \( v \not\in F \), \( F \) contains at most \( f \) incoming neighbors of node \( v \). That is, for every \( v \in \mathcal{V} - F \), \( |N_v^- \cap F| \leq f \).

We now derive the necessary condition on the network topology.
Theorem 1. Suppose that CPA is correct in graph \( G(\mathcal{V}, \mathcal{E}) \) under the \( f \)-local fault model. Let sets \( F, L, R \) form a partition\(^2\) of \( \mathcal{V} \), such that (i) source \( s \in L \), (ii) \( R \) is non-empty, and (iii) \( F \) is a feasible \( f \)-local fault set. Then

- \( L \Rightarrow R \), or
- \( R \) contains an outgoing neighbor of \( s \), i.e., \( N_s^+ \cap R \neq \emptyset \).

Proof. The proof is by contradiction. Consider any partition \( F, L, R \) such that \( s \in L \), \( R \) is non-empty, and \( F \) is a feasible \( f \)-local fault set. Suppose that the input at \( s \) is \( x_s \). Consider any single execution of the CPA algorithm such that the nodes in \( F \) behave as if they have crashed.

By assumption, CPA is correct in the given network under such a behavior by the faulty nodes. Thus, all the fault-free nodes eventually commit their output to \( x_s \). Let round \( r \) \((r > 0)\), be the earliest round in which at least one of the nodes in \( R \) commits to \( x_s \). Let \( v \) be one of the node in \( R \) that commits in round \( r \). Such a node \( v \) must exist since \( R \) is non-empty, and it does not contain source node \( s \). For node \( v \) to be able to commit, as per specification of the CPA algorithm, either node \( v \) should receive the message \( x_s \) directly from the source \( s \), or node \( v \) must have \( f + 1 \) distinct incoming neighbors that have already committed to \( x_s \). By definition of node \( v \), nodes that have committed to \( x_s \) prior to \( v \) must be outside \( R \); since nodes in \( F \) behave as crashed, these \( f + 1 \) nodes must be in \( L \). Thus, either \((s, v) \in \mathcal{E}\), or node \( v \) has at least \( f + 1 \) distinct incoming neighbors in set \( L \).

\( \square \)

\(^2\)Sets \( X_1, X_2, X_3, ..., X_p \) are said to form a partition of set \( X \) provided that (i) \( \cup_{1 \leq i \leq p} X_i = X \), and (ii) \( X_i \cap X_j = \emptyset \) if \( i \neq j \).
Sufficiency. We now show that the condition in Theorem 1 is also sufficient.

**Theorem 2.** If $G(\mathcal{V}, \mathcal{E})$ satisfies the condition in Theorem 1, then CPA is correct in $G(\mathcal{V}, \mathcal{E})$ under the $f$-local fault model.

**Proof.** Suppose that $G(\mathcal{V}, \mathcal{E})$ satisfies the condition in Theorem 1. Let $F'$ be the set of faulty nodes. By assumption, $F'$ is a feasible local fault set. Let $x_s$ be the input at source node $s$. We will show that, (i) fault-free nodes do not commit to any value other than $x_s$ (Validity), and, (ii) until all the fault-free nodes have committed, in each round of CPA, at least one additional fault-free node commits to value $x_s$ (Termination). The proof is by induction.

*Induction basis:* Source node $s$ commits in round 0 to output equal to its input $x_s$. No other fault-free nodes commit in round 0.

*Induction:* Suppose that $L$ is the set of fault-free nodes that have committed to $x_s$ through round $r$, $r \geq 0$. Thus, $s \in L$. Define $R = \mathcal{V} - L - F'$. If $R = \emptyset$, then the proof is complete. Let us now assume that $R \neq \emptyset$.

Now consider round $r + 1$.

- **Validity:**
  Consider any fault-free node $u$ that has not committed prior to round $r + 1$ (i.e., $u \in R$). All the nodes in $L$ have committed to $x_s$ by the end of round $r$. Thus, in round $r + 1$ or earlier, node $u$ may receive messages containing values different from $x_s$ only from nodes in $F'$. Since there are at most $f$ incoming neighbors of $u$ in $F'$, node $u$ cannot commit to any value different from $x_s$ in round $r + 1$.

- **Termination:**
By the condition in Theorem 1, there exists a node $w$ in $R$ such that (i) node $w$ has an incoming link from $s$, or (ii) node $w$ has incoming links from $f + 1$ nodes in $L$. In case (i), node $w$ will commit to $x_s$ on receiving $x_s$ from node $s$ in round $r + 1$ (in fact, $r + 1$ in this case must be 1). In case (ii), first observe that all the nodes in $L$ from whom node $w$ has incoming links have committed to $x_s$ (by definition of $L$). Then, node $w$ will be able to commit to $x_s$ after receiving messages from at least $f + 1$ incoming neighbors in $L$, since all nodes in $L$ have committed to $x_s$ by the end of round $r$ by the definition of $L$.\[3^3\] Thus, node $w$ will commit to $x_s$ in round $r + 1$.

This completes the proof. \[\Box\]

3. Discussion

This section presents extensions and complexity of verifying the condition. Due to space limitation, please refer to [11] for details.

CPA without prior knowledge of $f$. Given a graph $G$ that can tolerate $f$-local faults (where $f$ is unknown), we construct a broadcast algorithm in $G$ without usage of $f$. The core idea is for each node to exhaustively test all possible parameters by running $n + 1$ instances of CPA algorithm in parallel.

Other Communication Model. In the broadcast model [6, 1], when a node transmits a value, all of its outgoing neighbors receive this value identically.

\[3^3\]Since node $w$ did not commit prior to round $r + 1$, it follows that at least one node in $L$ must have committed in round $r$. 

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Thus, no node can transmit mismatching values to different outgoing neighbors. In the asynchronous model \[2\], the algorithm may not proceed in rounds, but a node still commits to value \( x \) either on receiving the value directly from \( s \), or from \( f + 1 \) nodes. Under both models, condition in Theorem 1 is both necessary and sufficient for the correctness of CPA. The claim for asynchronous model may seem to contradict the FLP result \[4\]. However, our claim assumes that the source node is fault-free, unlike \[4\].

**Complexity.** [7] proved that it is NP-hard to examine whether CPA is correct in a given undirected graph. The condition in [7] is indeed equivalent to our condition (condition in Theorem 1) in undirected graphs. Therefore, it is NP-hard to examine whether a given graph satisfies our condition or not.

4. Conclusion

In this paper, we explore broadcast in arbitrary network using the CPA algorithm in \( f \)-local fault model. In particular, we provide a tight necessary and sufficient condition on the underlying network for the correctness of CPA.

References


