On Processing Hexagonally Sampled Images

SOAR2 Review
12-15 JULY 2011

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**Report Documentation Page**

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<table>
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<tr>
<th>a. REPORT</th>
<th>b. ABSTRACT</th>
<th>c. THIS PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>unclassified</td>
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<td>unclassified</td>
</tr>
</tbody>
</table>

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**Standard Form 298 (Rev. 8-98)**  
**Prescribed by ANSI Std Z39-18**
Outline

- Hexagonal sampling
- Array set addressing (ASA)
- Processing with ASA
  - Gradient estimation, convolution, downsampling, wavelet decomposition, and hexagonal DFT
  - Comparison with spiral addressing
- Hex-Rect sensor
- Fourier transform experiment
- Conclusion / questions
Hexagonal vs. Rectangular

Hexagonal Grid

• Optimal representation
• Consistent connectivity
• Angular resolution is 60 degrees
• Equidistant Spacing
• 6-fold symmetry
• Mimics nature

Rectangular Grid

• Non-optimal representation
• Connectivity ambiguity: 4-way vs. 8-way
• Angular resolution is 90 degrees
• Unequal spacing
• 4-fold symmetry
• Man-made
Natural Systems

Compound eye of the blowfly (*Calliphora Vomitoria*)

Distribution of cones in the fovea of a human retina showing high peak density (A) and low peak density (B) (bar is 10 microns).

Reproduced from [http://www.bath.ac.uk/ceos/Insects1.html](http://www.bath.ac.uk/ceos/Insects1.html) © University of Bath

Reprinted from Curcio et al. (1987) © AAAS
Why is Hex Optimal?

The spatial sampling geometry determines the spectral tiling, and the density of the spatial samples determines the area of the tile.

Because it provides the most efficient packing of circles in the frequency domain.

\[ \frac{A_{\text{gray}}}{A_{\text{green}}} = \frac{\sqrt{3}}{2} \]

\[ A_{\text{hex}} = A_{\text{rect}} \]
Addressing Schemes

Oblique Addressing

Spiral Addressing
(GBT, HIP, etc.)

Her’s Approach

x + y + z = 0
Array Set Addressing (ASA)

- ASA separates the hexagonal grid into two rectangular arrays.
- A three coordinate system addresses the individual points on the grid – a binary array coordinate followed by the familiar row and column coordinates: \((a, r, c) \in \{0,1\} \times \mathbb{Z} \times \mathbb{Z}\)
Hexagonal Neighbors

For any pixel \((a, r, c)\), its neighbors are:

\[
\begin{align*}
(1-a, r-(1-a), c-(1-a)) & \quad (1-a, r-(1-a), c+a) \\
(a, r, c-1) & \quad (a, r, c) & \quad (a, r, c+1) \\
(1-a, r+a, c-(1-a)) & \quad (1-a, r+a, c+a)
\end{align*}
\]

- Finding a neighbor’s address is an \(O((\log N)^2)\) operation using spiral addressing.
- No connectedness ambiguity – a neighbor is a neighbor.
Distance Measures

Converting ASA to Cartesian is a simple matrix multiplication:

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} =
\begin{bmatrix}
  1/2 & 0 & 1 \\
  \sqrt{3}/2 & \sqrt{3} & 0
\end{bmatrix}
\begin{bmatrix}
  a \\
  r \\
  c
\end{bmatrix} =
\begin{bmatrix}
  (a/2 + c) \\
  (\sqrt{3})(a/2 + r)
\end{bmatrix}
\]

Euclidean distance (on the image plane) between two points \( p_1 = (a_1, r_1, c_1) \) and \( p_2 = (a_2, r_2, c_2) \):

\[
d(p_1, p_2) = \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 + (c_1 - c_2)^2 + (3)\left(\frac{a_1 - a_2}{2}\right)^2 + (r_1 - r_2)^2}
\]

“City-Block” distance (on the image plane) between two points \( p_1 = (a_1, r_1, c_1) \) and \( p_2 = (a_2, r_2, c_2) \):

\[
U = (c_1 - c_2) - (r_1 - r_2) \\
V = (a_1 - a_2) + (2)(r_1 - r_2)
\]

\[
d_6(p_1, p_2) = \begin{cases} 
|U| + |V| & \text{if } U \text{ and } V \text{ have the same sign} \\
\max(|U|, |V|) & \text{otherwise}
\end{cases}
\]
Vector Operations

Let \( \mathbf{p}_i = \begin{pmatrix} a_i \\ r_i \\ c_i \end{pmatrix} \in ASA \)

<table>
<thead>
<tr>
<th>Operation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>( \mathbf{p}_1 + \mathbf{p}_2 = \begin{pmatrix} a_1 \oplus a_2 \ r_1 + r_2 + (a_1 \land a_2) \ c_1 + c_2 + (a_1 \land a_2) \end{pmatrix} )</td>
</tr>
<tr>
<td>Negation</td>
<td>( -\mathbf{p} = \begin{pmatrix} a \ -r - a \ -c - a \end{pmatrix} )</td>
</tr>
<tr>
<td>Subtraction</td>
<td>( \mathbf{p}_1 - \mathbf{p}_2 = \mathbf{p}_1 + (-\mathbf{p}_2) )</td>
</tr>
<tr>
<td>Scalar Multiplication</td>
<td>( k\mathbf{p} = \begin{pmatrix} (ak) \mod 2 \ kr + (a) \lfloor k/2 \rfloor \ kc + (a) \lfloor k/2 \rfloor \end{pmatrix}, \ k \in \mathbb{N} \quad \text{and} \quad -k\mathbf{p} = k(-\mathbf{p}) )</td>
</tr>
</tbody>
</table>
ASA is a Z-Module

ASA satisfies the 8 properties of a Z-module:

<table>
<thead>
<tr>
<th>Property</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutativity of addition</td>
<td>$p_1 + p_2 = p_2 + p_1$</td>
</tr>
<tr>
<td>Associativity of addition</td>
<td>$p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3$</td>
</tr>
<tr>
<td>Identity element of addition</td>
<td>$\exists 0 \in ASA: \ p + 0 = p, \ \forall p \in ASA$</td>
</tr>
<tr>
<td>Inverse elements of addition</td>
<td>$\exists q \in ASA: \ p + q = 0, \ \forall p \in ASA$</td>
</tr>
<tr>
<td>Distributivity of scalar multiplication (wrt vector addition)</td>
<td>$k(p + q) = kp + kq$</td>
</tr>
<tr>
<td>Distributivity of scalar multiplication (wrt scalar addition)</td>
<td>$(k + j)p = kp + jp$</td>
</tr>
<tr>
<td>Compatibility of scalar multiplication (with multiplication of scalars)</td>
<td>$k(jp) = (kj)p$</td>
</tr>
<tr>
<td>Identity element of scalar multiplication</td>
<td>$1p = p$</td>
</tr>
</tbody>
</table>
Gradient Estimation

\[ z = ax + by + c \]

By symmetry, we can estimate gradients along 6 axes:

- x direction (0°)
  -1 1
  -2 0 2
  -1 1

- y direction (90°)
  -1 -1
  0 0 0
  1 1

120° 90° 60° 150° 210°

0°
Performing Convolutions
Convection Complexity

Assumptions:

- Hexagonal and rectangular images are each M x N pixels
- Image borders are padded to allow each pixel to use the full convolution mask
- Let $C_{ij}$ be the convolution of the $i$-array of the image with the $j$-array of the convolution mask

<table>
<thead>
<tr>
<th>Step</th>
<th>Multiplications</th>
<th>Additions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate $C_{00}$</td>
<td>$(3)(M/2)(N)$</td>
<td>$(2)(M/2)(N)$</td>
</tr>
<tr>
<td>Calculate $C_{01}$</td>
<td>$(4)(M/2)(N)$</td>
<td>$(3)(M/2)(N)$</td>
</tr>
<tr>
<td>Calculate $C_{10}$</td>
<td>$(3)(M/2)(N)$</td>
<td>$(2)(M/2)(N)$</td>
</tr>
<tr>
<td>Calculate $C_{11}$</td>
<td>$(4)(M/2)(N)$</td>
<td>$(3)(M/2)(N)$</td>
</tr>
<tr>
<td>Sum of $C_{00}$ and $C_{11}$</td>
<td>0</td>
<td>$(M/2)(N)$</td>
</tr>
<tr>
<td>Sum of $C_{01}$ and $C_{10}$</td>
<td>0</td>
<td>$(M/2)(N)$</td>
</tr>
<tr>
<td>TOTALS:</td>
<td>7MN</td>
<td>6MN</td>
</tr>
</tbody>
</table>

ASA convolution (7 point mask):

<table>
<thead>
<tr>
<th>Step</th>
<th>Multiplications</th>
<th>Additions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate $C_{00}$</td>
<td>$(3)(M/2)(N)$</td>
<td>$(2)(M/2)(N)$</td>
</tr>
<tr>
<td>Calculate $C_{01}$</td>
<td>$(4)(M/2)(N)$</td>
<td>$(3)(M/2)(N)$</td>
</tr>
<tr>
<td>Calculate $C_{10}$</td>
<td>$(3)(M/2)(N)$</td>
<td>$(2)(M/2)(N)$</td>
</tr>
<tr>
<td>Calculate $C_{11}$</td>
<td>$(4)(M/2)(N)$</td>
<td>$(3)(M/2)(N)$</td>
</tr>
<tr>
<td>Sum of $C_{00}$ and $C_{11}$</td>
<td>0</td>
<td>$(M/2)(N)$</td>
</tr>
<tr>
<td>Sum of $C_{01}$ and $C_{10}$</td>
<td>0</td>
<td>$(M/2)(N)$</td>
</tr>
<tr>
<td>TOTALS:</td>
<td>9MN</td>
<td>8MN</td>
</tr>
</tbody>
</table>

Rectangular convolution (9 point mask):
Canny Edge Detector

Hexagonally sampled

Rectangularly sampled
Angular Resolution

The increased angular resolution of the hexagonal grid may account for the increased performance of the Canny edge detector.
Downsampling

We want to use ½ of each of the neighboring pixels since they are shared with adjacent “superpixels”. So we are averaging together \((6)(1/2) + 1 = 4\) pixels, resulting in the above averaging mask.

After convolving the image with the averaging mask, the light blue pixels form the downsampled 0-array and the dark blue pixels form the downsampled 1-array. The resulting arrays are 1/4 the size of the original arrays (i.e. \((N/2) \times N\) => \((N/4) \times (N/2))\).

Anti-aliasing Mask
Wavelet HPF Advantage

Idealized Frequency Domain Regions of Support

Rectangularly Sampled

Hexagonally Sampled
Perfect Reconstruction (PR) Example

ASA implementation of Allen PR wavelet, runtime = 0.5017 (0.0077) sec

Rect. implementation of CDF 9/7 wavelet, runtime = 0.5484 (0.008) sec
Mersereau's HDFT:

\[
X(k_1, k_2) = \sum_{n_1} \sum_{n_2} x(n_1, n_2) \exp \left[ -j\pi \left( \frac{1}{2N_1 + N_2} (2n_1 - n_2)(2k_1 - k_2) + \frac{1}{N_2} (n_2 k_2) \right) \right]
\]

\[
x(n_1, n_2) = \frac{1}{N_2(2N_1 + N_2)} \sum_{k_1} \sum_{k_2} X(k_1, k_2) \exp \left[ j\pi \left( \frac{1}{2N_1 + N_2} (2n_1 - n_2)(2k_1 - k_2) + \frac{1}{N_2} (n_2 k_2) \right) \right]
\]

Mersereau encountered an “insurmountable difficulty” when attempting to develop a fast algorithm to compute the hexagonal DFT, due to the product of mixed coordinates in the exponential.
The HDFT in ASA becomes:

\[
X(b, s, d) = \sum_a \sum_r \sum_c x(a, r, c) \exp \left[ -j\pi \left( \frac{1}{2m} (a+2c)(b+2d) + \frac{1}{n} (a+2r)(b+2s) \right) \right]
\]

\[
x(a, r, c) = \frac{1}{2mn} \sum_b \sum_s \sum_d X(b, s, d) \exp \left[ j\pi \left( \frac{1}{2m} (a+2c)(b+2d) + \frac{1}{n} (a+2r)(b+2s) \right) \right]
\]

The Fourier kernel is separable in ASA space!
Fourier Transform of Allen’s Filter Bank

Low-Pass filter

\[
\begin{bmatrix}
-14 & -14 \\
-84 & +276 & +259 \\
-14 & +276 & +259 \\
-84 & -14 & -14 \\
\end{bmatrix}
\]

High-Pass filter

\[
\begin{bmatrix}
+50 \\
+500 & +50 \\
-2 & +84 & -925 \\
-12 & +37 & +231 & +84 \\
-2 & +84 & +37 \\
-12 & -2 \\
-2 \\
\end{bmatrix}
\]

The values given are exact. They must be divided by 1014 to achieve normalization. The other two filters can be visualized by rotating the High-pass filter 120° and 240°.

### ASA vs. HIP

<table>
<thead>
<tr>
<th>Operation</th>
<th>HIP</th>
<th>ASA</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Address (Vector) Addition</td>
<td>23.85 (3.15)</td>
<td>2.11 (0.97)</td>
<td>11.28</td>
</tr>
<tr>
<td>Address (Vector) Subtraction</td>
<td>33.98 (3.56)</td>
<td>2.56 (0.47)</td>
<td>13.28</td>
</tr>
<tr>
<td>Scalar Multiplication</td>
<td>6652.08 (4076.89)</td>
<td>3.73 (0.73)</td>
<td>1782.20</td>
</tr>
<tr>
<td>Calculate Euclidean Distance</td>
<td>15.83 (2.43)</td>
<td>2.73 (0.56)</td>
<td>5.79</td>
</tr>
<tr>
<td>Calculate 6 Nearest Neighbor Addresses</td>
<td>118.94 (10.49)</td>
<td>3.31 (0.75)</td>
<td>35.89</td>
</tr>
<tr>
<td>Convert From Cartesian</td>
<td>9189.68 (3784.79)</td>
<td>4.48 (1.13)</td>
<td>2052.31</td>
</tr>
</tbody>
</table>

Each result is the mean of 10,000 operations on randomly selected addresses (μs, mean (std))

<table>
<thead>
<tr>
<th>Operation</th>
<th>HIP</th>
<th>ASA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Address (Vector) Addition / Subtraction</td>
<td>(O((\log N)^2))</td>
<td>(O(1))</td>
</tr>
<tr>
<td>Scalar Multiplication</td>
<td>(O(N(\log N)^2))</td>
<td>(O(1))</td>
</tr>
<tr>
<td>Calculate Euclidean Distance</td>
<td>(O(\log N))</td>
<td>(O(1))</td>
</tr>
<tr>
<td>Calculate 6 Nearest Neighbor Addresses</td>
<td>(O((\log N)^2))</td>
<td>(O(1))</td>
</tr>
<tr>
<td>Convert From Cartesian</td>
<td>(O(N(\log N)^2))</td>
<td>(O(1))</td>
</tr>
</tbody>
</table>
Hex-Rect Imager

Developed by Centeye, Inc.
Experiment Results

<table>
<thead>
<tr>
<th>Spatial</th>
<th>Fourier</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangularly</td>
<td>Rectangularly</td>
</tr>
<tr>
<td>Sampled</td>
<td>Sampled</td>
</tr>
<tr>
<td>F = 0.227</td>
<td>F = 0.268</td>
</tr>
<tr>
<td>Rectangularly</td>
<td>Rectangularly</td>
</tr>
<tr>
<td>Sampled</td>
<td>Sampled</td>
</tr>
<tr>
<td>F = 0.268</td>
<td>F = 0.309</td>
</tr>
<tr>
<td>Rectangularly</td>
<td>Rectangularly</td>
</tr>
<tr>
<td>Sampled</td>
<td>Sampled</td>
</tr>
<tr>
<td>F = 0.309</td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{0.268}{0.309} \approx 0.867 \approx \frac{\sqrt{3}}{2} \approx 0.866
\]
Conclusion

• There are several advantages to sampling digital images hexagonally rather than rectangularly.

• ASA is a tri-coordinate system for addressing a hexagonal grid that provides support for efficient image processing.

• Efficient ASA methods were shown for gradient estimation, convolution, downsampling, wavelet decomposition, and hexagonal DFT.

• The Hex-Rect imager can be used to quantitatively compare hexagonal and rectangular sampling.
Questions?
Backup Slides Follow
On-FPA Processing with Difference of Gaussians

Input Signal

HRes Smoothed Signal

Difference Signal (Input – HRes)

Zero-Crossings

0 Ref

"Local Contrast Enhancement"

Addition Processing with User-Adjustable Gain

0 Ref
Neuromorphic Infrared Sensor (NIFS)
Hexagonal Imagers

• Carver Mead’s Silicon Retina
• Hauschild’s Prototype
• Gaber’s Design
• Centeye’s Hex-Rect
• More to come...
Hex-Rect Unit Cells

Photosensitive area

Center of trapezoid

\[ \sqrt{3} \]

2
# Hex-Rect Specs

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Drawn chip size</strong></td>
<td>6.1mm x 11.1mm</td>
</tr>
<tr>
<td><strong>Focal plane size</strong></td>
<td>4.7mm x 9.2mm</td>
</tr>
</tbody>
</table>
| **Focal plane resolution** | Raw trapezoid pixels: 304 x 512  
                          | Hexagonal array: 152 x 255 (even rows have 256 hex pixels)  
                          | Rectangular array: 151 x 256 |
| **Pixel type**          | 3-transistor active pixel, with support for both logarithmic response and linear response |
| **Pixel pitch**         | 18 microns wide by 15.6 microns high for raw pixels |
| **Post-pixel circuitry** | 8-bit flash ADC |
| **Interface**           | PIO12B parallel interface:  
                          | 8 bidirectional digital, 2 digital in, 1 analog out  
                          | 12-bit command bus in two 6-bit words  
                          | 8-bit digital out  
                          | Optional 3 input chip select  
                          | Optional analog out  
                          | Alternative 12 bit input / 8 bit output parallel interface |
| **Process**             | ON-Semi C5N 3 metal 2 poly 0.5 micron process |
| **Chip operating voltage** | 4V to 5V preferred |
| **Digital input 0/1 threshold** | About 0.95V |
| **Voltage regulation**  | On-chip voltage regulator for analog circuits and bias generators |
Hex-Rect Interface

Note: GND < VDDI ≤ VDD
IR Readout Considerations

- Typical readouts (ROICs) are designed to read out rectangular arrays.
- Slight modifications should allow hexagonally sampled images to be read out into the ASA data structure.
- Images from the prototype on the right could have been processed directly using ASA.

IR Detector Materials and Bump Bonding Considerations

- **Indium Gallium Arsenide (InGaAs)**
  - NIR (0.4 – 1.6 um), Uncooled or slightly cooled
- **Indium Antimonide (InSb)**
  - MWIR (3-5 um), Cooled to 77K
- **Mercury Cadmium Telluride (HgCdTe)**
  - MWIR (3-5 um), Cooled to 77K or 120K+
  - LWIR (8-12 um), Cooled to 77K or 120K+
- **QWIP**
- **Strained Layer Superlattice**
Retesselating the Image

Image Patch

Pixel Dimensions

Resample to resize pixels

\( \frac{\sqrt{3}}{2} \) 1

Conglomerate

\( \frac{\sqrt{3}}{2} \) \( \sqrt{3} \)

30 pixels + 12 half-pixels = 36 pixels

Total Area: \( (36) \left( \frac{\sqrt{3}}{2} \right) = 18 \sqrt{3} \)

36 pixels

Total Area: \( (36) \left( \frac{\sqrt{3}}{2} \right) = 18 \sqrt{3} \)
Image Formation Results

Original Image  Hexagonally Sampled  Rectangularly Sampled
Pixel Geometries

“Pixel Geometries”, P. Halasz
Reproduced from:
http://commons.wikimedia.org/wiki/File:Pixel_geometry_01_Pengo.jpg
Use memory addresses as indices:

Assume an \( N \times 2^j \) ASA image and a 32 bit address space

- Column index = \( j \)
- Row index = \( \text{ceil}(\log_2(N/2)) \) bits = \( m \)
- Array index = 1 bit
- Base address = \( 32-(j+m+1) \)

For example, a 443x512 format provides a 4:3 aspect ratio (in Cartesian space) and the address format is

\[ \text{XXXXXXXXXXXXXXXXRRRRRRRRACCCCCCCC} \]

- \( \text{XXXXXXXXXXXXXXXX} \) Base
- \( \text{RRRRRRRR} \) Row
- \( \text{ACCCCCCCC} \) Array
- \( \text{Column} \)

Yields row-major order storage
Converting ASA to Cartesian

For a regular hexagonal grid described by

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} =
\begin{bmatrix}
  d & d/2 \\
  0 & d\sqrt{3}/2
\end{bmatrix}
\begin{bmatrix}
  n_1 \\
  n_2
\end{bmatrix}
\]

where \( x \) and \( y \) are Cartesian coordinates, \( n_1 \) and \( n_2 \) are integers (oblique coordinates), the conversion from ASA to Cartesian coordinates is a simple matrix multiplication:

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} =
\begin{bmatrix}
  d/2 & 0 & d \\
  d\sqrt{3}/2 & d\sqrt{3} & 0
\end{bmatrix}
\begin{bmatrix}
  a \\
  r \\
  c
\end{bmatrix} =
\begin{bmatrix}
  (d)(a/2 + c) \\
  (d\sqrt{3})(a/2 + r)
\end{bmatrix}
\]

The parameter \( d \) is the distance between any two adjacent grid points. Assume that \( d=1 \) for the remainder of the presentation.
Converting Cartesian to ASA

Convert the Cartesian coordinates \((x,y)\) into integers \((x_r,y_r)\) by first scaling each dimension, then rounding to the nearest integer:

\[
\begin{align*}
x_s &= 2x \\
y_s &= \frac{2y}{\sqrt{3}}
\end{align*}
\]

\[
\begin{align*}
x_r &= \text{round}(x_s) \\
y_r &= \text{round}(y_s)
\end{align*}
\]
Converting Cartesian to ASA (Cont.)

- Determine which quadrant \((x_s, y_s)\) is in by comparing to \((x_r, y_r)\)
- Using the known point and slope determine if \((x_s, y_s)\) is above or below the line
- Adjust \((x_r, y_r)\) to correct hexagon center
- Convert \((x_r, y_r)\) to ASA using:

\[
\begin{align*}
a &= y_r \mod 2 \\
r &= \frac{y_r - a}{2} \\
c &= \frac{x_r - a}{2}
\end{align*}
\]
Downsampling Example
Sampling Densities
Hex Characteristics

The spacing is important to maintaining the natural symmetry of the hexagonal grid.