

# A Method to Easily Visualize and Solve a Convolution Integral by Direct Integration

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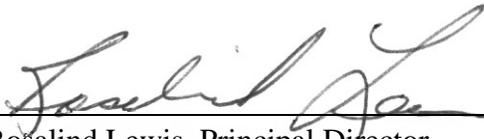
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# A Method to Easily Visualize and Solve a Convolution Integral by Direct Integration

Approved by:



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Rosalind Lewis, Principal Director  
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Systems Engineering Division  
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## Foreword

Anyone who has had to take upper-division courses in the hard sciences, engineering, or applied mathematics has probably encountered convolution. The typical methods used to solve a convolution integral—the graphical method, the direct integration method, or the use of Fourier or Laplace transforms—can be very difficult to actually set up and solve, even when worked examples are available as guides. This report shows a method of solving the convolution integral that should make it easy for anyone, including anyone using numerical methods, to visualize the convolution and then determine the integrals needed to solve the problem.

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## Theoretical Development of a Method of Evaluating Convolutions

The convolution  $h(x)$  of two real-valued functions  $f(x)$  and  $g(x)$  is indicated by the shorthand notation  $h(x)=f(x)*g(x)$  and is defined by the integral:

$$h(x) = \int_{-\infty}^{\infty} f(\alpha)g(x - \alpha)d\alpha = \int_{-\infty}^{\infty} f(x - \alpha)g(\alpha)d\alpha^1$$

Anyone who has had to take upper-division courses in the hard sciences, engineering or applied mathematics has probably encountered convolution. The typical methods used to solve a convolution integral—the graphical method, the direct integration method or the use of Fourier or Laplace transforms—can be very difficult to actually set-up and solve, even when worked examples are available as guides. This report shows a method of solving the convolution integral that should make it easy for anyone, including anyone using numerical methods, to visualize the convolution and then determine the integrals needed to solve the problem.

Assume we want to convolve two functions:  $f(x)$  which exists only on the closed interval  $[a_1, b_1]$ ; and  $g(x)$  which exists only on the closed interval  $[a_2, b_2]$ . This method starts by placing the two functions  $f$  and  $g$  in a three-dimensional rectangular Cartesian coordinate system. Let  $z_1(x)=f(x)$  and  $z_2(y)=g(y)$ . Since the convolution process is commutative, it does not matter in this method if  $f$  is a function of  $x$  and  $g$  is a function of  $y$  or vice-versa. Let  $z(x,y)=z_1(x)z_2(y)$  or  $z(x,y)=f(x)g(y)$ . So,  $z(x,y)$  will only exist in the closed rectangular domain  $[x,y]=[a_1 \leq x \leq b_1, a_2 \leq y \leq b_2]$ , as seen in Figure 1.

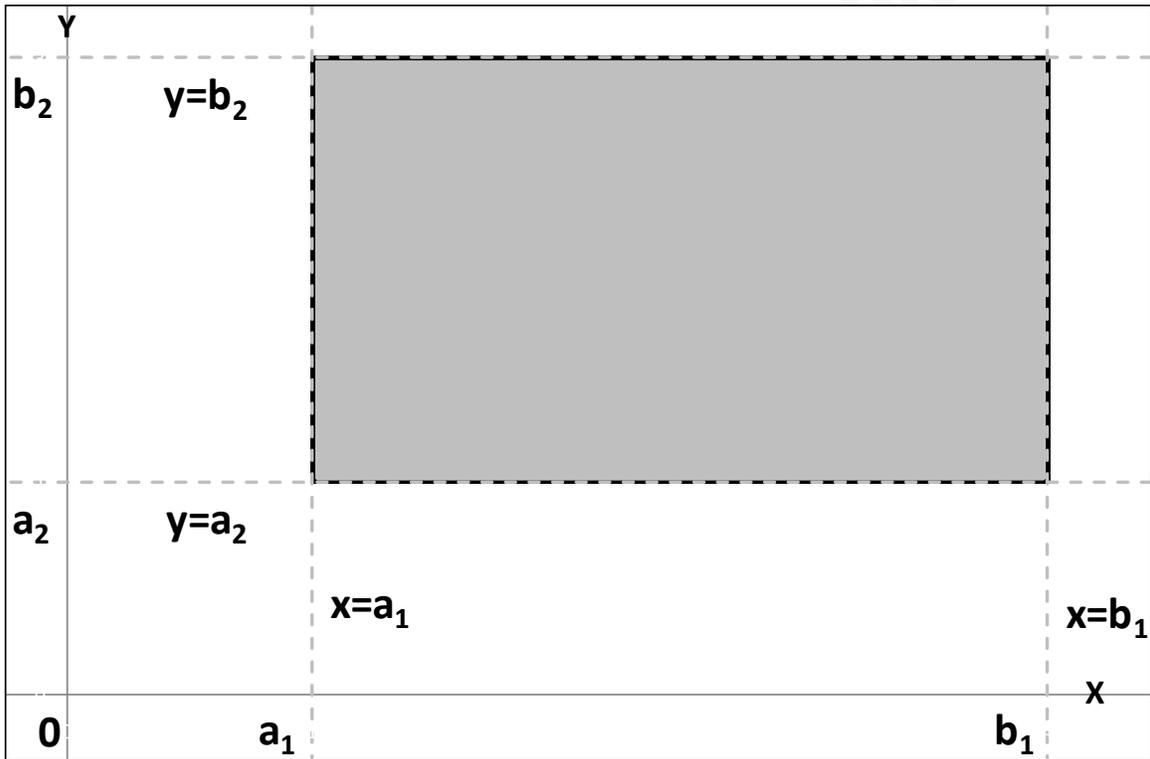
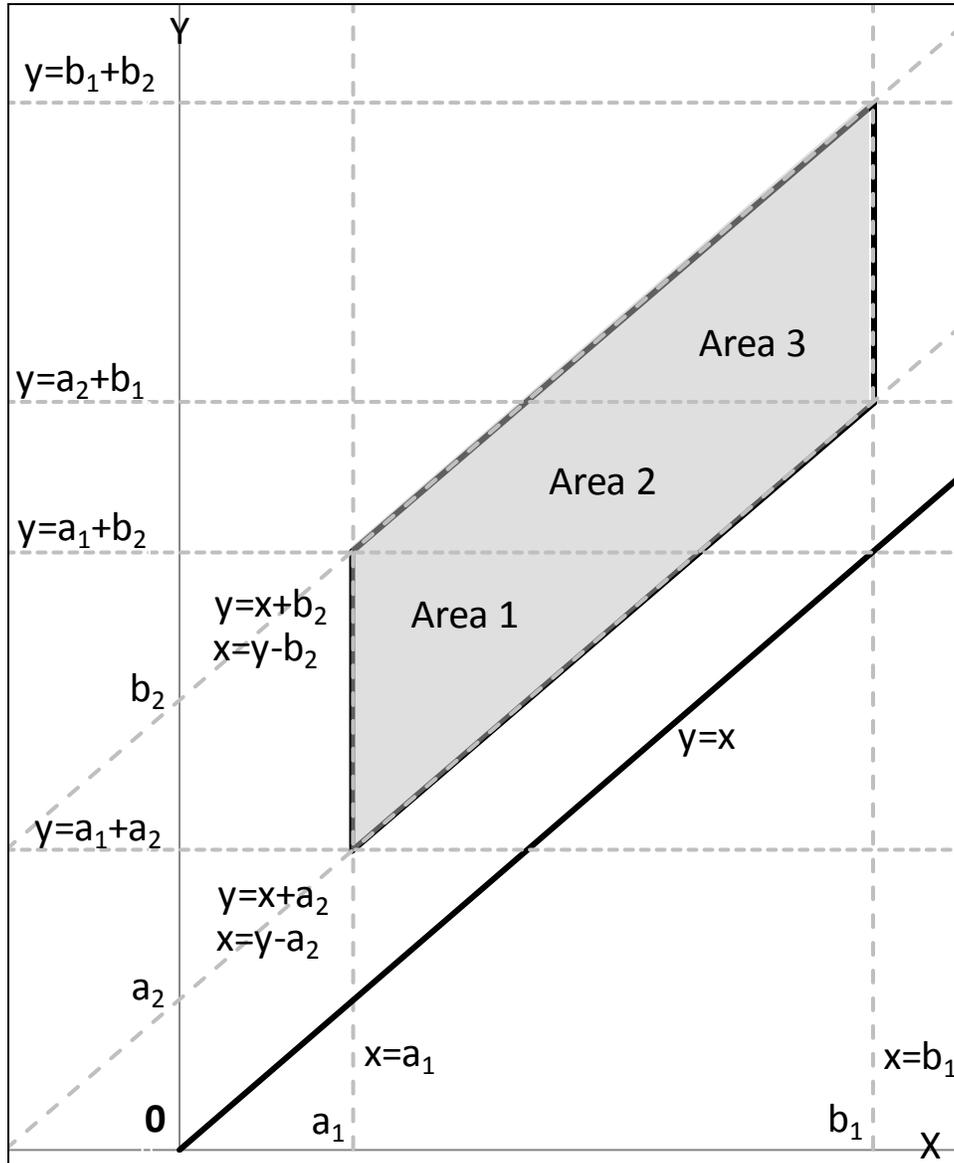


Figure 1: Domain  $[a_1 \leq x \leq b_1, a_2 \leq y \leq b_2]$  of the Function  $z(x,y)=f(x)g(y)$

<sup>1</sup> Linear Systems, Fourier Transforms and Optics, Jack D. Gaskill, John Wiley & Sons, 1978, Chapter 6, Section 6-1, Page 150

Now translate the function  $z(x,y)$  in the  $y$ -direction by the amount  $x$ . In other words, move the rectangle shown in Figure 1 up in the positive  $y$  direction so that the bottom boundary of the domain is a distance  $a_2$  above the line  $y=x$ . This changes the function  $z(x,y)$  to the function  $z(x,y-x)$ , as seen in Figure 2.



**Figure 2: Domain  $[a_1 \leq x \leq b_1, a_2 \leq y-x \leq b_2] = [a_1 \leq x \leq b_1, x+a_2 \leq y \leq x+b_2]$  of the Function  $z(x,y-x) = f(x)g(y-x)$**

Let a plane perpendicular to the  $xy$ -plane and the  $yz$ -plane and parallel to the  $xz$ -plane intersect the function  $z(x,y-x)$  shown in Figure 2 at some point  $y' \in [a_1+a_2, b_1+b_2]$ . The cross-sectional area formed by this plane and the function  $z(x,y-x)$  is:

$$A(y') = \int_{x_1}^{x_2} f(x)g(y' - x)dx$$

where: if  $a_1+a_2 \leq y' \leq a_1+b_2$  (Area 1), then  $x_1 = a_1$  and  $x_2 = y' - a_2$   
 if  $a_1+b_2 \leq y' \leq a_2+b_1$  (Area 2), then  $x_1 = y' - b_2$  and  $x_2 = y' - a_2$   
 if  $a_2+b_1 \leq y' \leq b_1+b_2$  (Area 3), then  $x_1 = y' - b_2$  and  $x_2 = b_1$

But, since the function  $z(x,y-x)$  does not exist outside of the closed interval  $[a_1, b_1]$ , we can extend the limits of integration to  $x_1=-\infty$  and to  $x_2=+\infty$  without changing the value of the integral. This now gives us the following equation for the cross-sectional area:

$$A(y') = \int_{-\infty}^{\infty} f(x)g(y' - x)dx$$

which is exactly the definition of the convolution.

The following Examples show how to apply this method.

### Example 1: Convolution of a Right Triangle Function & a Rectangle Function<sup>2</sup>

Suppose we want to convolve two functions: the first function is a right triangle with endpoints (0,0), (3,0) and (3,2); the second function is a rectangle with endpoints (-1,0), (3,0), (3,1) and (-1,1). Let the first function be  $f(x)$  such that:

$$f(x) = \begin{cases} \frac{2x}{3}, & 0 \leq x \leq 3 \\ 0, & x < 0, x > 3 \end{cases}$$

Let the second function by  $g(y)$  such that:

$$g(y) = \begin{cases} 1, & -1 \leq y \leq 3 \\ 0, & y < -1, y > 3 \end{cases}$$

The two functions are shown in Figure 3. Their domain in the  $xy$ -plane is shown in Figure 4, and their translated domain with respect to the line  $y=x$  is shown in Figure 5

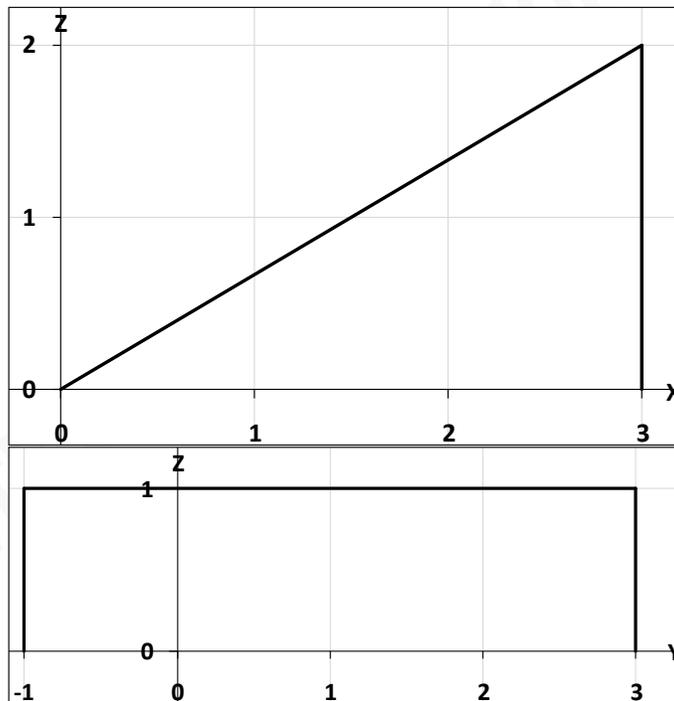
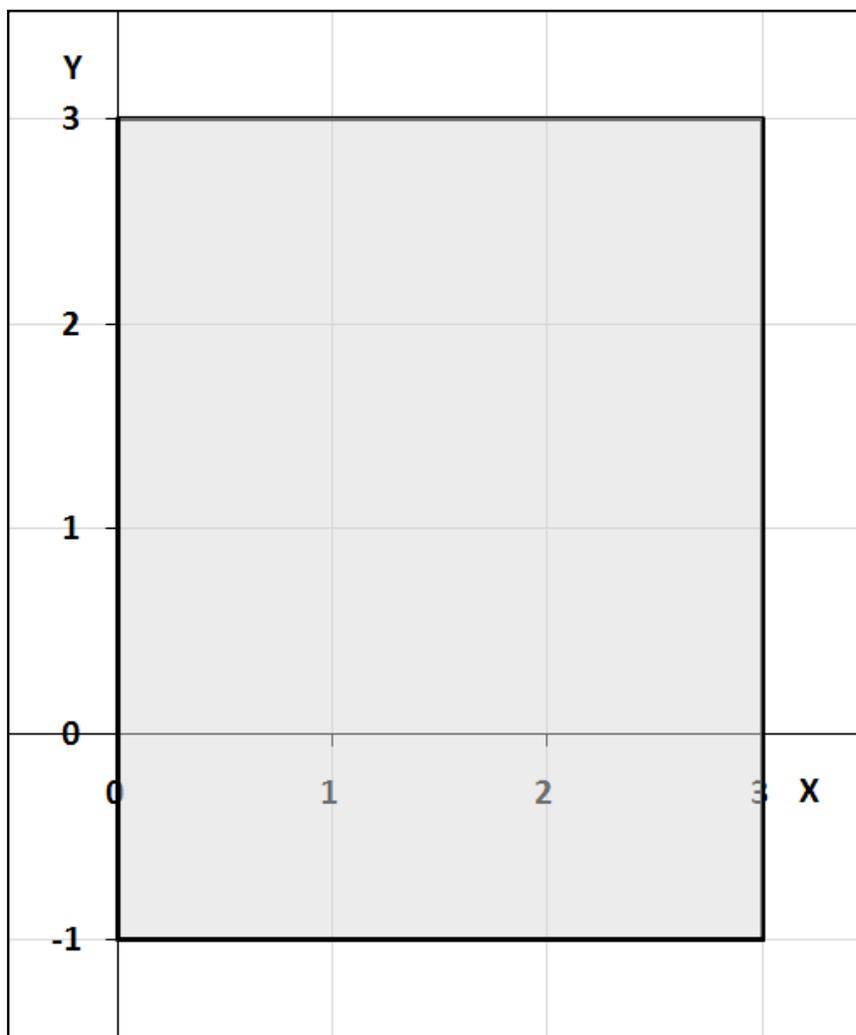


Figure 3: Two Functions to Convolve in Example 1: Right Triangle  $f(x)$  and Rectangle  $g(y)$

<sup>2</sup> This example is derived from the example shown in Linear Systems, Fourier Transforms and Optics, Jack D. Gaskill, John Wiley & Sons, 1978, Chapter 6, Section 6-1, Pages 154-155



**Figure 4: Domain  $[0 \leq x \leq 3, -1 \leq y \leq 3]$  of the Function  $z(x,y)=f(x)g(y)$**

In Figure 5, there are three different sets of limits of integration of the integral: in Area 1, where  $-1 \leq y \leq 2$ , the value of  $x$  can vary from 0 to  $y+1$ ; in Area 2, where  $2 \leq y \leq 3$ , the value of  $x$  can vary from 0 to 3; and in Area 3, where  $3 \leq y \leq 6$ , the value of  $x$  can vary from  $y-3$  to 3. For all  $y < -1$  and  $y > 6$ , the integral is equal to zero since the function  $z$  does not exist there. In the closed  $y$ -interval  $[-1, 6]$ , we can evaluate the integral in the three intervals associated with Areas 1, 2, and 3 to get a piecewise solution. Plug  $f(x)$  into the equation for  $A(y')$ . For the function  $g(y)$ , substitute  $y-x$  for  $y$  throughout the function and then plug this function  $g(y-x)$  into the equation for  $A(y')$ . Since  $g(y)$  is constant in the interval  $[-1, 3]$ , it will not change when translated from  $y$  to  $y-x$ .

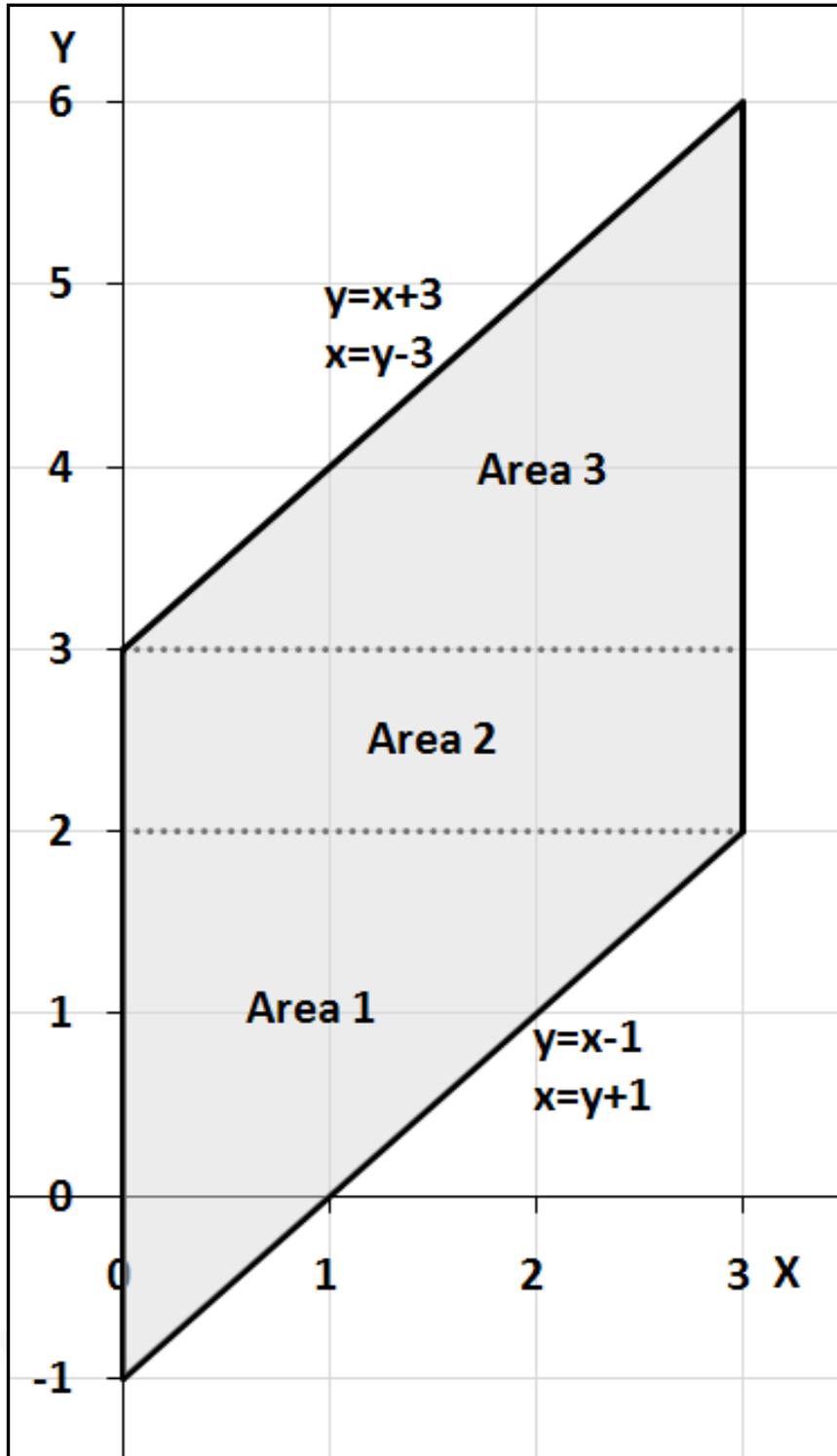


Figure 5: Domain  $[1 \leq x \leq 3, -1 \leq y-x \leq 3] = [1 \leq x \leq 3, x-1 \leq y \leq x+3]$  of the Function  $z(x,y-x)=f(x)g(y-x)$

In the  $y$ -interval associated with Area 1,  $-1 \leq y \leq 2$ , we can evaluate the integral as follows:

$$h_1(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_1(y) = \int_0^{y+1} \left(\frac{2}{3}x\right) (1)dx = \frac{2}{3} \int_0^{y+1} x dx = \frac{x^2}{3} \Big|_0^{y+1} = \frac{(y+1)^2}{3}$$

In the  $y$ -interval associated with Area 2,  $2 \leq y \leq 3$ , we can evaluate the integral as follows:

$$h_2(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_2(y) = \int_0^3 \left(\frac{2}{3}x\right) (1)dx = \frac{2}{3} \int_0^3 x dx = \frac{x^2}{3} \Big|_0^3 = 3$$

In the  $y$ -interval associated with Area 3,  $3 \leq y \leq 6$ , we can evaluate the integral as follows:

$$h_3(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_3(y) = \int_{y-3}^3 \left(\frac{2}{3}x\right) (1)dx = \frac{2}{3} \int_{y-3}^3 x dx = \frac{x^2}{3} \Big|_{y-3}^3 = 3 - \frac{(y-3)^2}{3}$$

Finally, we can specify the entire solution of the convolution as:<sup>3</sup>

$$h(x) = \begin{cases} 0, & x < -1 \\ \frac{(x+1)^3}{3}, & -1 \leq x \leq 2 \\ 3, & 2 \leq x \leq 3 \\ 3 - \frac{(x-3)^2}{3}, & 3 \leq x \leq 6 \\ 0, & x > 6 \end{cases}$$

Since convolution is commutative, we will get the same answer if we switch the functions around and let the first function be the rectangle and the second function be the right triangle such that:

$$f(x) = \begin{cases} 1, & -1 \leq x \leq 3 \\ 0, & x < -1, x > 3 \end{cases}$$

and

$$g(y) = \begin{cases} \frac{2y}{3}, & 0 \leq y \leq 3 \\ 0, & y < 0, y > 3 \end{cases}$$

For this switch, the domain in the  $xy$ -plane is shown in Figure 6, and the translated domain with respect to the line  $y=x$  is shown in Figure 7.

<sup>3</sup> Linear Systems, Fourier Transforms and Optics, Jack D. Gaskill, John Wiley & Sons, 1978, Chapter 6, Section 6-1, Pages 154-155

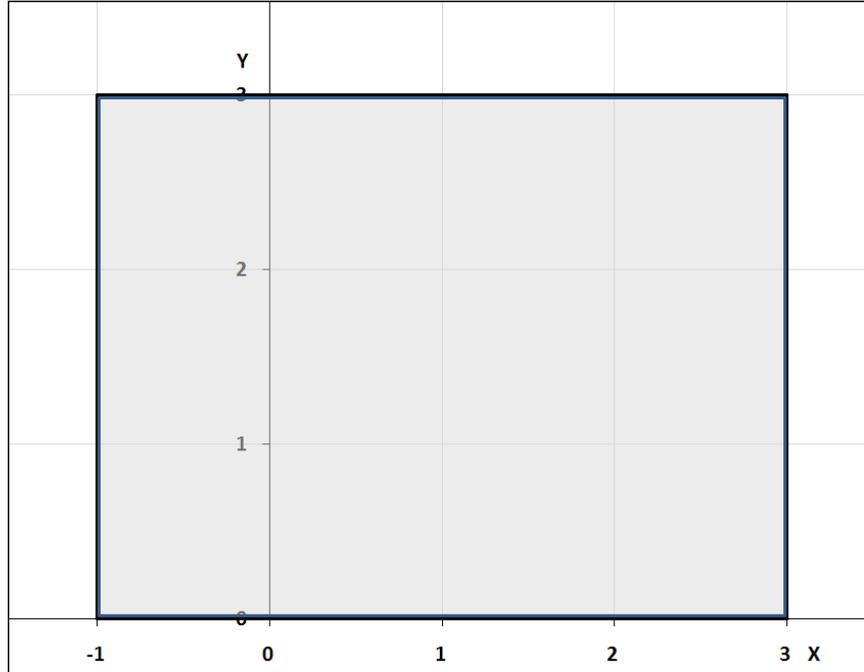


Figure 6: Domain  $[-1 \leq x \leq 3, 0 \leq y \leq 3]$  of the Switched Function  $z(x,y)=f(x)g(y)$

In Figure 7, there are three different sets of limits of integration of the integral: in Area 1, where  $-1 \leq y \leq 2$ , the value of  $x$  can vary from 0 to  $y$ ; in Area 2, where  $2 \leq y \leq 3$ , the value of  $x$  can vary from  $y-3$  to  $y$ ; and in Area 3, where  $3 \leq y \leq 6$ , the value of  $x$  can vary from  $y-3$  to 3. For all  $y < -1$  and  $y > 6$ , the integral is equal to zero since the function  $z$  does not exist there. In the closed  $y$ -interval  $[-1, 6]$ , we can evaluate the integral in the three intervals associated with Areas 1, 2, and 3 to get a piecewise solution. Plug  $f(x)$  into the equation for  $A(y)$ . For the function  $g(y)$ , substitute  $y-x$  for  $y$  throughout the function and then plug this function  $g(y-x)$  into the equation for  $A(y)$ .

In the  $y$ -interval associated with Area 1,  $-1 \leq y \leq 2$ , we can evaluate the integral as follows:

$$h_1(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_1(y) = \int_{-1}^y (1) \left( \frac{2}{3}(y-x) \right) dx = \frac{2}{3} \int_{-1}^y (y-x) dx = \frac{2}{3} \left( xy - \frac{x^2}{2} \right) \Big|_{-1}^y = \frac{2}{3} \left[ \left( y^2 - \frac{y^2}{2} \right) - \left( -y - \frac{1}{2} \right) \right]$$

$$h_1(y) = \frac{2}{3} \left( \frac{y^2}{2} + y + \frac{1}{2} \right) = \frac{1}{3} (y^2 + 2y + 1) = \frac{(y+1)^2}{3}$$

In the  $y$ -interval associated with Area 2,  $2 \leq y \leq 3$ , we can evaluate the integral as follows:

$$h_2(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_2(y) = \int_{y-3}^y (1) \left( \frac{2}{3}(y-x) \right) dx = \frac{2}{3} \int_{y-3}^y (y-x) dx = \frac{2}{3} \left( xy - \frac{x^2}{2} \right) \Big|_{y-3}^y$$

$$h_2(y) = \frac{2}{3} \left[ \left( y^2 - \frac{y^2}{2} \right) - \left( y(y-3) - \frac{(y-3)^2}{2} \right) \right] = \frac{2}{3} \left( \frac{y^2}{2} - \left( y^2 - 3y - \frac{y^2}{2} + \frac{6y}{2} - \frac{9}{2} \right) \right)$$

$$h_2(y) = \frac{2}{3} \left( \frac{y^2}{2} - \frac{y^2}{2} + 3y - 3y + \frac{9}{2} \right) = 3$$

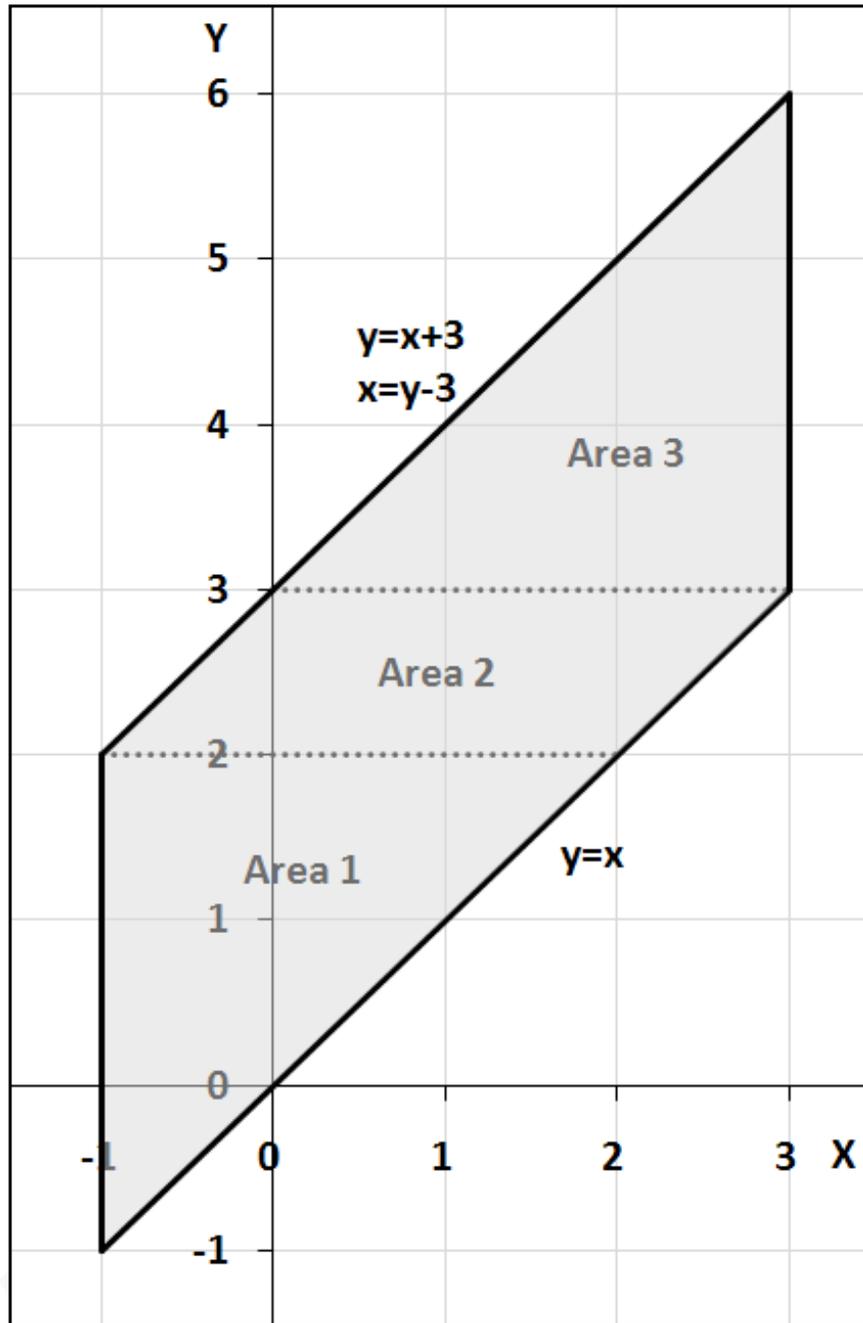


Figure 7: Domain  $[-1 \leq x \leq 3, 0 \leq y-x \leq 3] = [1 \leq x \leq 3, x \leq y \leq x+3]$  of the Switched Function  $z(x, y-x) = f(x)g(y-x)$

In the  $y$ -interval associated with Area 3,  $3 \leq y \leq 6$ , we can evaluate the integral as follows:

$$h_3(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_3(y) = \int_{y-3}^3 (1) \left( \frac{2}{3}(y-x) \right) dx = \frac{2}{3} \int_{y-3}^3 (y-x) dx = \frac{2}{3} \left( xy - \frac{x^2}{2} \right) \Big|_{y-3}^3$$

$$h_3(y) = \frac{2}{3} \left[ \left( 3y - \frac{9}{2} \right) - \left( y(y-3) - \frac{(y-3)^2}{2} \right) \right] = \frac{2}{3} \left( 3y - \frac{9}{2} - \left( y^2 - 3y - \frac{y^2}{2} + \frac{6y}{2} - \frac{9}{2} \right) \right)$$

$$h_3(y) = \frac{2}{3} \left( 3y - \frac{9}{2} - \frac{y^2}{2} + \frac{9}{2} \right) = \frac{2}{3} (3y - y^2) = 2y - \frac{y^2}{3}$$

By completing the square, we can change this equation as follows:

$$h_3(y) = -\frac{1}{3}(y^2 - 6y) = -\frac{1}{3}(y^2 - 6y + 9 - 9) = -\frac{1}{3}((y-3)^2 - 9) = 3 - \frac{(y-3)^2}{3}$$

Finally, we can again specify the entire solution of the convolution exactly as before:

$$h(x) = \begin{cases} 0, & x < -1 \\ \frac{(x+1)^3}{3}, & -1 \leq x \leq 2 \\ 3, & 2 \leq x \leq 3 \\ 3 - \frac{(x-3)^2}{3}, & 3 \leq x \leq 6 \\ 0, & x > 6 \end{cases}$$

This is exactly the same as the previous solution. The graph of this solution is shown in Figure 8.

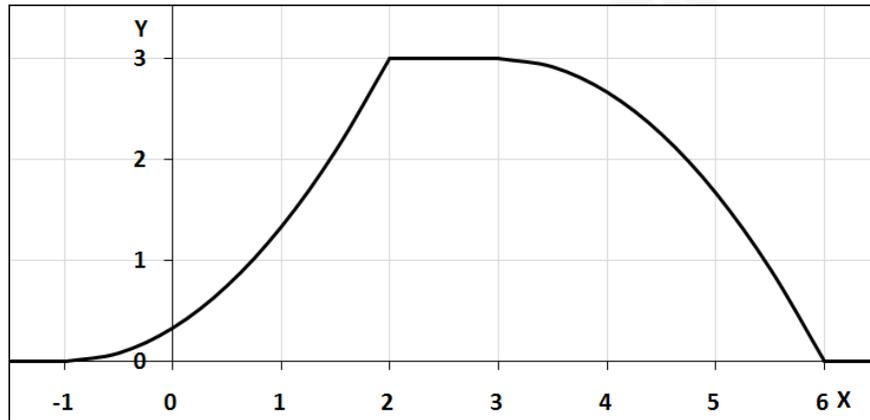


Figure 8: Results of the Convolution of the Functions in Example 1

### Example 2: Convolution of a Right Triangle Function with Itself

Suppose we want to convolve the following probability density function (PDF) with itself:

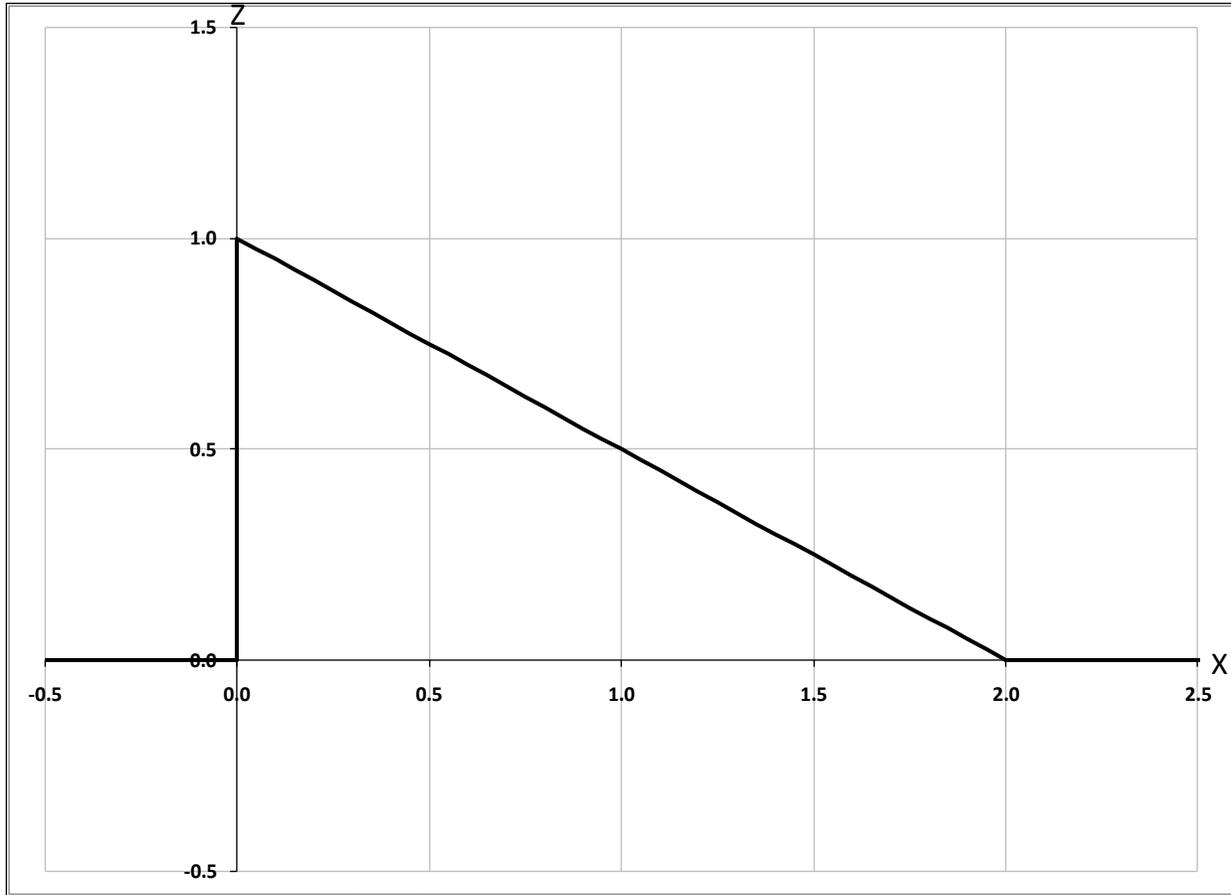
$$f(x) = \begin{cases} 1 - \frac{x}{2}, & 0 \leq x \leq 2 \\ 0, & x < 0, x > 2 \end{cases}$$

This PDF is a right triangle with endpoints (0,0), (2,0) and (0,1) and is shown in Figure 9. To convolve this PDF with itself, the second function will then be  $g(y)$  such that:

$$g(y) = \begin{cases} 1 - \frac{y}{2}, & 0 \leq y \leq 2 \\ 0, & y < 0, y > 2 \end{cases}$$

The domain of these two PDF s in the  $xy$ -plane is shown in Figure 10, and their translated domain with respect to the line  $y=x$  is shown in Figure 11.

In Figure 11, there are two different sets of limits of integration of the integral: in Area 1, where  $0 \leq y \leq 2$ , the value of  $x$  can vary from 0 to  $y$ ; in Area 2, where  $2 \leq y \leq 4$ , the value of  $x$  can vary from  $y-2$  to 2. For all  $y < 0$  and  $y > 4$ , the integral is equal to zero since the function  $z$  does not exist there. In the closed  $y$ -interval  $[0,4]$ , we can evaluate the integral in the two intervals associated with Areas 1 and 2 to get a piecewise solution. Plug  $f(x)$  into the equation for  $A(y')$ . For the function  $g(y)$ , substitute  $y-x$  for  $y$  throughout the function and then plug this function  $g(y-x)$  into the equation for  $A(y')$ .



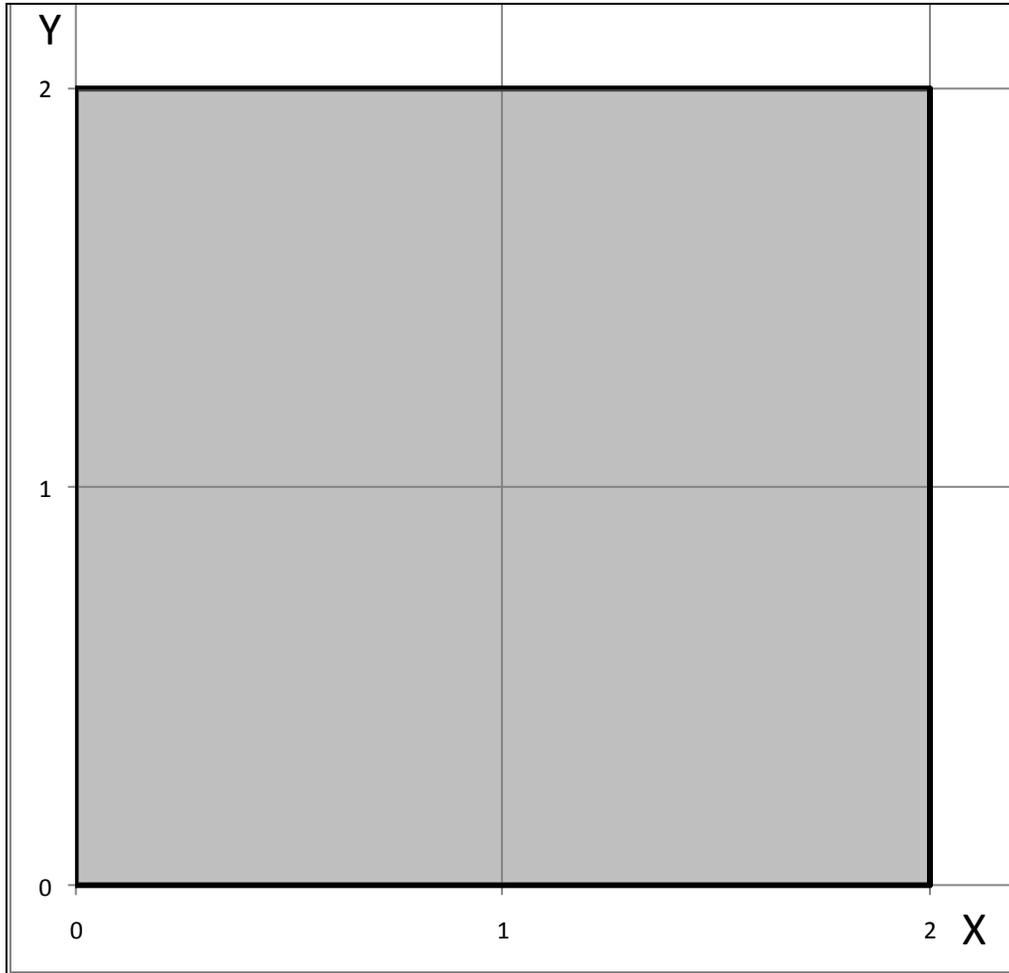
**Figure 9: The Right Triangle Function to Convolve with Itself in Example 2**

In the  $y$ -interval associated with Area 1,  $0 \leq y \leq 2$ , we can evaluate the integral as follows:

$$h_1(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_1(y) = \int_0^y \left(1 - \frac{x}{2}\right) \left(1 - \frac{y-x}{2}\right) dx = \frac{1}{4} \int_0^{y+1} (2-x)(2-y+x) dx = \frac{1}{4} \int_0^{y+1} (4-2y+xy-x^2) dx$$

$$h_1(y) = \frac{1}{4} \left(4y - 2y^2 + \frac{y^3}{2} - \frac{y^3}{3}\right) \Big|_0^y = \frac{y}{4} \left(4 - 2y + \frac{y^2}{6}\right) = \frac{y}{24} (24 - 12y + y^2)$$



**Figure 10: Domain  $[0 \leq x \leq 2, 0 \leq y \leq 2]$  of the Function  $z(x,y)=f(x)g(y)$**

In the  $y$ -interval associated with Area 2,  $2 \leq y \leq 4$ , we can evaluate the integral as follows:

$$h_2(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_2(y) = \int_{y-2}^2 \left(1 - \frac{x}{2}\right) \left(1 - \frac{y-x}{2}\right) dx = \frac{1}{4} \left(4y - 2y^2 + \frac{y^3}{2} - \frac{y^3}{3}\right) \Big|_{y-2}^2$$

$$h_2(y) = \frac{1}{4} \left(8 - 4y + 2y - \frac{8}{3} - \left(4(y-2) - 2y(y-2) + \frac{y(y-2)^2}{2} - \frac{(y-2)^3}{3}\right)\right)$$

$$h_2(y) = \frac{1}{4} \left(\frac{16}{3} - 2y - \left(4y - 8 - 2y^2 + 4y + \frac{y^3 - 4y^2 + 4y}{2} - \frac{y^3 - 6y^2 + 12y - 8}{3}\right)\right)$$

$$h_2(y) = \frac{1}{4} \left(\frac{32 - 60y + 48 + 12y^2 - 3y^3 + 12y^2 - 12y + 2y^3 - 12y^2 + 2y^3 - 12y^2 + 24y - 16}{6}\right)$$

$$h_2(y) = \frac{64 - 48y + 12y^2 - y^3}{24}$$

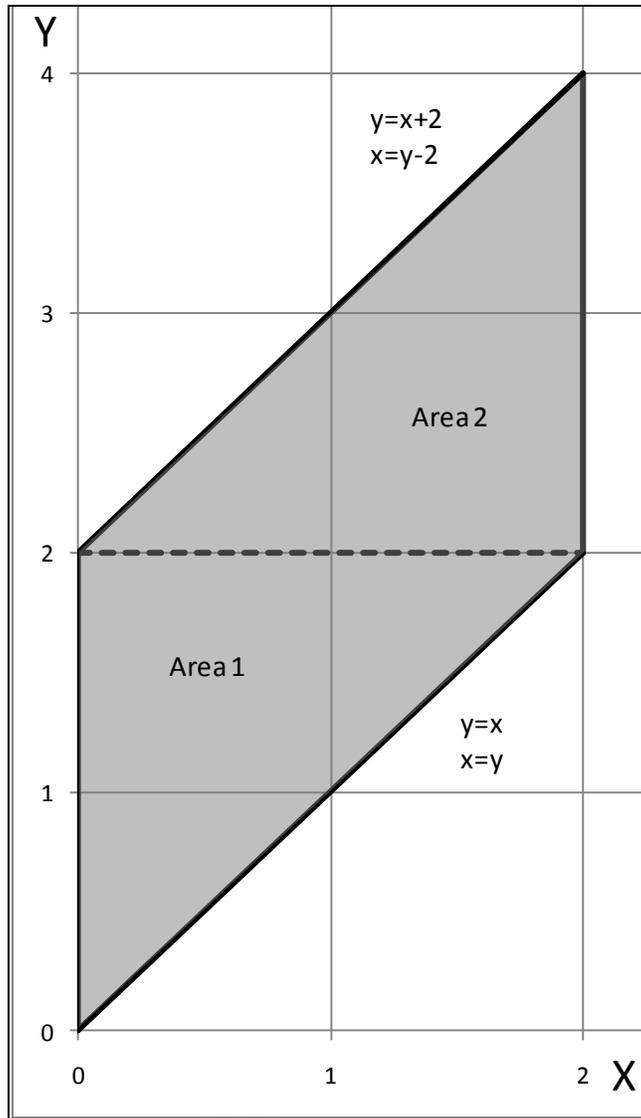
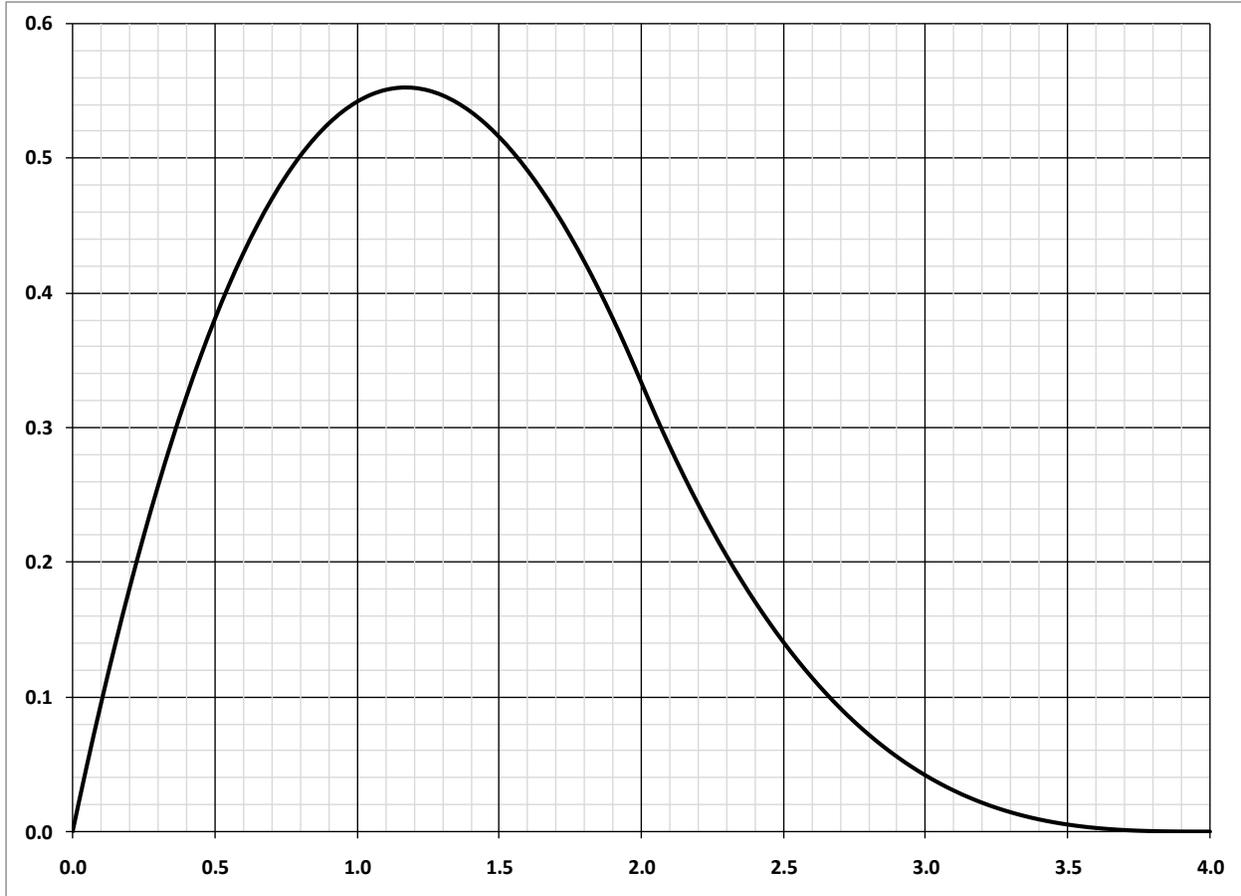


Figure 11: Domain  $[0 \leq x \leq 2, 0 \leq y-x \leq 2] = [0 \leq x \leq 2, x \leq y \leq x+2]$  of the Function  $z(x, y-x) = f(x)g(y-x)$

Finally, we can specify the entire solution of the convolution as:

$$h(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{24}(24 - 12x + x^2), & 0 \leq x < 2 \\ \frac{64 - 48x + 12x^2 - x^3}{24}, & 2 \leq x \leq 4 \\ 0, & x > 4 \end{cases}$$

The graph of this solution is shown in Figure 12.



**Figure 12: Results of the Convolution of the Function in Example 2 with Itself**

Another advantage of this method of solving convolutions by direct integration is that it can allow people to solve convolutions of functions that have variable shapes, provided that the functions being convolved are independent of each other.<sup>4</sup> The following example shows how to do this.

### **Example 3: Convolution of a Right Triangle Function & a Varying Isosceles Triangle Function**

Suppose we want to find the convolution of two probability density functions (PDFs). Let the first PDF be:

$$f(x) = \begin{cases} \frac{3-x}{2}, & 1 \leq x \leq 3 \\ 0, & x < 1, x > 3 \end{cases}$$

This PDF is a right triangle with vertices at (1,0), (1,1) and (3,0). Let the second PDF,  $g(x)$ , be an isosceles triangle centered (and having its mode) at any point on the positive  $x$ -axis. The base of this triangle extends to  $\pm 50\%$  of the central value on the  $x$ -axis, and the probability goes to zero at the ends of this base. Since the length of the base increases as the central value on the  $x$ -axis increases, the height of the triangle must decrease as the central value on the  $x$ -axis increases to keep the area of the PDF equal to

---

<sup>4</sup> An example of a varying shape would be the PDF associated with a multiplicative estimating error. Such an error is typically stated as a plus-or-minus percent of a given value with the PDF mode at the given value and the PDF going to zero at the maximum plus or minus percent.

one. So, the length of the base,  $b$ , of the isosceles triangle PDF centered at a point  $x_c$  on the  $x$ -axis will be  $b=2(x_c/2)=x_c$ . Using the standard equation for the area of a triangle, we can then calculate the height of the isosceles triangle PDF:

$$A = \frac{bh}{2} = 1$$

$$h = \frac{2}{b}$$

To find the convolution of these two PDFs, keep the first function, the right-triangle PDF, as a function of  $x$ ,  $f(x)$ . Make the second function, the isosceles triangle PDF a function of  $y$ ,  $g(y)$ . For the second PDF, the isosceles triangle is still centered at any point  $x$  on the positive  $x$ -axis, but this triangle is now in a plane parallel to the  $yz$ -plane and perpendicular to the  $xz$ -plane with endpoints at  $(x, x/2, 0)$ ,  $(x, -x/2, 0)$  and  $(x, 0, 2/x)$ . For the side of the triangle in the positive  $y$  region, the slope of the line segment formed by this side is  $m_+ = (2/x - 0)/(0 - x/2) = (-4/x^2)$ , and, for the side of the triangle in the negative  $y$  region, the slope of the line segment formed by this side is  $m_- = (2/x - 0)/(0 - (-x/2)) = (4/x^2)$ . So, the equation of the line segment in the positive  $y$  region is  $z - 0 = (-4/x^2)(y - x/2) = -2(2y - x)/x^2$ , and the equation of the line segment in the negative  $y$  region is  $z - 0 = (4/x^2)(y - (-x/2)) = 2(2y + x)/x^2$ .

The two functions are shown in Figure 13. The domain of these two functions in the  $xy$ -plane is shown in Figure 14, and Figure 15 shows what the function  $z(x, y) = f(x)g(y)$  looks like. The translated domain of the two functions with respect to the line  $y = x$  is shown in Figure 16.

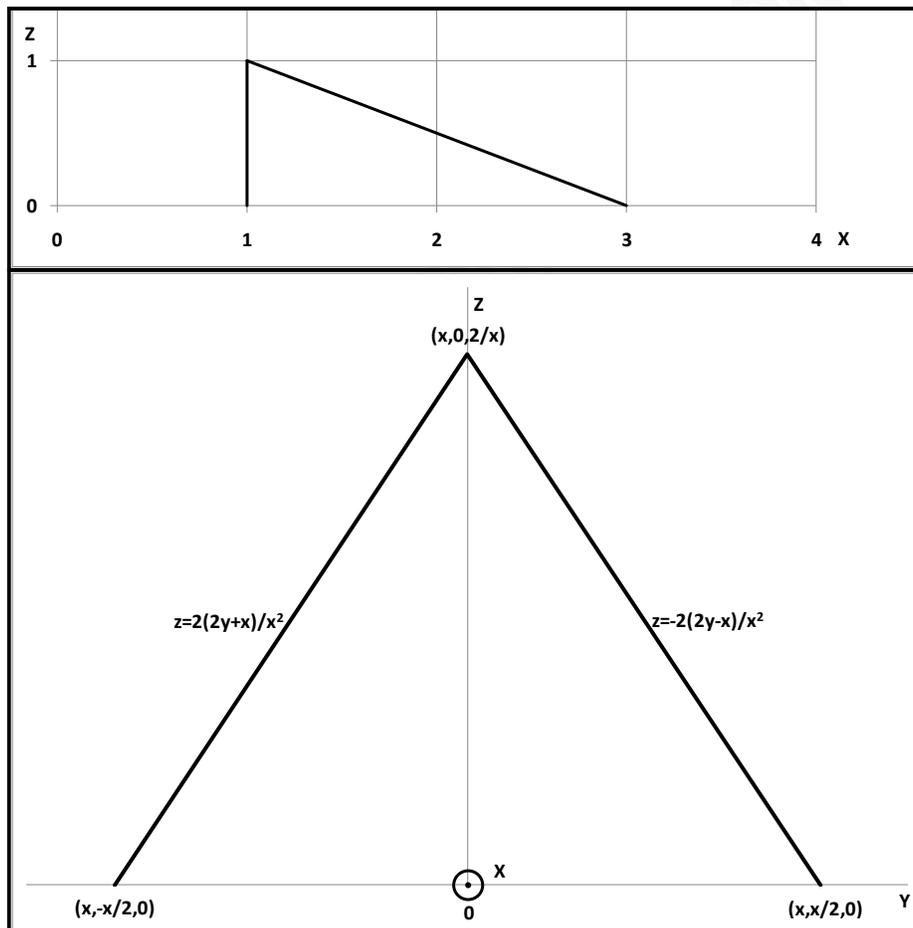


Figure 13: The Two Functions to Convolve in Example 3

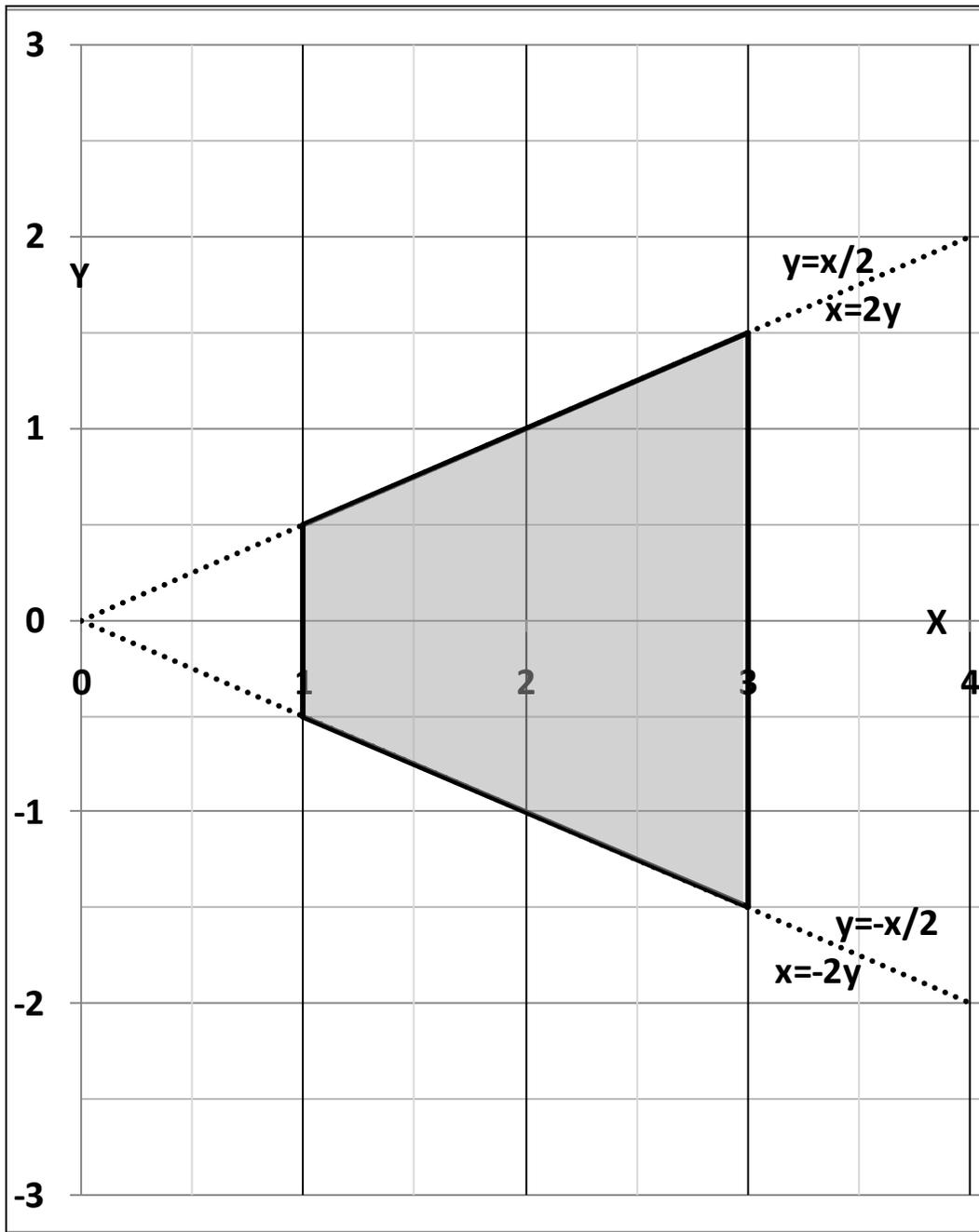
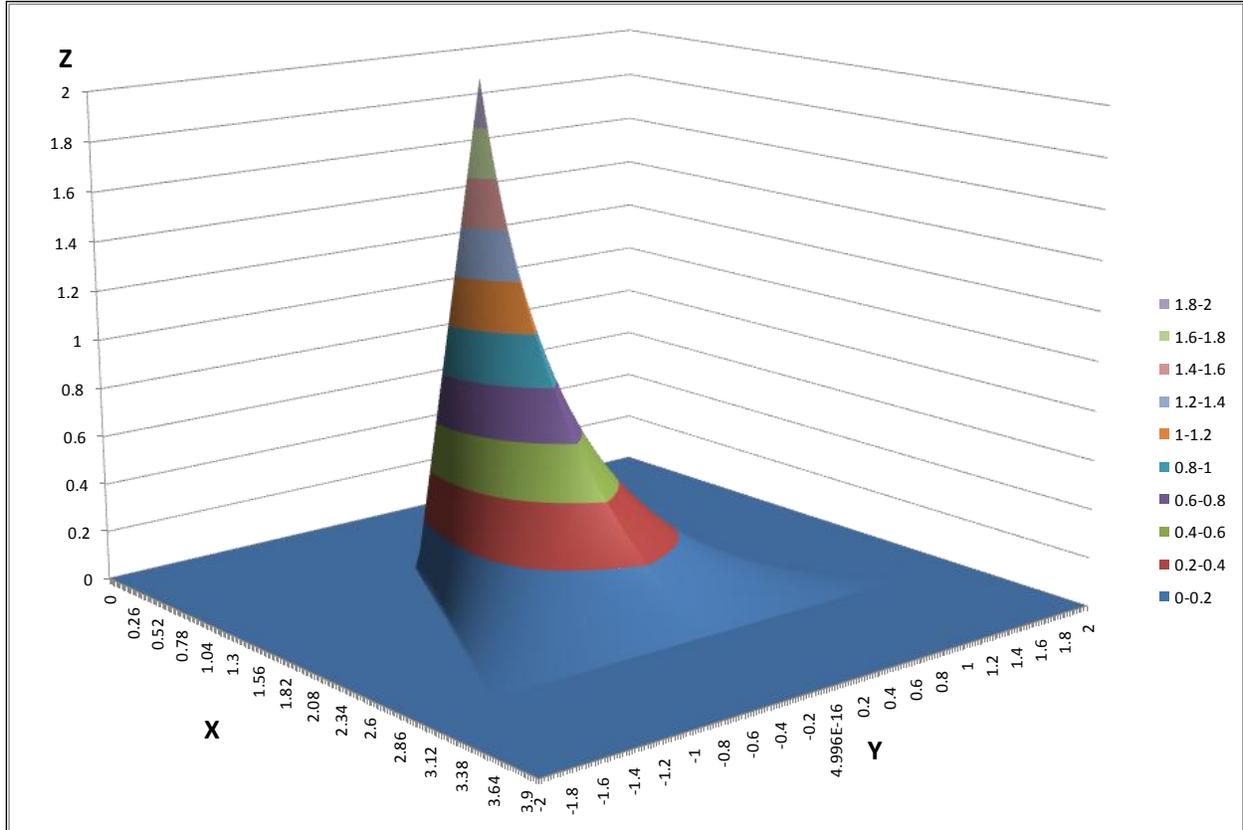


Figure 14: Domain  $[1 \leq x \leq 3, -x/2 \leq y-x \leq x/2] = [1 \leq x \leq 3, x/2 \leq y \leq 3x/2]$  of the Function  $z(x, y-x) = f(x)g(y-x)$

Since  $g(y)$  is a piece-wise continuous function described by one function when  $y > 0$  and a different function when  $y < 0$ , then  $g(y-x)$  is also a piece-wise continuous function described by one function—call it  $g(y-x)_{y>x}$ —when  $y > x$  and a different function—call it  $g(y-x)_{y<x}$ —when  $y < x$ . Also, as shown in Figure 16, there are four different sets of limits of integration of the integrals: in Area 1, where  $0.5 \leq y \leq 1$ , the value of  $x$  can vary from 1 to  $2y$  for the integrand  $f(x)g(y-x)_{y<x}$ ; in Area 2, where  $1 \leq y \leq 1.5$ , the value of  $x$  can vary from 1 to  $y$  for the integrand  $f(x)g(y-x)_{y>x}$  (in Subarea 2,1) and from  $y$  to  $2y$  for the integrand  $f(x)g(y-x)_{y<x}$  (in Subarea 2,2); in Area 3, where  $1.5 \leq y \leq 3$ , the value of  $x$  can vary from  $2y/3$  to  $y$  for the integrand  $f(x)g(y-x)_{y>x}$  (in Subarea 3,1) and from  $y$  to  $3$  for the integrand  $f(x)g(y-x)_{y<x}$  (in Subarea 3,2);

and in Area 4, where  $3 \leq y \leq 4.5$ , the value of  $x$  can vary from  $2y/3$  to  $3$  for the integrand  $f(x)g(y-x)_{y>x}$ . For all  $y < 0.5$  and  $y > 4.5$ , the integral is equal to zero since the function  $z$  does not exist there. In the closed  $y$ -interval  $[0.5, 4.5]$ , we can evaluate the integral in the four intervals associated with Areas 1, 2, 3 and 4 to get a piecewise solution. Plug  $f(x)$  into the equation for  $A(y')$ . For the function  $g(y)$ , substitute  $y-x$  for  $y$  throughout the function and then plug this function  $g(y-x)$  into the equation for  $A(y')$ .



**Figure 15: The Surface of the Product of the Two Functions  $z(x,y)=f(x)g(y)$  in Example 3**

In the  $y$ -interval associated with Area 1,  $0.5 \leq y \leq 1$ , we can evaluate the integral as follows:

$$h_1(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_1(y) = \int_1^{2y} \left( \frac{3-x}{2} \right) \left( \frac{2(2(y-x)+x)}{x^2} \right) dx = \int_1^{2y} \frac{(3-x)(2y-x)}{x^2} dx$$

$$h_1(y) = \int_1^{2y} \left( \frac{6y}{x^2} - \frac{3+2y}{x} + 1 \right) dx = \left( -\frac{6y}{x} - (3+2y) \ln(x) + x \right) \Big|_1^{2y}$$

$$h_1(y) = (-3 - (3+2y) \ln(2y) + 2y) - (-6 - 0 + 1) = 8y - (3+2y) \ln(2y) - 4$$

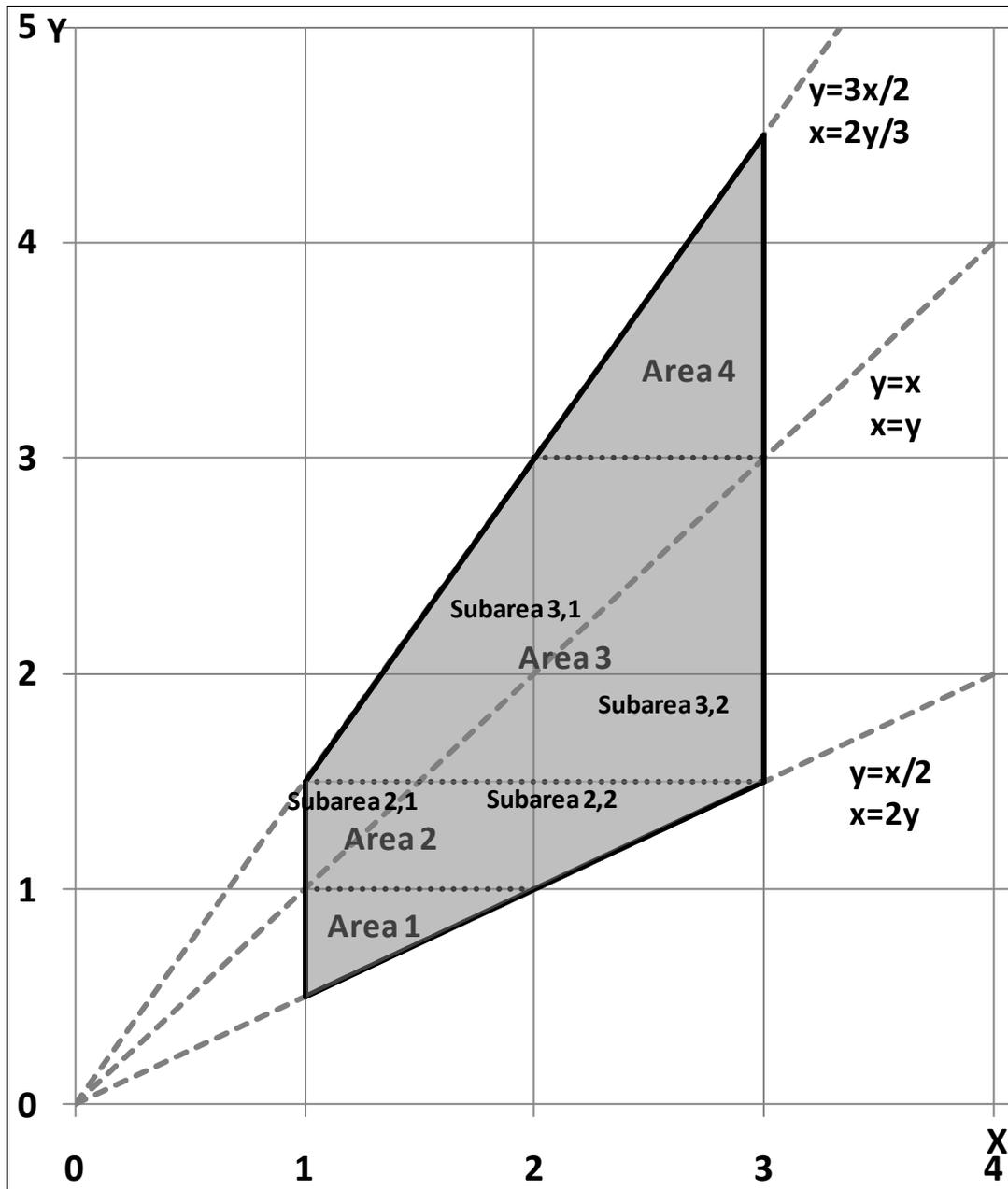


Figure 16: Domain  $[1 \leq x \leq 3, -x/2 \leq y-x \leq x/2] = [1 \leq x \leq 3, x/2 \leq y \leq 3x/2]$  of the Function  $z(x, y-x) = f(x)g(y-x)$

In the  $y$ -interval associated with Area 2,  $1 \leq y \leq 1.5$ , we can evaluate the integral as follows:

$$h_2(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_2(y) = \int_1^y \left(\frac{3-x}{2}\right) \left(-\frac{2(2(y-x)-x)}{x^2}\right) dx + \int_y^{2y} \left(\frac{3-x}{2}\right) \left(\frac{2(2(y-x)+x)}{x^2}\right) dx$$

$$h_2(y) = \int_1^y \frac{(x-3)(2y-3x)}{x^2} dx + \int_y^{2y} \frac{(3-x)(2y-x)}{x^2} dx$$

$$h_2(y) = \int_1^y \left( -\frac{6y}{x^2} + \frac{9+2y}{x} - 3 \right) dx + \int_y^{2y} \left( \frac{6y}{x^2} - \frac{3+2y}{x} + 1 \right) dx$$

$$h_2(y) = \left( \frac{6y}{x} + (9+2y) \ln(x) - 3x \right) \Big|_1^y + \left( -\frac{6y}{x} - (3+2y) \ln(x) + x \right) \Big|_y^{2y}$$

$$h_2(y) = (6 + (9+2y) \ln(y) - 3y) - (6y - 3) + (-3 - (3+2y) \ln(2y) + 2y) - (-6 - (3+2y) \ln(y) + y)$$

$$h_2(y) = 12 - 8y + 4(3+y) \ln(y) - (3+2y) \ln(2y)$$

In the  $y$ -interval associated with Area 3,  $1.5 \leq y \leq 3$ , we can evaluate the integral as follows:

$$h_3(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_3(y) = \int_{\frac{2y}{3}}^y \left( \frac{3-x}{2} \right) \left( -\frac{2(2(y-x)-x)}{x^2} \right) dx + \int_y^3 \left( \frac{3-x}{2} \right) \left( \frac{2(2(y-x)+x)}{x^2} \right) dx$$

$$h_3(y) = \int_{\frac{2y}{3}}^y \frac{(x-3)(2y-3x)}{x^2} dx + \int_y^3 \frac{(3-x)(2y-x)}{x^2} dx$$

$$h_3(y) = \int_{\frac{2y}{3}}^y \left( -\frac{6y}{x^2} + \frac{9+2y}{x} - 3 \right) dx + \int_y^3 \left( \frac{6y}{x^2} - \frac{3+2y}{x} + 1 \right) dx$$

$$h_3(y) = \left( \frac{6y}{x} + (9+2y) \ln(x) - 3x \right) \Big|_{\frac{2y}{3}}^y + \left( -\frac{6y}{x} - (3+2y) \ln(x) + x \right) \Big|_y^3$$

$$h_3(y) = (6 + (9+2y) \ln(y) - 3y) - \left( 9 + (9+2y) \ln\left(\frac{2y}{3}\right) - 2y \right) + (-2y - (3+2y) \ln(3) + 3) - (-6 - (3+2y) \ln(y) + y)$$

$$h_3(y) = 4(3+y) \ln(y) - (9+2y) \ln\left(\frac{2y}{3}\right) - (3+2y) \ln(3) + 6 - 4y$$

In the  $y$ -interval associated with Area 4,  $3 \leq y \leq 4.5$ , we can evaluate the integral as follows:

$$h_4(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_4(y) = \int_{\frac{2y}{3}}^3 \left( \frac{3-x}{2} \right) \left( -\frac{2(2(y-x)-x)}{x^2} \right) dx = \left( \frac{6y}{x} + (9+2y) \ln(x) - 3x \right) \Big|_{\frac{2y}{3}}^3$$

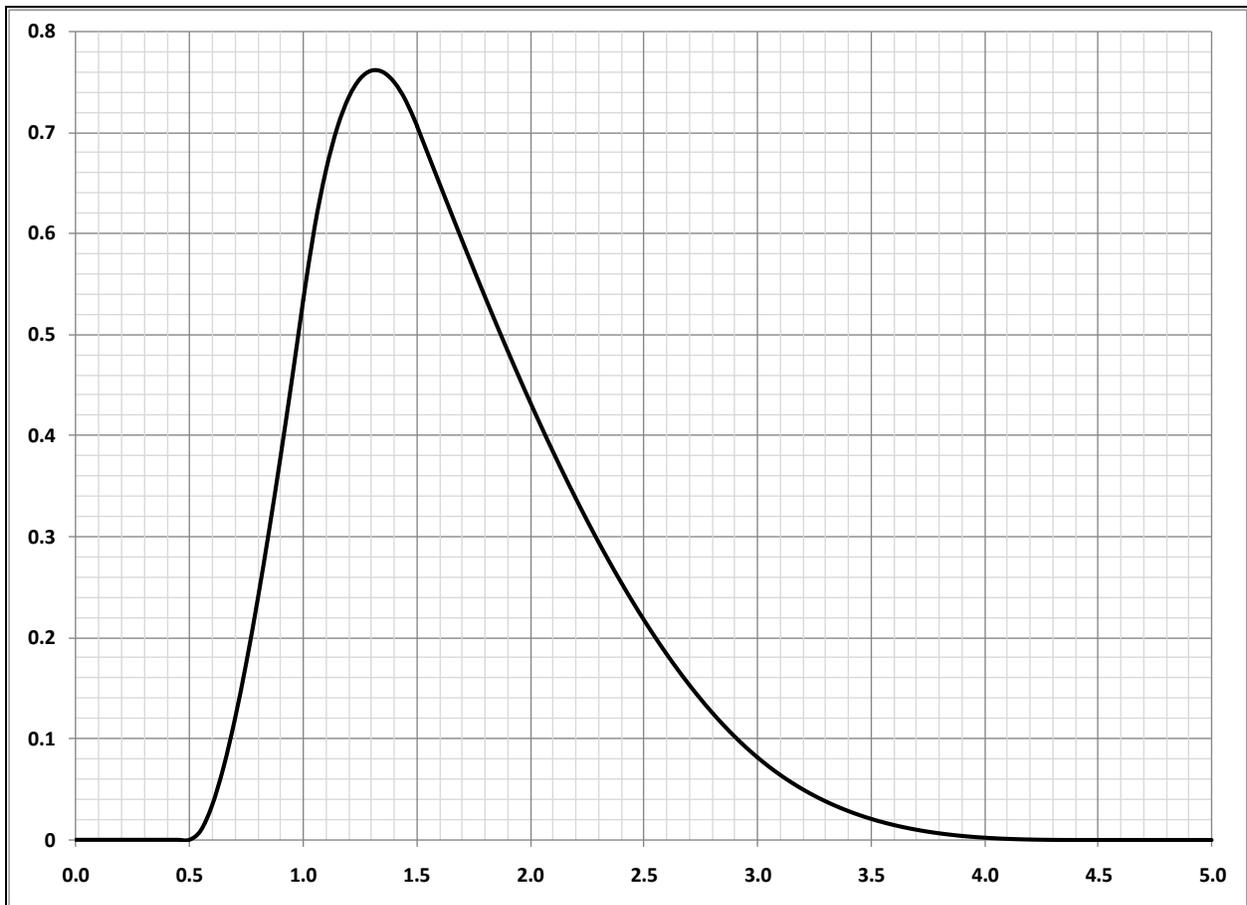
$$h_4(y) = (2y + (9+2y) \ln(3) - 9) - \left( 9 + (9+2y) \ln\left(\frac{2y}{3}\right) - 2y \right)$$

$$h_4(y) = (9+2y) \ln\left(\frac{9}{2y}\right) + 4y - 18$$

Finally, we can specify the entire solution of the convolution as:

$$h(x) = \begin{cases} 0, & x < 0.5 \\ 8x - (3 + 2x) \ln(2x) - 4, & 0.5 \leq x < 1 \\ 12 - 8x + 4(3 + x) \ln(x) - (3 + 2x) \ln(2x), & 1 \leq x \leq 1.5 \\ 4(3 + x) \ln(x) - (9 + 2x) \ln\left(\frac{2x}{3}\right) - (3 + 2x) \ln(3) + 6 - 4x, & 1.5 \leq x \leq 3 \\ (9 + 2x) \ln\left(\frac{9}{2x}\right) + 4x - 18, & 3 \leq x \leq 4.5 \\ 0, & x > 4.5 \end{cases}$$

The graph of this solution is shown in Figure 17.



**Figure 17: The Results of the Convolution of the Functions in Example 3**

## Repeated Convolution

Given the convolution  $\mathbf{h(x)=f(x)*g(x)}$ , it is generally true that  $\mathbf{h(x)}$  will be smoother than either  $\mathbf{f(x)}$  or  $\mathbf{g(x)}$ , and the convolution of a large number of functions generally yields a function that is much smoother than any of the individual functions involved in the convolution.<sup>5</sup> In addition, as the number of functions convolved becomes larger and larger, the resulting convolution often begins to look more and more like a Gaussian function. For example, consider the convolution:

$$h(x) = f_1(x) * f_2(x) * f_3(x) * \dots * f_n(x)$$

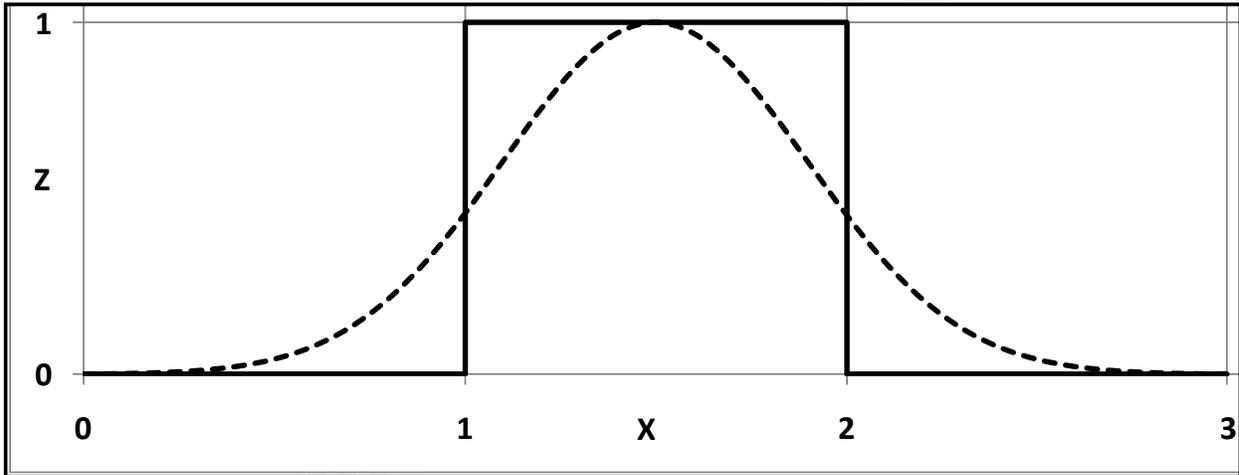
It can be shown that, if the functions  $\mathbf{f_1(x), f_2(x), f_3(x), \dots, f_n(x)}$  satisfy certain conditions, then as  $\mathbf{n \rightarrow \infty}$ ,  $\mathbf{h(x)}$  will tend towards a Gaussian function. This is a non-rigorous statement of the Central Limit Theorem.<sup>6</sup> Example 4 shows these behaviors.

### Example 4: Repeated Convolution of a Rectangle Function with Itself

Suppose we want to find the results of the repeated convolution of the following rectangle PDF:

$$f(x) = \begin{cases} 1, & 1 \leq x \leq 2 \\ 0, & x < 1, x > 2 \end{cases}$$

The PDF is shown in Figure 18 along with its corresponding Gaussian function, i.e., a Gaussian function with the same mean and variance as a given PDF. Addendum 1 shows how to calculate corresponding normal and lognormal functions.



**Figure 18: Rectangle Function in Example 4 & Corresponding (Dotted) Gaussian Function**

To convolve this PDF with itself, the second function will then be  $\mathbf{g(y)}$  such that:

$$g(y) = \begin{cases} 1, & 1 \leq y \leq 2 \\ 0, & y < 1, y > 2 \end{cases}$$

The domain of these two functions in the  $\mathbf{xy}$ -plane is shown in Figure 19, and their translated domain with respect to the line  $\mathbf{y=x}$  is shown in Figure 20.

<sup>5</sup> Linear Systems, Fourier Transforms and Optics, Jack D. Gaskill, John Wiley & Sons, 1978, Chapter 6, Section 6-3, Page 162, Paragraph 2 and Page 163 Paragraph 2.

<sup>6</sup> Ibid, Page 164, Paragraph 1.

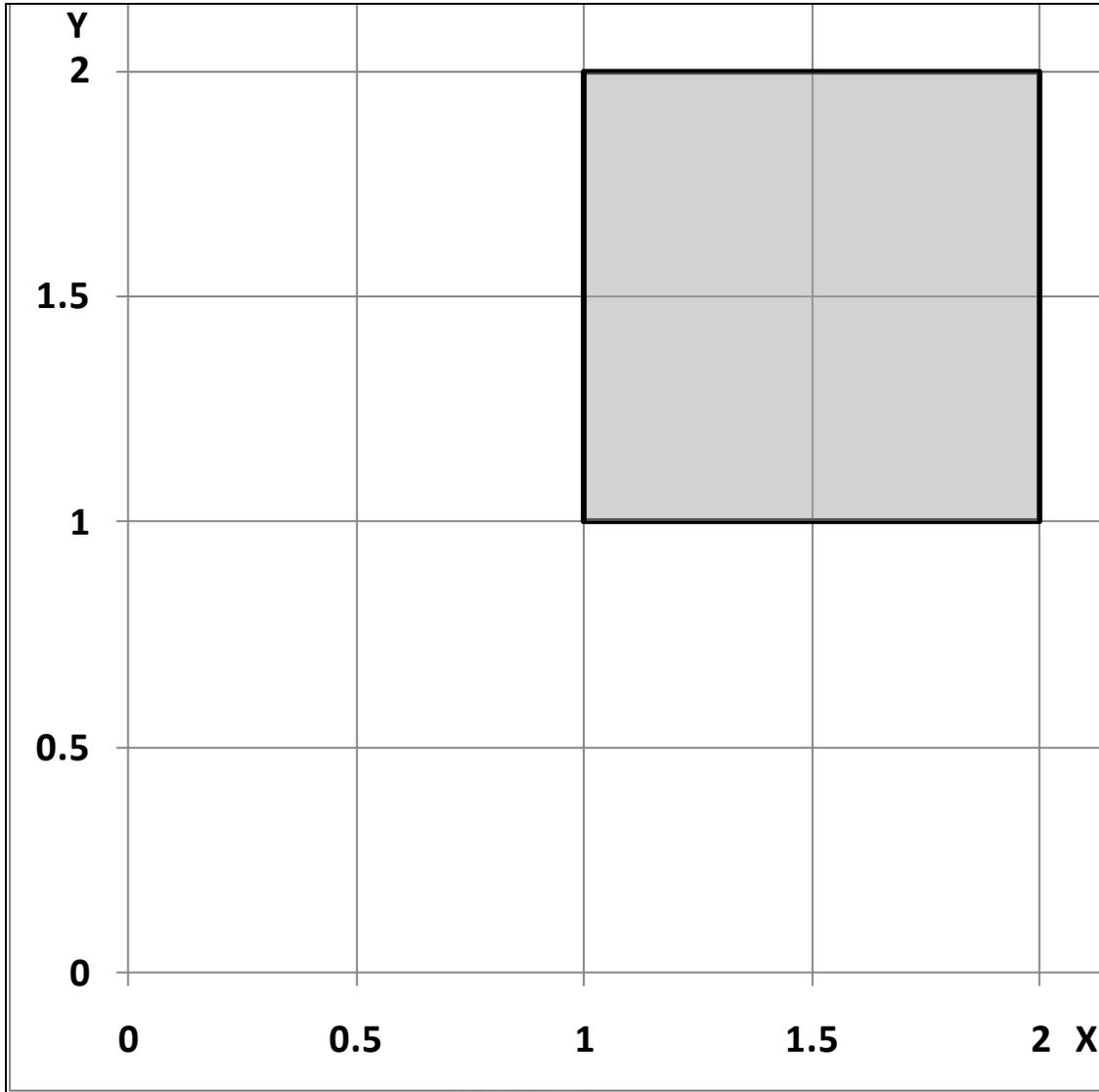


Figure 19: Domain  $[1 \leq x \leq 2, 1 \leq y \leq 2]$  of the Function  $z(x,y)=f(x)g(y)$

In Figure 20, there are two different sets of limits of integration of the integral: in Area 1, where  $2 \leq y \leq 3$ , the value of  $x$  can vary from 1 to  $y-1$ ; in Area 2, where  $3 \leq y \leq 4$ , the value of  $x$  can vary from  $y-2$  to 2. For all  $y < 2$  and  $y > 4$ , the integral is equal to zero since the function  $z$  does not exist there. In the closed  $y$ -interval  $[2,4]$ , we can evaluate the integral in the two intervals associated with Areas 1 and 2 to get a piecewise solution. Plug  $f(x)$  into the equation for  $A(y)$ . For the function  $g(y)$ , substitute  $y-x$  for  $y$  throughout the function and then plug this function  $g(y-x)$  into the equation for  $A(y)$ .

In the  $y$ -interval associated with Area 1,  $2 \leq y \leq 3$ , we can evaluate the integral as follows:

$$h_1(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_1(y) = \int_1^{y-1} (1)(1)dx = \int_1^{y-1} dx = x|_1^{y-1} = y-1-1 = y-2$$

In the  $y$ -interval associated with Area 2,  $3 \leq y \leq 4$ , we can evaluate the integral as follows:

$$h_2(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_2(y) = \int_{y-2}^2 (1)(1)dx = \int_{y-2}^2 dx = x|_{y-2}^2 = 2 - (y-2) = 4-y$$

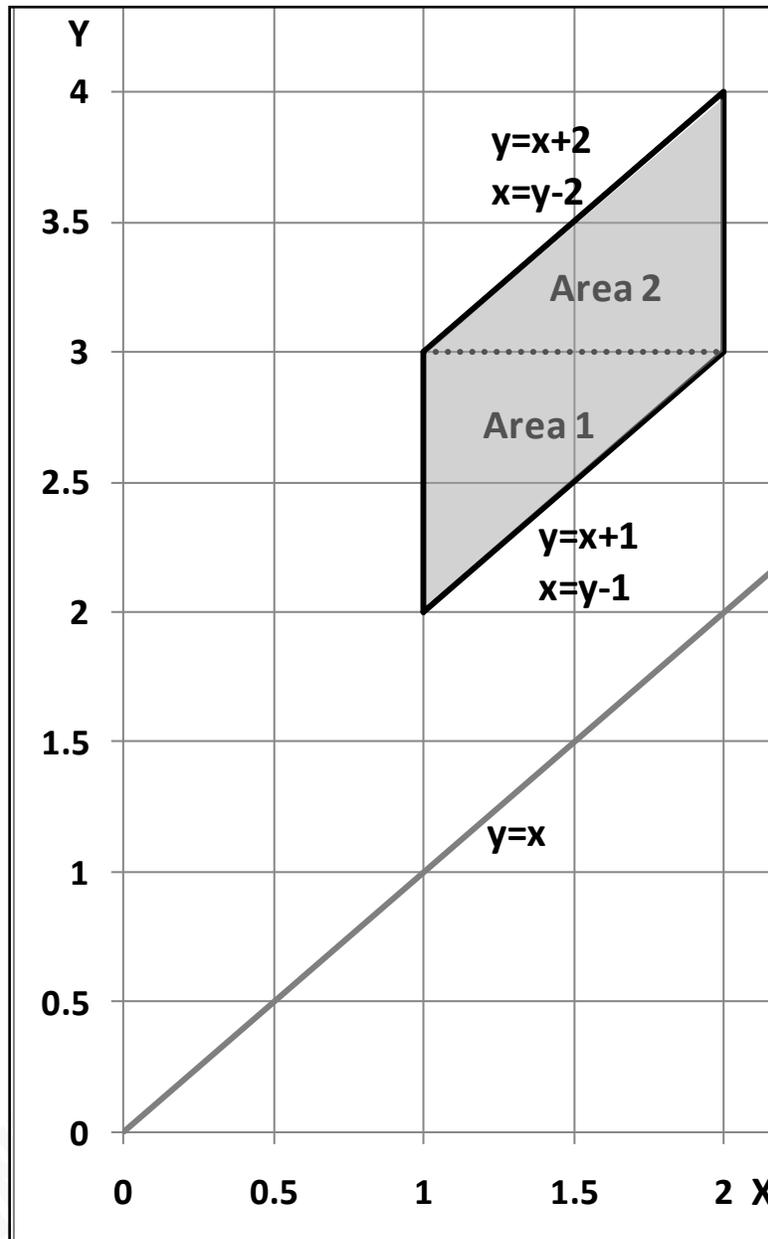


Figure 20: Domain  $[1 \leq x \leq 2, 1 \leq y-x \leq 2] = [1 \leq x \leq 2, x+1 \leq y \leq x+2]$  of the Function  $z(x,y-x)=f(x)g(y-x)$

The entire solution of the first convolution is:

$$h(x) = \begin{cases} x-2, & 2 \leq x \leq 3 \\ 4-x, & 3 \leq x \leq 4 \\ 0, & y < 2, y > 4 \end{cases}$$

This function is shown in Figure 21 along with its corresponding Gaussian function, which makes a fairly good fit to the convolution function. If two functions are not correlated, i.e., if they are independent functions, then the mean of the convolution is the sum of the means of the two functions being convolved, and the variance of the convolution is the sum of the variances of the two functions being convolved. Addendum 2 shows the proofs of these statements. So, for repeated convolutions, the mean and variance of the corresponding Gaussian function is just the respective sums of the means and variances of the functions being convolved.

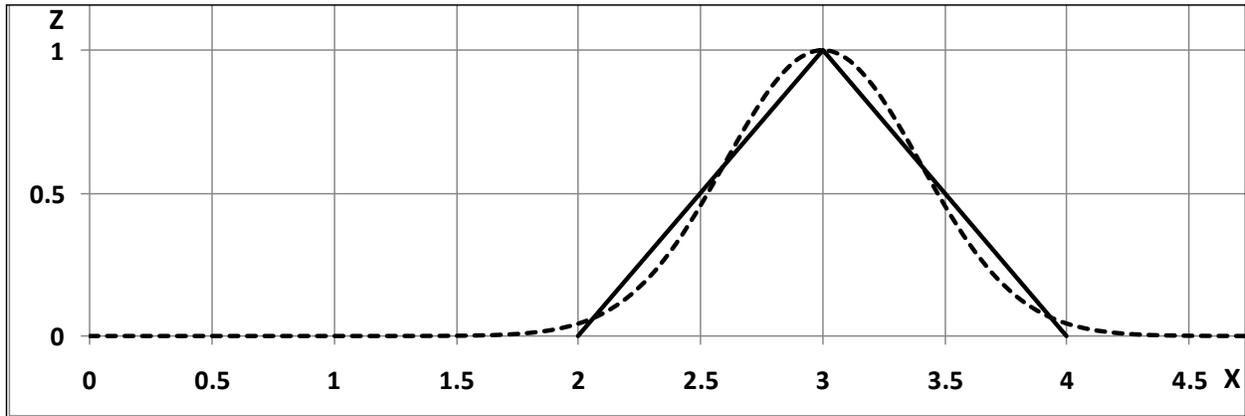


Figure 21: Results of 1<sup>st</sup> Example 4 Convolution & Corresponding (Dotted) Gaussian Function

For the second convolution, let this function  $h(x)$  from the first convolution be the new  $f(x)$  and convolve it with the original  $g(y)$ . In this case,  $f(x)$  is a piece-wise continuous function described by one function when  $2 \leq x \leq 3$ —call it  $f_1(x)$ —and a different function when  $3 \leq x \leq 4$ —call it  $f_2(x)$ . The domain of these two functions in the  $xy$ -plane is shown in Figure 22, and their translated domain with respect to the line  $y=x$  is shown in Figure 23.

In Figure 23, there are three different sets of limits of integration of the integral: in Area 1, where  $3 \leq y \leq 4$ , the value of  $x$  can vary from 2 to  $y-1$  for the integrand  $f_1(x)g(y-x)$ ; in Area 2, where  $4 \leq y \leq 5$ , the value of  $x$  can vary from  $y-2$  to 3 for the integrand  $f_1(x)g(y-x)$  (in Subarea 2,1) and from 3 to  $y-1$  for the integrand  $f_2(x)g(y-x)$  (in Subarea 2,2); and in Area 3, where  $5 \leq y \leq 6$ , the value of  $x$  can vary from  $y-2$  to 4 for the integrand  $f_2(x)g(y-x)$ . For all  $y < 3$  and  $y > 6$ , the integral is equal to zero since the function  $z$  does not exist there. In the closed  $y$ -interval  $[3,6]$ , we can evaluate the integral in the three intervals associated with Areas 1, 2 and 3 to get a piecewise solution. Plug  $f(x)$  into the equation for  $A(y')$ . For the function  $g(y)$ , substitute  $y-x$  for  $y$  throughout the function and then plug this function  $g(y-x)$  into the equation for  $A(y')$ .

In the  $y$ -interval associated with Area 1,  $3 \leq y \leq 4$ , we can evaluate the integral as follows:

$$h_1(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_1(y) = \int_2^{y-1} (x-2)(1)dx = \left( \frac{x^2}{2} - 2x \right) \Big|_2^{y-1} = \left[ \frac{(y-1)^2}{2} - 2(y-1) \right] - [2-4]$$

$$h_1(y) = \frac{y^2 - 2y + 1}{2} - 2y + 2 + 2 = \frac{y^2 - 6y + 9}{2} = \frac{(y-3)^2}{2}$$

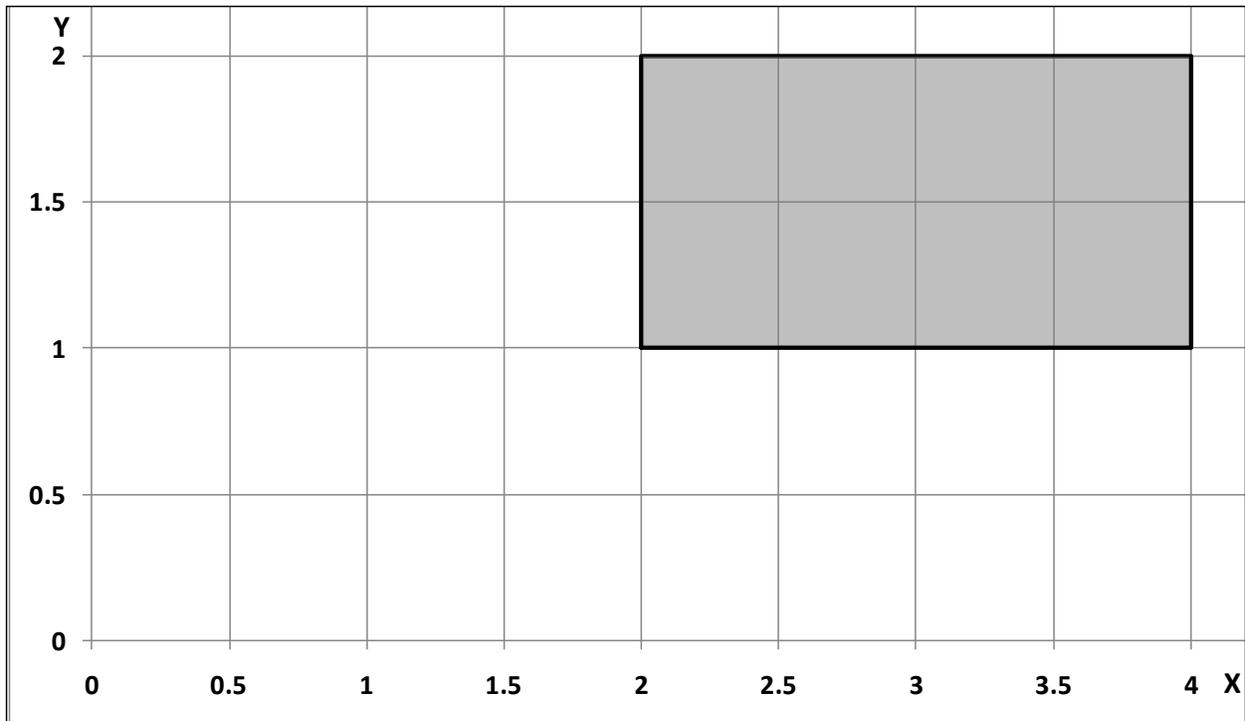


Figure 22: Domain  $[2 \leq x \leq 4, 1 \leq y \leq 2]$  of the Function  $z(x,y)=f(x)g(y)$

In the  $y$ -interval associated with Area 2,  $4 \leq y \leq 5$ , we can evaluate the integral as follows:

$$h_2(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_2(y) = \int_{y-2}^3 (x-2)(1)dx + \int_3^{y-1} (4-x)(1)dx = \left(\frac{x^2}{2} - 2x\right)\Big|_{y-2}^3 + \left(4x - \frac{x^2}{2}\right)\Big|_3^{y-1}$$

$$h_2(y) = \left[\frac{9}{2} - 6\right] - \left[\frac{(y-2)^2}{2} - 2(y-2)\right] + \left[4(y-1) - \frac{(y-1)^2}{2}\right] - \left[12 - \frac{9}{2}\right]$$

$$h_2(y) = 9 - 18 - \frac{y^2 - 4y + 4}{2} + 2y - 4 + 4y - 4 - \frac{y^2 - 2y + 1}{2}$$

$$h_2(y) = -17 + 6y - \frac{2y^2 - 6y + 5}{2} = \frac{-34 + 12y - 2y^2 + 6y - 5}{2} = -\frac{2y^2 - 18y + 39}{2}$$

In the  $y$ -interval associated with Area 3,  $5 \leq y \leq 6$ , we can evaluate the integral as follows:

$$h_3(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_3(y) = \int_{y-2}^4 (4-x)(1)dx = \left(4x - \frac{x^2}{2}\right)\Big|_{y-2}^4 = [16 - 8] - \left[4(y-2) - \frac{(y-2)^2}{2}\right]$$

$$h_3(y) = 8 - 4y + 8 + \frac{y^2 - 4y + 4}{2} = \frac{y^2 - 12y + 36}{2} = \frac{(y-6)^2}{2}$$

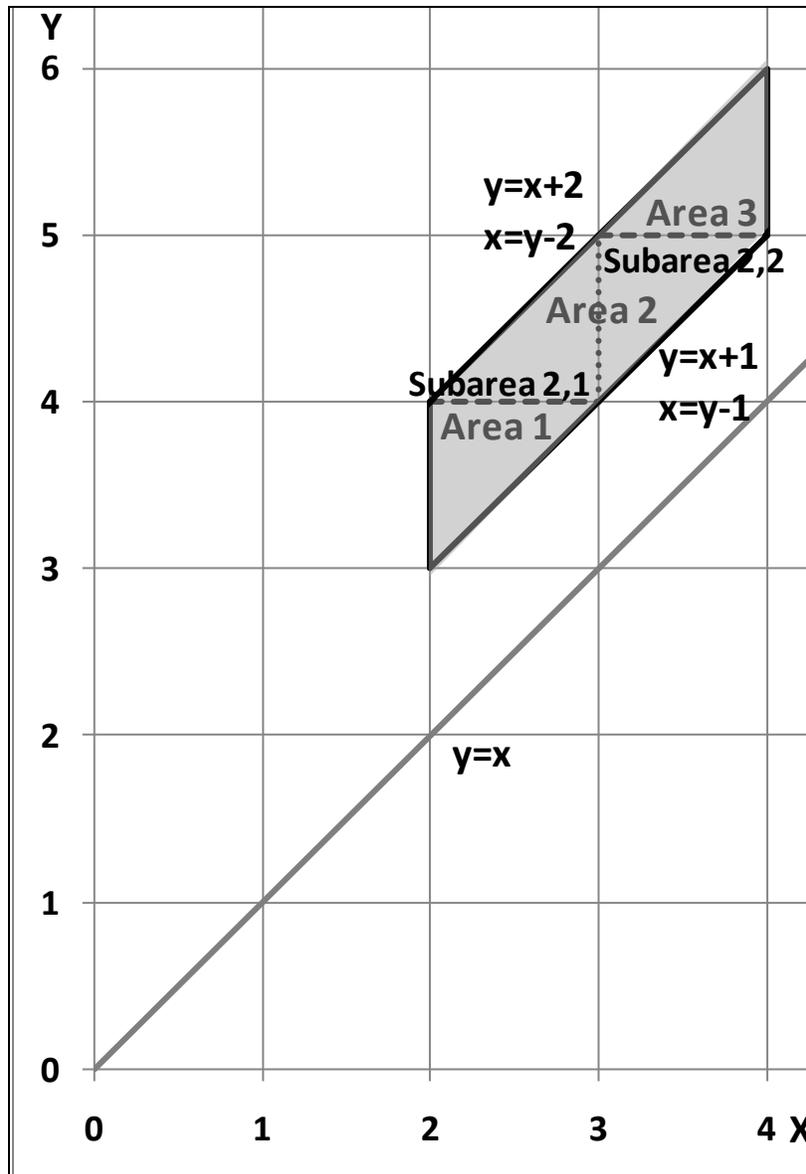


Figure 23: Domain  $[2 \leq x \leq 4, 1 \leq y-x \leq 2] = [2 \leq x \leq 4, x+1 \leq y \leq x+2]$  of the Function  $z(x,y-x) = f(x)g(y-x)$

The entire solution of the second convolution is:

$$h(x) = \begin{cases} \frac{(y-3)^2}{2}, & 3 \leq x \leq 4 \\ -\frac{2y^2 - 18y + 39}{2}, & 4 \leq x \leq 5 \\ \frac{(y-6)^2}{2}, & 5 \leq x \leq 6 \\ 0, & y < 3, y > 6 \end{cases}$$

This function is shown in Figure 24 along with its corresponding Gaussian function, which makes a good fit to the convolution function.

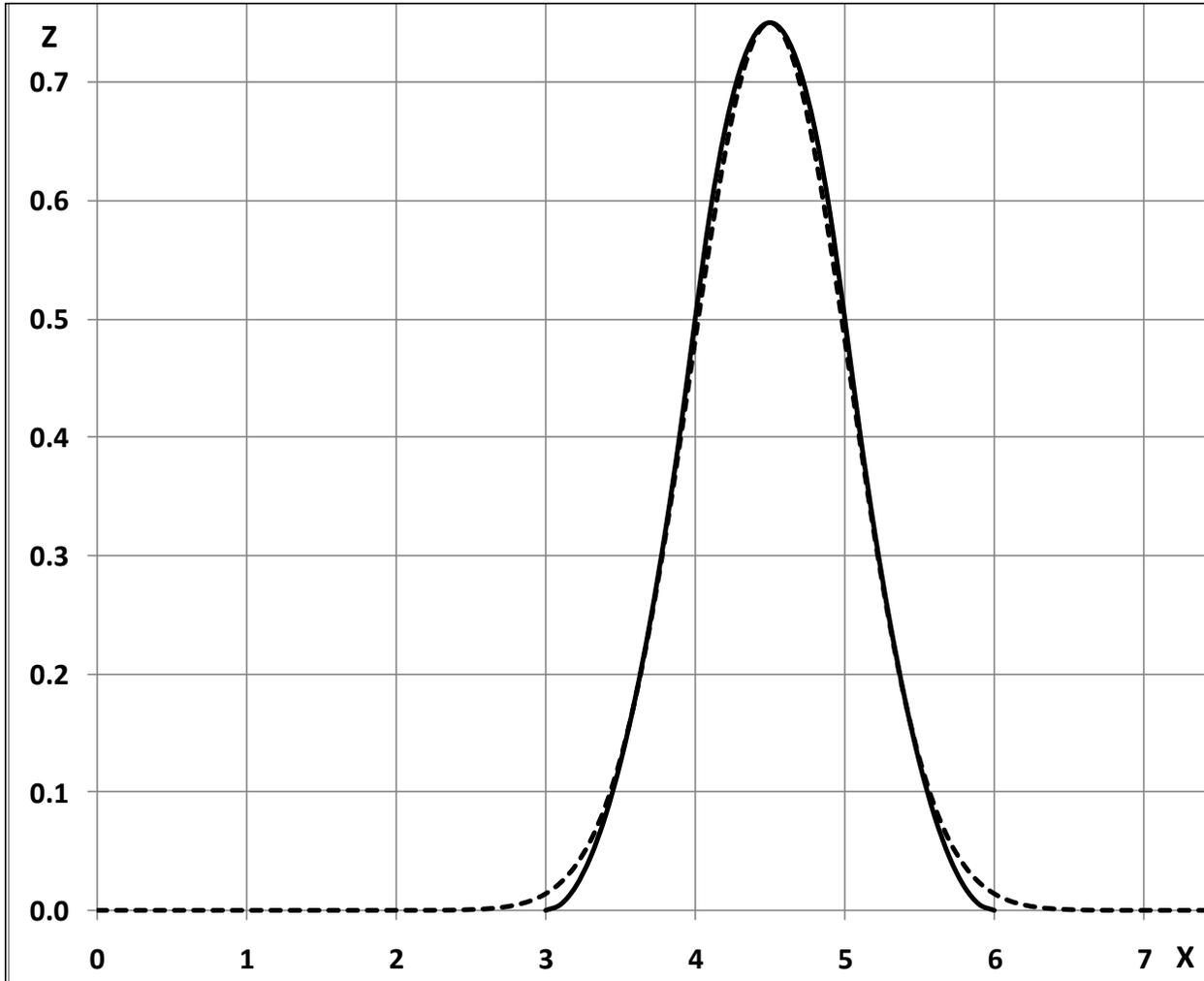


Figure 24: Results of 2<sup>nd</sup> Example 4 Convolution & Corresponding (Dotted) Gaussian Function

For the third convolution, let this function  $h(x)$  from the second convolution be the new  $f(x)$  and convolve it with the original  $g(y)$ . In this case,  $f(x)$  is a piece-wise continuous function described by one function when  $3 \leq x \leq 4$ —call it  $f_1(x)$ —a second function when  $4 \leq x \leq 5$ —call it  $f_2(x)$ —and a third function when  $5 \leq x \leq 6$ —call it  $f_3(x)$ . The domain of these three functions in the  $xy$ -plane is shown in Figure 25, and their translated domain with respect to the line  $y=x$  is shown in Figure 26.

In Figure 26, there are four different sets of limits of integration of the integral: in Area 1, where  $4 \leq y \leq 5$ , the value of  $x$  can vary from 3 to  $y-1$  for the integrand  $f_1(x)g(y-x)$ ; in Area 2, where  $5 \leq y \leq 6$ , the value of  $x$  can vary from  $y-2$  to 4 for the integrand  $f_1(x)g(y-x)$  (in Subarea 2,1) and from 4 to  $y-1$  for the integrand  $f_2(x)g(y-x)$  (in Subarea 2,2); in Area 3, where  $6 \leq y \leq 7$ , the value of  $x$  can vary from  $y-2$  to 5 for the integrand  $f_2(x)g(y-x)$  (in Subarea 3,1) and from 5 to  $y-1$  for the integrand  $f_3(x)g(y-x)$  (in Subarea 3,2); and in Area 4, where  $7 \leq y \leq 8$ , the value of  $x$  can vary from  $y-2$  to 6 for the integrand  $f_3(x)g(y-x)$ . For all  $y < 4$  and  $y > 8$ , the integral is equal to zero since the function  $z$  does not exist there. In the closed  $y$ -interval  $[4,8]$ , we can evaluate the integral in the four intervals associated with Areas 1, 2, 3 and 4 to get a piecewise solution. Plug  $f(x)$  into the equation for  $A(y)$ . For the function  $g(y)$ , substitute  $y-x$  for  $y$  throughout the function and then plug this function  $g(y-x)$  into the equation for  $A(y)$ .

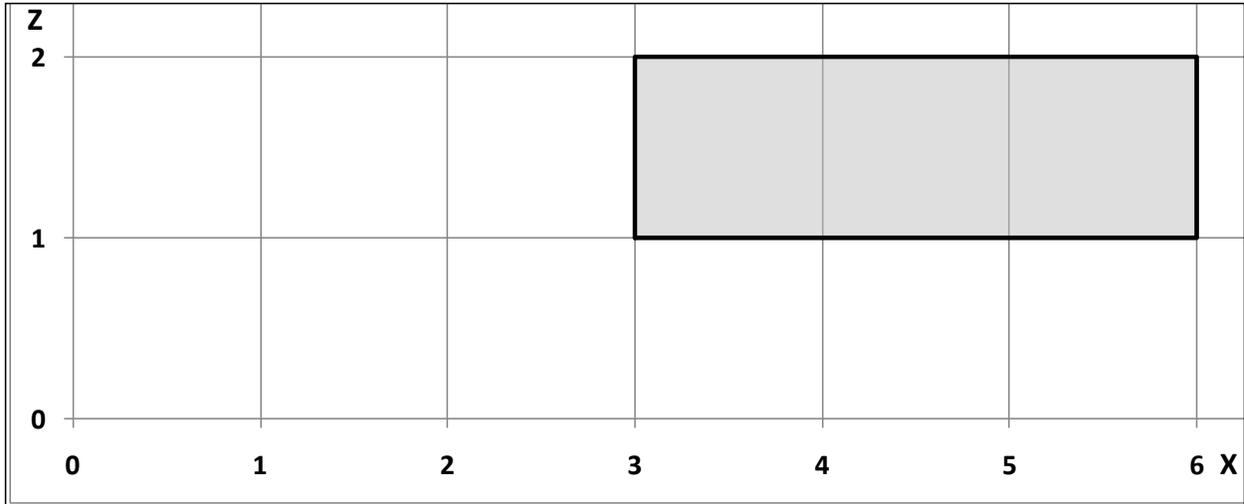


Figure 25: Domain  $[3 \leq x \leq 6, 1 \leq y \leq 2]$  of the Function  $z(x,y)=f(x)g(y)$

In the  $y$ -interval associated with Area 1,  $4 \leq y \leq 5$ , we can evaluate the integral as follows:

$$h_1(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_1(y) = \int_3^{y-1} \left( \frac{(x-3)^2}{2} \right) (1) dx = \frac{1}{2} \int_3^{y-1} (x^2 - 6x + 9) dx = \frac{1}{2} \left( \frac{x^3}{3} - 3x^2 + 9x \right) \Big|_3^{y-1}$$

$$h_1(y) = \frac{1}{2} \left( \left[ \frac{(y-1)^3}{3} - 3(y-1)^2 + 9(y-1) \right] - [9 - 27 + 27] \right)$$

$$h_1(y) = \frac{1}{2} \left( \left[ \frac{y^3 - 3y^2 + 3y - 1 - 9y^2 + 18y - 9 + 27y - 27}{3} \right] - 9 \right)$$

$$h_1(y) = \frac{1}{2} \left( \frac{y^3 - 12y^2 + 48y - 64}{3} \right) = \frac{(y-4)^3}{6}$$

In the  $y$ -interval associated with Area 2,  $5 \leq y \leq 6$ , we can evaluate the integral as follows:

$$h_2(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_2(y) = \int_{y-2}^4 \left( \frac{(x-3)^2}{2} \right) (1) dx + \int_4^{y-1} \left( -\frac{2x^2 - 18x + 39}{2} \right) (1) dx$$

$$h_2(y) = \frac{1}{2} \left( \frac{x^3}{3} - 3x^2 + 9x \right) \Big|_{y-2}^4 - \frac{1}{2} \left( \frac{2x^3}{3} - 9x^2 + 39x \right) \Big|_4^{y-1}$$

$$h_2(y) = \frac{1}{2} \left( \left[ \frac{64}{3} - 48 + 36 \right] - \left[ \frac{(y-2)^3 - 9(y-2)^2 + 27(y-2)}{3} \right] - \left[ \frac{2(y-1)^3 - 27(y-1)^2 + 117(y-1)}{3} \right] - \left[ \frac{128}{3} - 144 + 156 \right] \right)$$

$$h_2(y) = \frac{1}{6} (192 - y^3 + 6y^2 - 12y + 8 + 9y^2 - 36y + 36 - 27y + 54 - 2y^3 + 6y^2 - 6y + 2 + 27y^2 - 54y + 27 - 117y + 117)$$

$$h_2(y) = -\frac{1}{6} (3y^3 - 48y^2 + 252y - 436)$$

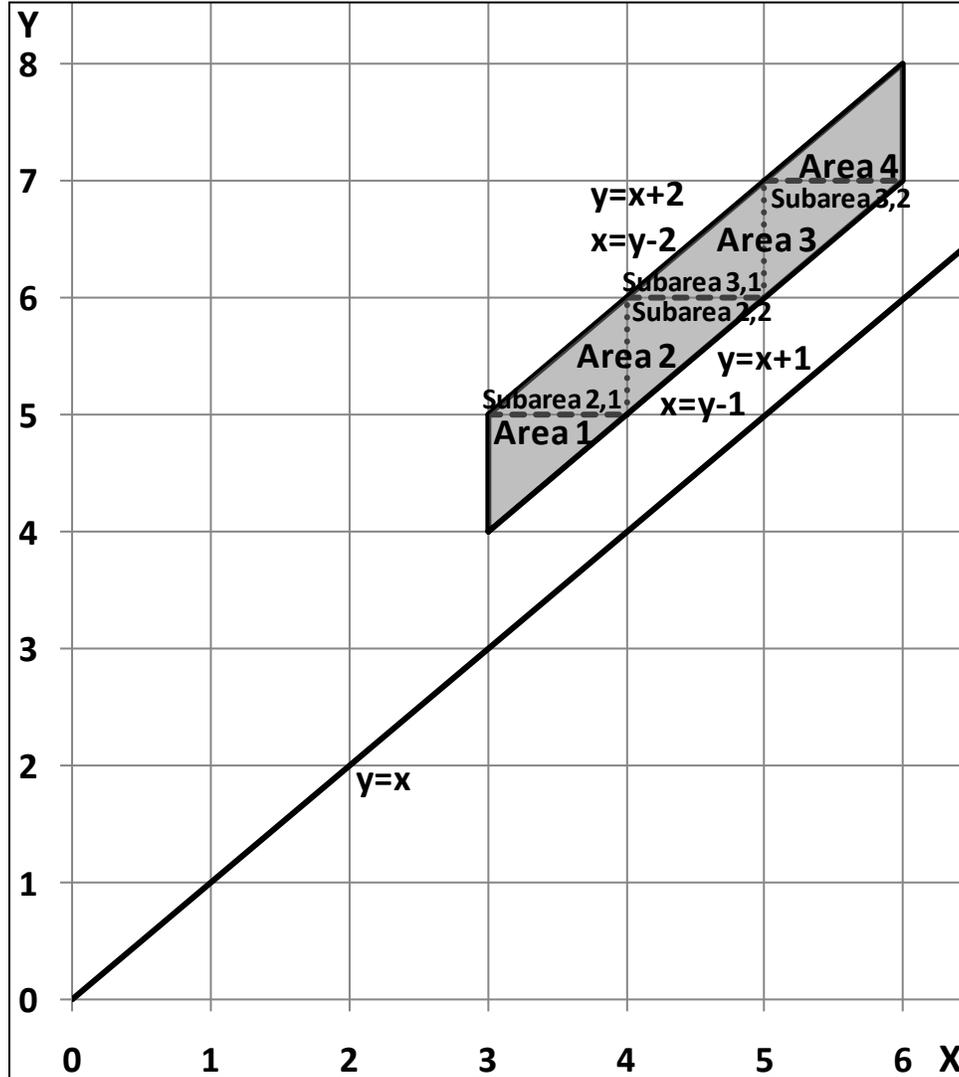


Figure 26: Domain  $[3 \leq x \leq 6, 1 \leq y-x \leq 2] = [3 \leq x \leq 6, x+1 \leq y \leq x+2]$  of the Function  $z(x,y-x) = f(x)g(y-x)$

In the  $y$ -interval associated with Area 3,  $6 \leq y \leq 7$ , we can evaluate the integral as follows:

$$h_3(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_3(y) = \int_{y-2}^5 \left( -\frac{2x^2 - 18x + 39}{2} \right) (1)dx + \int_5^{y-1} \left( \frac{(x-6)^2}{2} \right) (1)dx +$$

$$h_3(y) = -\frac{1}{2} \left( \frac{2x^3}{3} - 9x^2 + 39x \right) \Big|_{y-2}^5 + \frac{1}{2} \left( \frac{x^3}{3} - 6x^2 + 36x \right) \Big|_5^{y-1}$$

$$h_3(y) = \frac{1}{2} \left( - \left[ \frac{250}{3} - 225 + 195 \right] + \left[ \frac{2(y-2)^3 - 27(y-2)^2 + 117(y-2)}{3} \right] + \left[ \frac{(y-1)^3 - 18(y-1)^2 + 108(y-1)}{3} \right] - \left[ \frac{125}{3} - 150 + 180 \right] \right)$$

$$h_3(y) = \frac{1}{6} (2y^3 - 12y^2 + 24y - 16 - 27y^2 + 108y - 108 + 117y - 234 + y^3 - 3y^2 + 3y - 1 - 18y^2 + 36y - 18 + 108y - 375)$$

$$h_3(y) = \frac{1}{6} (3y^3 - 60y^2 + 396y - 860)$$

In the  $y$ -interval associated with Area 4,  $7 \leq y \leq 8$ , we can evaluate the integral as follows:

$$h_4(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_4(y) = \int_{y-2}^6 \left( \frac{(x-6)^2}{2} \right) (1) dx = \frac{1}{2} \left( \frac{x^3}{3} - 6x^2 + 36x \right) \Big|_{y-2}^6$$

$$h_4(y) = \frac{1}{2} \left( [72 - 216 + 216] - \left[ \frac{(y-2)^3}{3} - 6(y-2)^2 + 36(y-2) \right] \right)$$

$$h_4(y) = \frac{1}{6} (216 - y^3 + 6 - 12y + 8 + 18y^2 + 72y - 72 + 108y - 216)$$

$$h_4(y) = -\frac{1}{6} \left( \frac{y^3 - 24y^2 + 192y - 512}{3} \right) = -\frac{(y-8)^3}{6}$$

The entire solution of the third convolution is:

$$h(x) = \begin{cases} \frac{(y-4)^3}{6}, & 4 \leq x \leq 5 \\ -\frac{1}{6} (3y^3 - 48y^2 + 252y - 436), & 5 \leq x \leq 6 \\ \frac{1}{6} (3y^3 - 60y^2 + 396y - 860), & 6 \leq x \leq 7 \\ -\frac{(y-8)^3}{6}, & 7 \leq x \leq 8 \\ 0, & y < 4, y > 8 \end{cases}$$

This function is shown in Figure 27 along with its corresponding Gaussian function, which makes an extremely good fit to the convolution function.

Because the rectangle function in Example 4 is a symmetrical function about the line  $x=3/2$ , the repeated convolutions of this rectangle function converged quite rapidly—in just a few convolutions—to their corresponding Gaussian function<sup>7</sup>. But, the repeated convolution of highly asymmetrical functions, as shown in Example 5, may take much longer to converge well to a corresponding Gaussian function.

<sup>7</sup> Linear Systems, Fourier Transforms and Optics, Jack D. Gaskill, John Wiley & Sons, 1978, Chapter 6, Section 6-3, Page 165, Paragraph 2

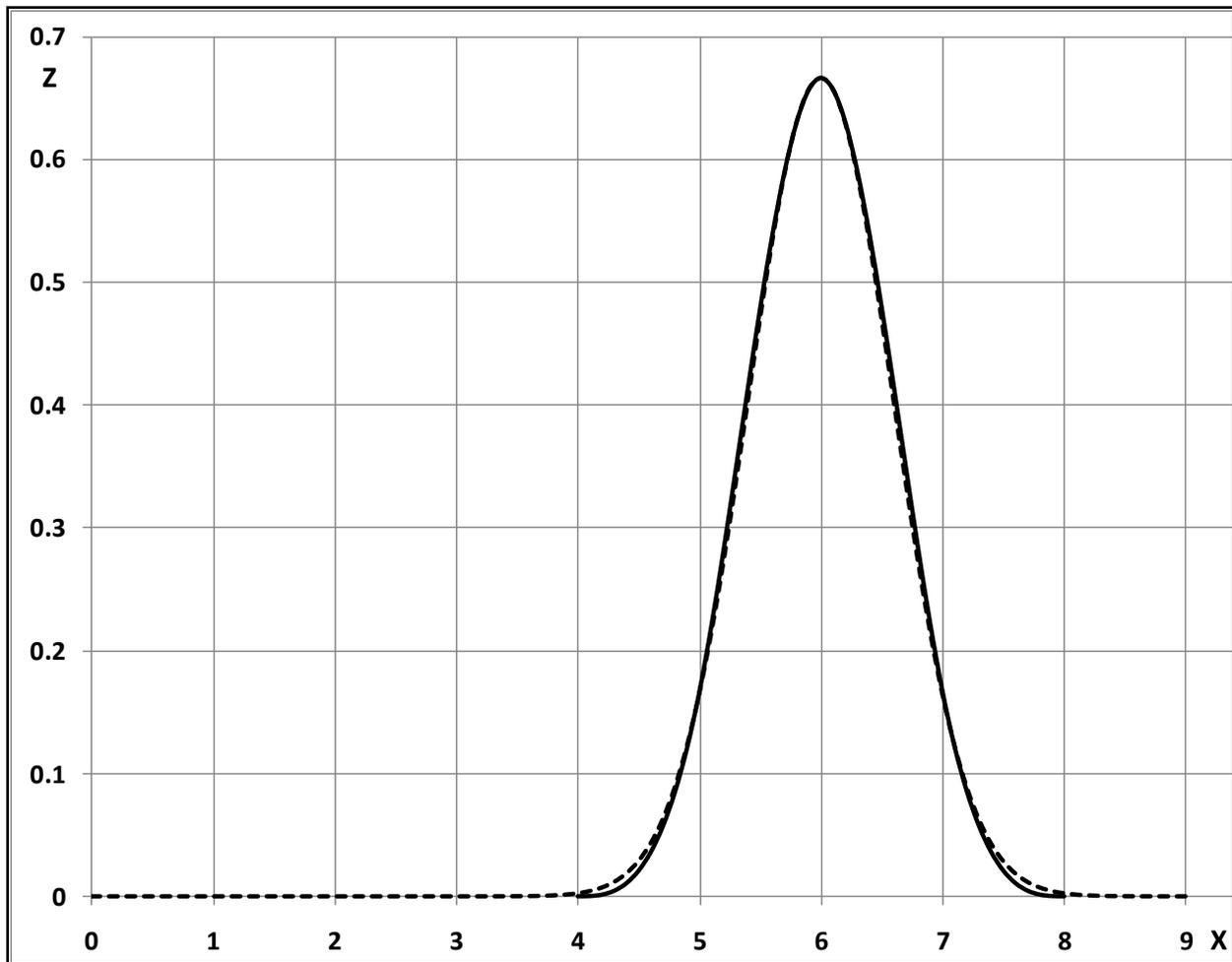


Figure 27: Results of 3<sup>rd</sup> Example 4 Convolution & Corresponding (Dotted) Gaussian Function

### Example 5: Repeated Convolution of an Exponential Function with Itself<sup>8</sup>

Suppose we want to find the results of the repeated convolution of the following exponential PDF:

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Then,  $g(y)$  is:

$$g(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

And,  $g(y-x)$  is:

$$g(y-x) = \begin{cases} e^{-(y-x)}, & y-x \geq 0 \\ 0, & y-x < 0 \end{cases}$$

The domain of the product  $f(x)g(y)$  in the  $xy$ -plane is the entire first quadrant, and the translated domain with respect to the line  $y=x$  of the product  $f(x)g(y-x)$  is the area in the first quadrant between the line  $y=x$  and the  $y$ -axis as shown in Figure 28.

<sup>8</sup> This example is derived from the example shown in Linear Systems, Fourier Transforms and Optics, Jack D. Gaskill, John Wiley & Sons, 1978, Chapter 6, Section 6-3, Page 165

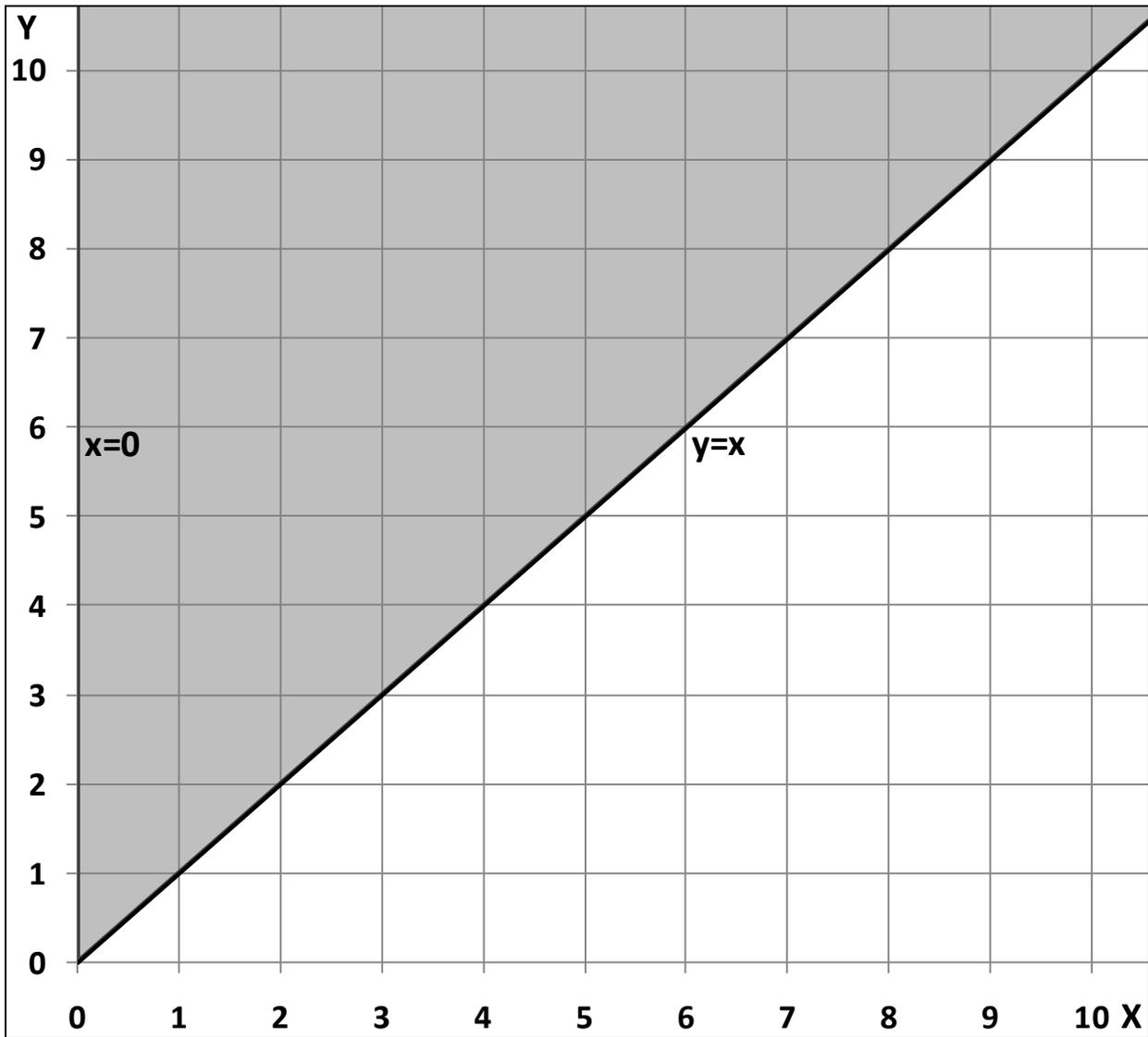


Figure 28: Domain  $[x \geq 0, y-x \geq 0] = [x \geq 0, y \geq x]$  of the Function  $z(x,y-x)=f(x)g(y-x)$

The  $y$ -interval for this entire domain is  $[0, \infty)$ , so we can evaluate the integral for  $h_n(y)$ , the  $n^{\text{th}}$  convolution of  $f(x)$ , as follows:

$$h_1(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_1(y) = \int_0^y (e^{-x})(e^{-(y-x)})dx = \int_0^y e^{-x-y+x} dx = \int_0^y e^{-y} dx = xe^{-y}|_0^y = ye^{-y}$$

If we repeat the convolution by always letting  $f(x)=h_n(x)$ , for all  $n$ , and using  $g(y)=e^{-y}$ , we will get:

$$h_2(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_2(y) = \int_0^y (xe^{-x})(e^{-(y-x)})dx = \int_0^y xe^{-x-y+x} dx = \int_0^y xe^{-y} dx = \frac{x^2}{2} e^{-y} \Big|_0^y = \frac{y^2}{2} e^{-y}$$

$$h_3(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_3(y) = \int_0^y \left(\frac{x^2}{2} e^{-x}\right) (e^{-(y-x)}) dx = \int_0^y \frac{x^2}{2} e^{-x-y+x} dx = \int_0^y \frac{x^2}{2} e^{-y} dx = \frac{x^3}{6} e^{-y} \Big|_0^y = \frac{y^3}{6} e^{-y}$$

$$h_4(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$h_4(y) = \int_0^y \left(\frac{x^3}{6} e^{-x}\right) (e^{-(y-x)}) dx = \int_0^y \frac{x^3}{6} e^{-x-y+x} dx = \int_0^y \frac{x^3}{6} e^{-y} dx = \frac{x^4}{24} e^{-y} \Big|_0^y = \frac{y^4}{24} e^{-y}$$

From these first four repeated convolutions, the resulting convolution seems to follow the following pattern, such that the  $\mathbf{n}^{\text{th}}$  convolution will be:

$$h_n(y) = \frac{y^n}{n!} e^{-y}$$

where  $\mathbf{n}=0, 1, 2, \dots$  (i.e.,  $\mathbf{n}$  can be all non-negative integers). If we plug in 0, 1, 2, 3 and 4 for  $\mathbf{n}$  in this equation  $\mathbf{h}_n(\mathbf{y})$ , we get the same results as we calculated above. To show this is true in general, we can complete a simple proof-by-induction. Assume this equation  $\mathbf{h}_n(\mathbf{y})$  is true and then show it will still be true for  $\mathbf{h}_{n+1}(\mathbf{y})$ :

$$h_{n+1}(y) = \int_{-\infty}^{\infty} h_n(x)g(y-x)dx$$

$$h_{n+1}(y) = \int_0^y \left(\frac{x^n}{n!} e^{-x}\right) (e^{-(y-x)}) dx = \int_0^y \frac{x^n}{n!} e^{-x-y+x} dx = \int_0^y \frac{x^n}{n!} e^{-y} dx = \frac{x^{n+1}}{n!(n+1)} e^{-y} \Big|_0^y$$

$$h_{n+1}(y) = \frac{y^{n+1}}{(n+1)!} e^{-y}$$

So the equation for  $\mathbf{h}_n(\mathbf{y})$  is true for all  $\mathbf{n}$ . The results of the first four repeated convolutions of the function in this example are shown in Figure 29 and Figure 30: Figure 29 shows the corresponding normal functions; and Figure 30 shows the corresponding lognormal functions. Also, Figure 31, Figure 32 and Figure 33 show the results after 10, 20 and 30 repeated convolutions, respectively. For the asymmetrical function used in this example, the convolutions have to be repeated many times before they truly come close to corresponding normal curves. Actually, the corresponding lognormal curves seem to be better approximations, but, eventually, after a few dozen repetitions of the convolution, both the convolution and its corresponding lognormal curve start getting fairly close to the corresponding normal curve.

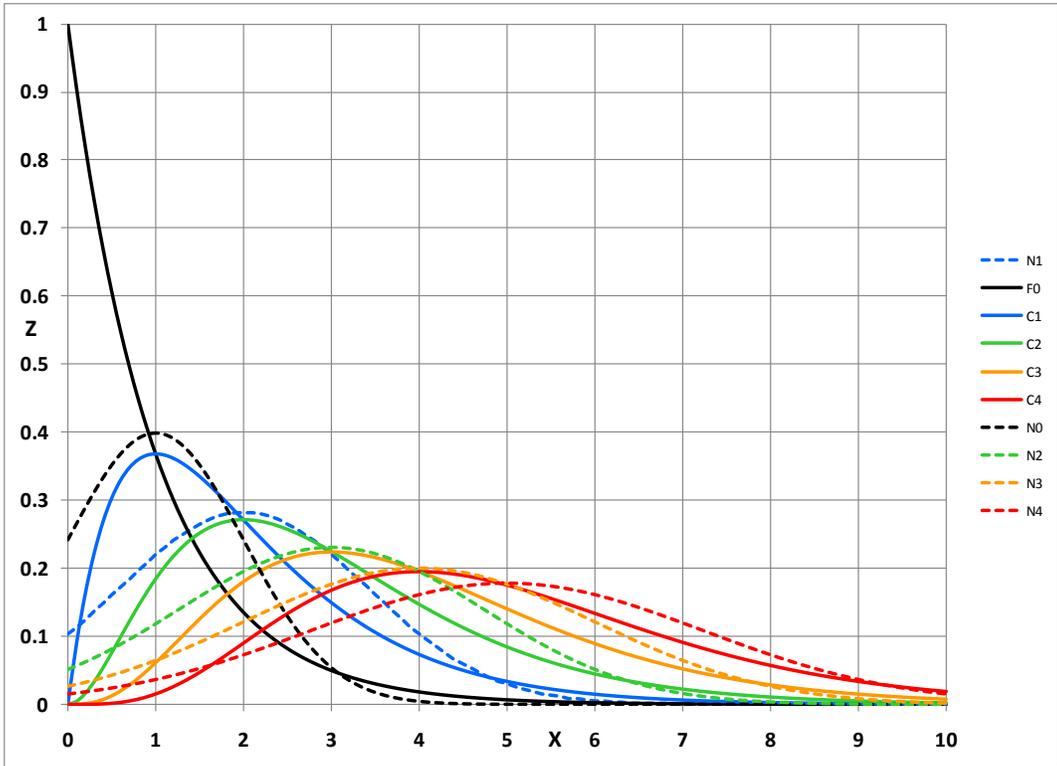


Figure 29: Example 5 Function, 4 Repeated Convolutions & Corresponding Normal Functions

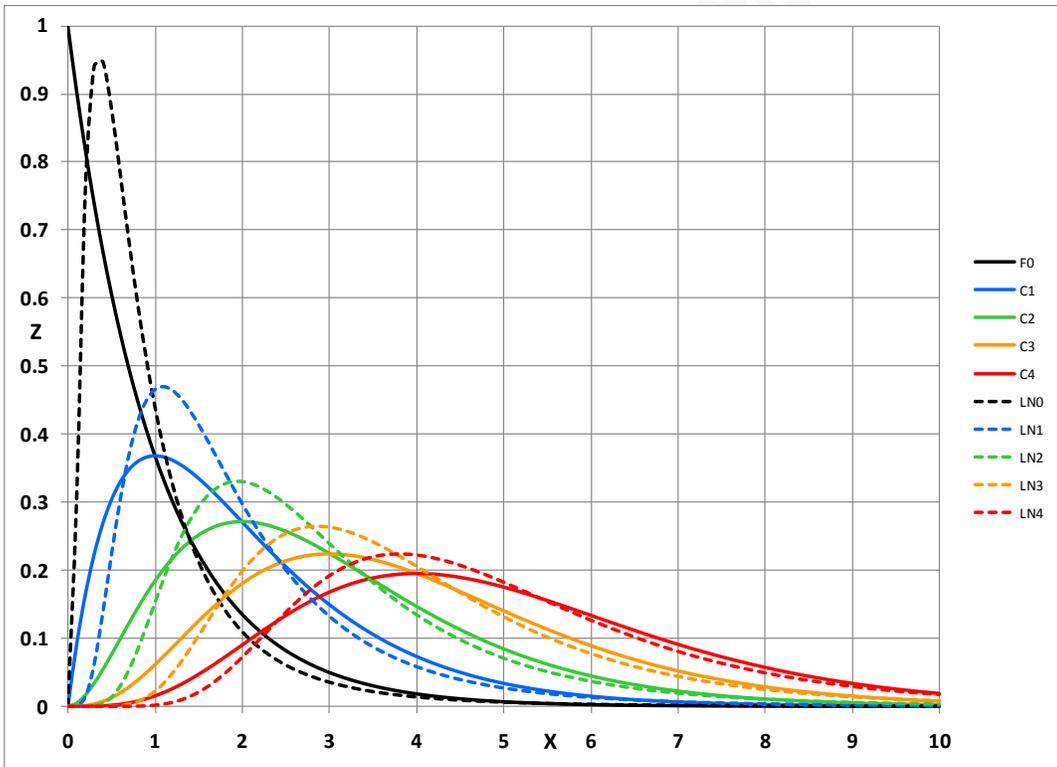


Figure 30: Example 5 Function, 4 Repeated Convolutions & Corresponding Lognormal Functions

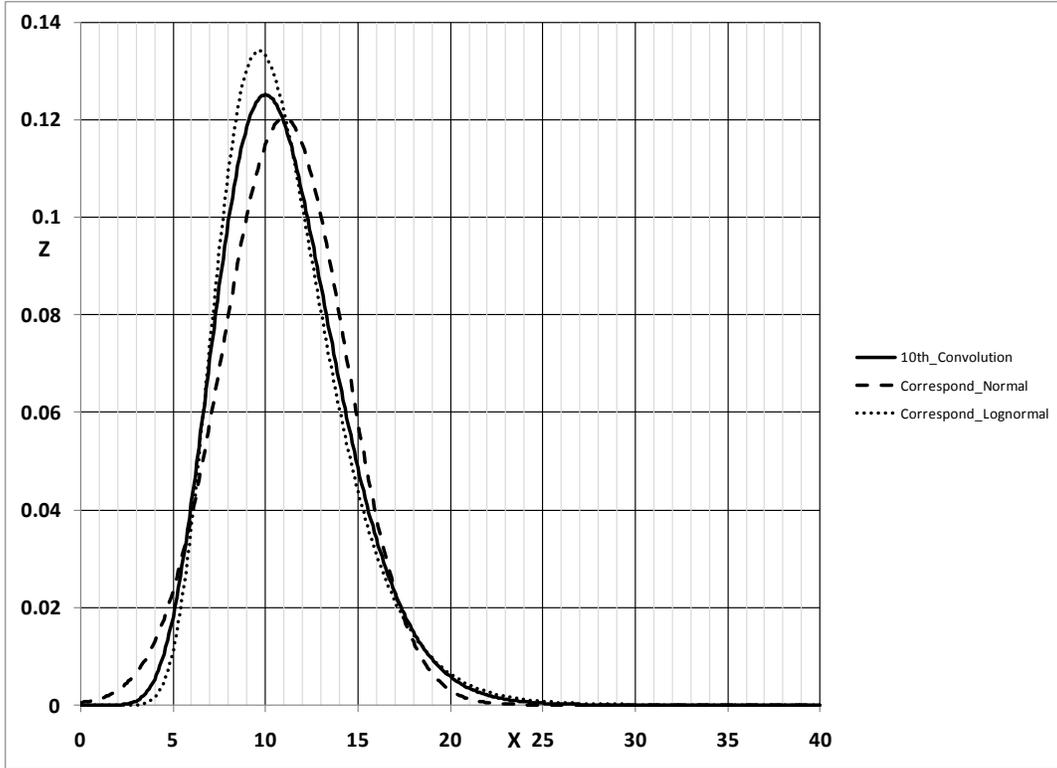


Figure 31: 10<sup>th</sup> Repeated Convolution & Corresponding Normal & Lognormal Functions

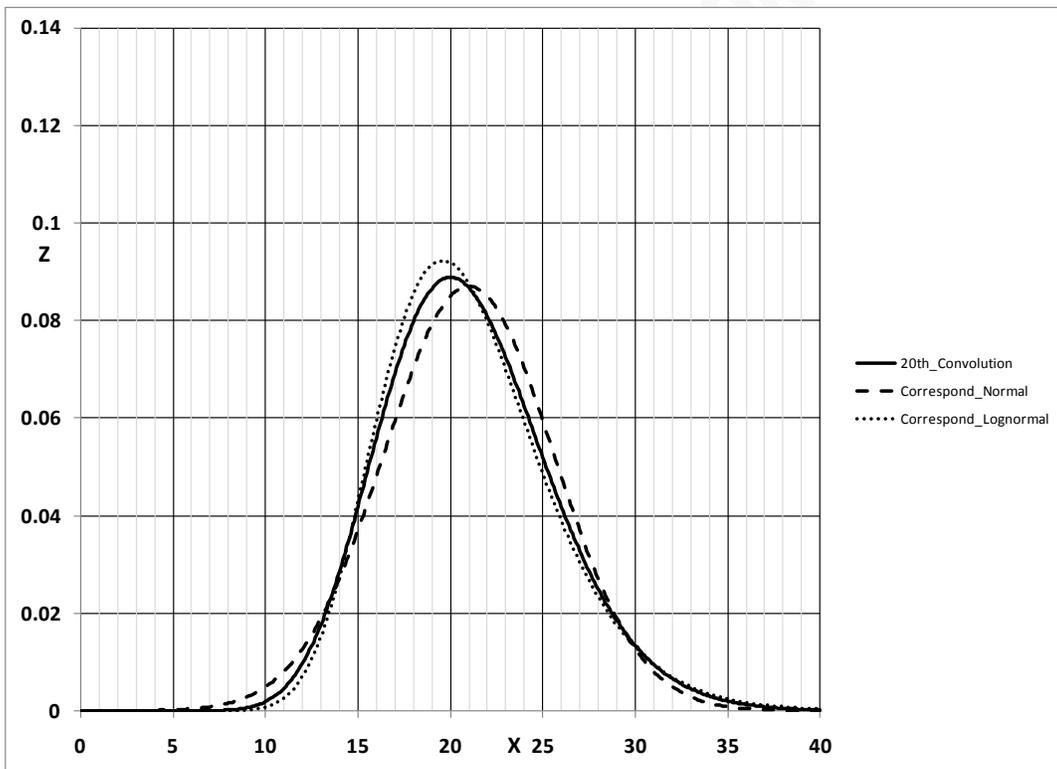


Figure 32: 20<sup>th</sup> Repeated Convolution & Corresponding Normal & Lognormal Functions

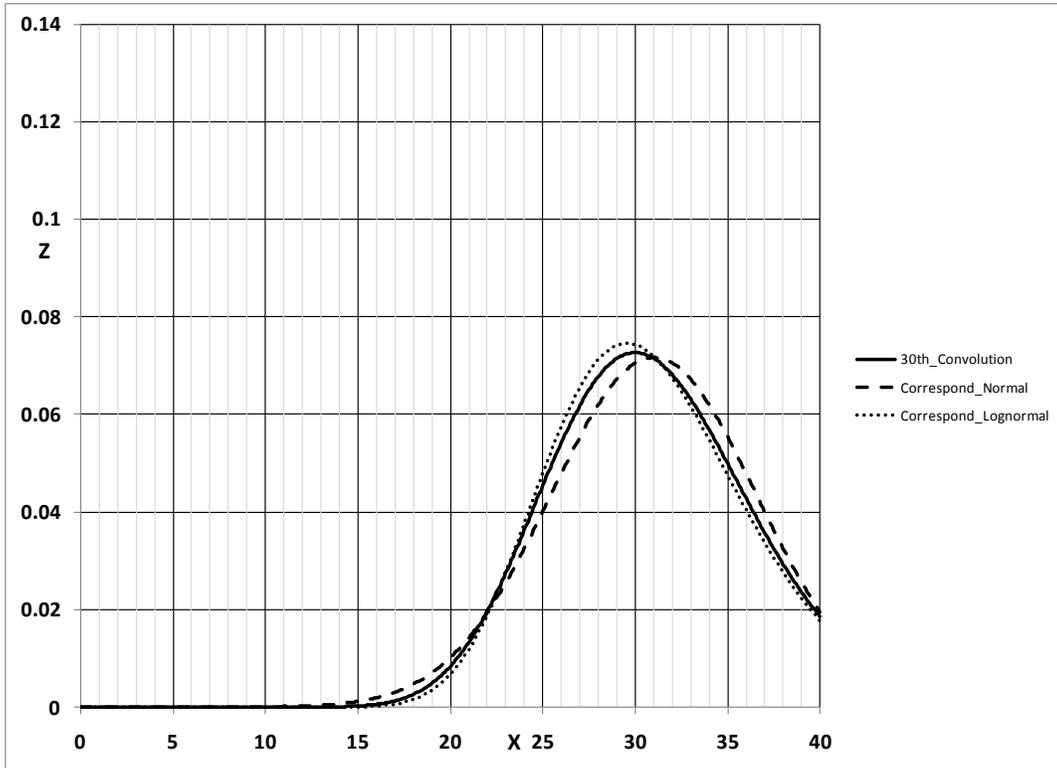


Figure 33: 30th Repeated Convolution & Corresponding Normal & Lognormal Functions

INTERNAL DISTRIBUTION

## Addendum 1: Determining the Corresponding Normal & Lognormal PDF

### Determining the Corresponding Normal PDF

To find the corresponding normal PDF that has the same mean,  $\mu$ , and variance,  $\sigma^2$ , of a given PDF, we can start by writing a normal equation in the general form:

$$f(x) = ae^{-b(x-c)^2} \quad (1)$$

where  $c$  and  $a$  are the abscissa and ordinate, respectively, of the mode of this corresponding normal PDF. For a normal equation, the mode, median and mean are identically the same, so we can use the above equation to get:

$$c = \mu$$

To find the other values, start by calculating the area under equation (1) by evaluating the integral

$$A = \int_{-\infty}^{\infty} ae^{-b(x-c)^2} dx$$

using the transformation  $m=x-c$ , with  $dm=dx$ , and where  $x=-\infty \rightarrow m=-\infty$  and  $x=\infty \rightarrow m=\infty$ . The transformed integral is:

$$A = a \int_{-\infty}^{\infty} e^{-bm^2} dm$$

The easiest way to evaluate  $A$  is by using the standard trick from upper-division Calculus in which we write  $A$  as a two-dimensional integral in the Cartesian  $mn$ -plane and then introduce polar coordinates:

$$A^2 = \left( a \int_{-\infty}^{\infty} e^{-bm^2} dm \right) \left( a \int_{-\infty}^{\infty} e^{-bn^2} dn \right) = a^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-b(m^2+n^2)} dmdn$$

$$A^2 = a^2 \int_0^{2\pi} \int_0^{\infty} e^{-br^2} r dr d\theta = 2\pi a^2 \int_0^{\infty} e^{-br^2} r dr = -\frac{\pi a^2}{b} \int_0^{\infty} e^{-br^2} (-2br) dr$$

$$A^2 = -\frac{\pi a^2}{b} e^{-br^2} \Big|_0^{\infty} = -\frac{\pi a^2}{b} (0 - 1) = \frac{\pi a^2}{b}$$

$$A = a \sqrt{\frac{\pi}{b}}$$

If our corresponding normal curve represents a normalized PDF, then  $A=1$ , and we get:

$$A = a \sqrt{\frac{\pi}{b}} = 1$$

$$a = \sqrt{\frac{b}{\pi}}$$

Next, look at the variance:

$$\sigma^2 = \frac{B}{A} = \frac{\int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx}{\int_{-\infty}^{\infty} f(x) dx}$$

Calculate the numerator  $B$  by evaluating the integral

$$B = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^{\infty} a(x - \mu)^2 e^{-b(x-\mu)^2} dx$$

Use the change-of-variable  $\mathbf{z=x-\mu}$ , with  $\mathbf{dz=dx}$ ,  $\mathbf{x=-\infty \rightarrow z=-\infty}$ , and  $\mathbf{x=\infty \rightarrow z=\infty}$ , to get:

$$B = a \int_{-\infty}^{\infty} z^2 e^{-bz^2} dz$$

Evaluate this integral using integration by parts:

$$\int u dv = uv - \int v du$$

with:

$$u = z \rightarrow du = dz$$

$$dv = ze^{-bz^2} dz = \left(-\frac{1}{2b}\right) (-2bz) e^{-bz^2} dz \rightarrow v = \left(-\frac{1}{2b}\right) e^{-bz^2}$$

The integral for **B** now becomes:

$$B = a \int_{-\infty}^{\infty} z^2 e^{-bz^2} dz = \left(-\frac{az}{2b}\right) e^{-bz^2} \Big|_{-\infty}^{\infty} + \frac{a}{2b} \int_{-\infty}^{\infty} e^{-bz^2} dz$$

The first part of the integral seems to be the product  $(\infty)(0)$ , but we can eliminate this problem by using L'Hopital's Rule as follows:

$$\lim_{z \rightarrow \infty} ze^{-bz^2} = \lim_{z \rightarrow \infty} \frac{z}{e^{bz^2}} = \lim_{z \rightarrow \infty} \frac{\frac{d}{dz}(z)}{\frac{d}{dz}(e^{bz^2})} = \lim_{z \rightarrow \infty} \frac{1}{2bze^{bz^2}} = \frac{1}{\infty} = 0$$

The same thing happens when  $\mathbf{z \rightarrow -\infty}$ , so the first part of the integral is zero. Except for the constants in front of the integral, the second part of the integral for **B** is exactly the same as the integral for **A**. So, substituting the constant  $\mathbf{a/2b}$  for **a**, we get:

$$B = \frac{a}{2b} \sqrt{\frac{\pi}{b}}$$

Using the values for **A** and **B** in the equation for the variance, we then have:

$$\sigma^2 = \frac{B}{A} = \frac{\frac{a}{2b} \sqrt{\frac{\pi}{b}}}{a \sqrt{\frac{\pi}{b}}} = \frac{1}{2b}$$

$$b = \frac{1}{2\sigma^2}$$

Using the equation for **a**, we get:

$$a = \sqrt{\frac{b}{\pi}} = \sqrt{\frac{1}{2\pi\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma^2}}$$

So, for a corresponding normal PDF, the values of the coefficients are shown in Table 1:

**Table 1: Equations for the Coefficients of a Corresponding Normal PDF**

Coefficient	Equation for the Coefficient
<b>a</b>	$\frac{1}{\sqrt{2\pi\sigma^2}}$
<b>b</b>	$\frac{1}{2\sigma^2}$
<b>c</b>	$\mu$

### Determining the Corresponding Lognormal PDF

To find the corresponding lognormal PDF that has the same mean,  $\mu$ , and variance,  $\sigma^2$ , of a given PDF, we can start by writing a lognormal equation in the general form:

$$y = f(x) = ae^{-b\left(\ln\left(\frac{x}{c}\right)\right)^2} \quad (2)$$

where **c** and **a** are the abscissa and ordinate, respectively, of the mode of the corresponding lognormal PDF and then determine the value of **b** so that the area under the normal function is equal to one. To find the values of **a**, **b** and **c**. Start by using the standard equation from Calculus for finding the mean **x**-value:

$$\mu = \frac{N}{D} = \frac{\int_0^{\infty} xf(x)dx}{\int_0^{\infty} f(x)dx}$$

**D**, the integral in the denominator, is simply the area under the lognormal equation. Calculate the denominator by evaluating the integral:

$$D = \int_0^{\infty} ae^{-b\left(\ln\left(\frac{x}{c}\right)\right)^2} dx$$

First use the transformation:

$$z = \ln\left(\frac{x}{c}\right)$$

$$\frac{x}{c} = e^z$$

$$x = ce^z$$

with:

$$dx = ce^z dz$$

and where  $x=0 \rightarrow z=-\infty$  and  $x=\infty \rightarrow z=\infty$ . The transformed integral is:

$$D = \int_{-\infty}^{\infty} ae^{-bz^2} ce^z dz = ac \int_{-\infty}^{\infty} e^{-bz^2+z} dz$$

Change the exponent by completing-the-square to get:

$$-bz^2 + z = -b\left(z^2 - \frac{z}{b}\right) = -b\left(z^2 - \frac{z}{b} + \frac{1}{4b^2} - \frac{1}{4b^2}\right) = -b\left(\left(z - \frac{1}{2b}\right)^2 - \frac{1}{4b^2}\right)$$

$$-bz^2 + z = -b\left(z - \frac{1}{2b}\right)^2 + \frac{1}{4b}$$

so, the integral now becomes:

$$D = ac \int_{-\infty}^{\infty} e^{-b\left(z-\frac{1}{2b}\right)^2 + \frac{1}{4b}} dz = ace^{\frac{1}{4b}} \int_{-\infty}^{\infty} e^{-b\left(z-\frac{1}{2b}\right)^2} dz$$

Now use the transformation:

$$m = z - \frac{1}{2b}$$

with  $d\mathbf{m}=\mathbf{d}z$ , and where  $z=-\infty \rightarrow m=-\infty$  and  $z=\infty \rightarrow m=\infty$ . The transformed integral is:

$$D = ace^{\frac{1}{4b}} \int_{-\infty}^{\infty} e^{-bm^2} dm$$

Except for the constants in front of the integral, this integral for  $\mathbf{D}$  is exactly the same as the integral for  $\mathbf{A}$  in the previous section on determining the corresponding normal PDF. So, substituting the constant  $ace^{1/(4b)}$  for  $\mathbf{a}$ , we get:

$$D = ace^{\frac{1}{4b}} \sqrt{\frac{\pi}{b}}$$

If the corresponding lognormal curve represents a normalized PDF, then  $\mathbf{D}=1$ :

$$D = 1 = ace^{\frac{1}{4b}} \sqrt{\frac{\pi}{b}}$$

Calculate  $\mathbf{N}$ , the numerator of the mean value equation, by evaluating the integral:

$$N = \int_0^{\infty} axe^{-b\left(\ln\left(\frac{x}{c}\right)\right)^2} dx$$

Again, use the transformation:

$$z = \ln\left(\frac{x}{c}\right)$$

$$\frac{x}{c} = e^z$$

$$x = ce^z$$

with:

$$dx = ce^z dz$$

and where  $x=0 \rightarrow z=-\infty$  and  $x=\infty \rightarrow z=\infty$ . The transformed integral is:

$$N = \int_{-\infty}^{\infty} ace^ze^{-bz^2} ce^z dz = ac^2 \int_{-\infty}^{\infty} e^{-bz^2+2z} dz$$

Change the exponent by completing-the-square to get:

$$-bz^2 + 2z = -b\left(z^2 - \frac{2z}{b}\right) = -b\left(z^2 - \frac{2z}{b} + \frac{1}{b^2} - \frac{1}{b^2}\right) = -b\left(\left(z - \frac{1}{b}\right)^2 - \frac{1}{b^2}\right)$$

$$-bz^2 + 2z = -b\left(z - \frac{1}{b}\right)^2 + \frac{1}{b}$$

The integral now becomes:

$$N = ac^2 \int_{-\infty}^{\infty} e^{-b\left(z-\frac{1}{b}\right)^2 + \frac{1}{b}} dz = ac^2 e^{\frac{1}{b}} \int_{-\infty}^{\infty} e^{-b\left(z-\frac{1}{b}\right)^2} dz$$

Now use the transformation:

$$m = z - \frac{1}{b}$$

with  $dm=dz$ , and where  $z=-\infty \rightarrow m=-\infty$  and  $z=\infty \rightarrow m=\infty$ . The transformed integral is:

$$N = ac^2 e^{\frac{1}{b}} \int_{-\infty}^{\infty} e^{-bm^2} dm$$

As with **A** and **D**, again use the standard trick to evaluate this integral and get:

$$N = ac^2 e^{\frac{1}{4b}} \sqrt{\frac{\pi}{b}}$$

So, using equations **N** and **D** and the equation for the overall mean value,  $\mu$ , we get:

$$\mu = \frac{N}{D} = \frac{ac^2 e^{\frac{1}{4b}} \sqrt{\frac{\pi}{b}}}{ace^{\frac{1}{4b}} \sqrt{\frac{\pi}{b}}} = ce^{\frac{3}{4b}}$$

Next, use the standard equation from Calculus for finding the variance:

$$\sigma^2 = \frac{M}{D} = \frac{\int_0^{\infty} (x - \mu)^2 f(x) dx}{\int_0^{\infty} f(x) dx}$$

**D**, the integral in the denominator, is simply the same integral as we evaluated above in the equation for the mean. Calculate the numerator **M** of the equation for the variance by evaluating the integral:

$$M = \int_0^{\infty} (x - \mu)^2 f(x) dx = \int_0^{\infty} (x - \mu)^2 a e^{-b\left(\ln\left(\frac{x}{c}\right)\right)^2} dx$$

$$M = a \int_0^{\infty} x^2 e^{-b\left(\ln\left(\frac{x}{c}\right)\right)^2} dx - 2a\mu \int_0^{\infty} x e^{-b\left(\ln\left(\frac{x}{c}\right)\right)^2} dx + a\mu^2 \int_0^{\infty} e^{-b\left(\ln\left(\frac{x}{c}\right)\right)^2} dx$$

with:

$$M_1 = a \int_0^{\infty} x^2 e^{-b\left(\ln\left(\frac{x}{c}\right)\right)^2} dx$$

$$M_2 = 2a\mu \int_0^{\infty} x e^{-b\left(\ln\left(\frac{x}{c}\right)\right)^2} dx$$

$$M_3 = a\mu^2 \int_0^{\infty} e^{-b\left(\ln\left(\frac{x}{c}\right)\right)^2} dx$$

Except for the constants in front of the integrals, the equations for **M<sub>2</sub>** and **M<sub>3</sub>** are exactly the same, respectively, as the equations **N** and **D**, above. Adjusting for these different constants, we get:

$$M_2 = 2\mu N = 2\mu ac^2 e^{\frac{1}{b}} \sqrt{\frac{\pi}{b}}$$

$$M_3 = \mu^2 D = \mu^2 a c e^{\frac{1}{4b}} \sqrt{\frac{\pi}{b}}$$

To solve for  $M_1$ , we again use the transformation:

$$z = \ln\left(\frac{x}{c}\right)$$

$$\frac{x}{c} = e^z$$

$$x = ce^z$$

with:

$$dx = ce^z dz$$

and where  $x=0 \rightarrow z=-\infty$  and  $x=\infty \rightarrow z=\infty$ . The transformed integral is:

$$M_1 = \int_{-\infty}^{\infty} ac^2 e^{2z} e^{-bz^2} ce^z dz = ac^3 \int_{-\infty}^{\infty} e^{-bz^2+3z} dz$$

Change the exponent by completing-the-square to get:

$$-bz^2 + 3z = -b\left(z^2 - \frac{3z}{b}\right) = -b\left(z^2 - \frac{3z}{b} + \frac{9}{4b^2} - \frac{9}{4b^2}\right) = -b\left(\left(z - \frac{3}{2b}\right)^2 - \frac{9}{4b^2}\right)$$

$$-bz^2 + 3z = -b\left(z - \frac{3}{2b}\right)^2 + \frac{9}{4b}$$

The integral now becomes:

$$M_1 = ac^3 \int_{-\infty}^{\infty} e^{-b\left(z - \frac{3}{2b}\right)^2 + \frac{9}{4b}} dz = ac^3 e^{\frac{9}{4b}} \int_{-\infty}^{\infty} e^{-b\left(z - \frac{3}{2b}\right)^2} dz$$

Now use the transformation:

$$m = z - \frac{3}{2b}$$

with  $dm=dz$ , and where  $z=-\infty \rightarrow m=-\infty$  and  $z=\infty \rightarrow m=\infty$ . The transformed integral is:

$$M_1 = ac^3 e^{\frac{9}{4b}} \int_{-\infty}^{\infty} e^{-bm^2} dm$$

As with **A** and **D**, again use the standard trick to evaluate this integral and get:

$$M_1 = ac^3 e^{\frac{9}{4b}} \sqrt{\frac{\pi}{b}}$$

Bringing together the equations for  $M_1$ ,  $M_2$  and  $M_3$ , we get:

$$M = M_1 - M_2 + M_3 = ac^3 e^{\frac{9}{4b}} \sqrt{\frac{\pi}{b}} - 2\mu ac^2 e^{\frac{1}{b}} \sqrt{\frac{\pi}{b}} + \mu^2 a c e^{\frac{1}{4b}} \sqrt{\frac{\pi}{b}}$$

$$M = a c e^{\frac{1}{4b}} \sqrt{\frac{\pi}{b}} \left( c^2 e^{\frac{2}{b}} - 2\mu a c e^{\frac{3}{4b}} + \mu^2 \right)$$

The coefficient term in front of the parentheses is equal to **D**. Using this, we get:

$$M = ace^{\frac{1}{4b}} \sqrt{\frac{\pi}{b}} \left( c^2 e^{\frac{2}{b}} - 2\mu ce^{\frac{3}{4b}} + \mu^2 \right) = D \left( c^2 e^{\frac{2}{b}} - 2\mu ce^{\frac{3}{4b}} + \mu^2 \right)$$

$$\frac{M}{D} = \sigma^2 = c^2 e^{\frac{2}{b}} - 2\mu ce^{\frac{3}{4b}} + \mu^2$$

Using the equation we derived, above, for the overall mean value,  $\mu$ :

$$\mu = ce^{\frac{3}{4b}}$$

we get:

$$\sigma^2 = c^2 e^{\frac{2}{b}} - 2\mu ce^{\frac{3}{4b}} + \mu^2 = c^2 e^{\frac{3+1}{2b}} - 2\mu \left( ce^{\frac{3}{4b}} \right) + \mu^2 = \mu^2 e^{\frac{1}{2b}} - 2\mu^2 + \mu^2 = \mu^2 \left( e^{\frac{1}{2b}} - 2 + 1 \right)$$

$$\sigma^2 = \mu^2 \left( e^{\frac{1}{2b}} - 1 \right)$$

We now have three equations—for  $\mu$ ,  $\sigma^2$  and  $D=1$ —to solve for three unknowns,  $a$ ,  $b$  and  $c$ . Starting with the equation for  $\sigma^2$ , we can solve for  $b$ :

$$\sigma^2 = \mu^2 \left( e^{\frac{1}{2b}} - 1 \right)$$

$$\frac{\sigma^2}{\mu^2} = e^{\frac{1}{2b}} - 1$$

$$e^{\frac{1}{2b}} = \frac{\sigma^2}{\mu^2} + 1$$

$$\frac{1}{2b} = \ln \left( \frac{\sigma^2}{\mu^2} + 1 \right)$$

$$b = \frac{1}{2 \ln \left( \frac{\sigma^2}{\mu^2} + 1 \right)}$$

Using equation for  $\mu$ , we get:

$$\mu = ce^{\frac{3}{4b}}$$

$$c = \mu e^{-\frac{3}{4b}} = \mu e^{-\frac{3}{4} \left( 2 \ln \left( \frac{\sigma^2}{\mu^2} + 1 \right) \right)} = \mu e^{\ln \left( \frac{\sigma^2}{\mu^2} + 1 \right) \cdot \frac{3}{2}} = \mu \left( \frac{\sigma^2}{\mu^2} + 1 \right)^{\frac{3}{2}} = \mu \left( \frac{\mu^2}{\mu^2 + \sigma^2} \right)^{\frac{3}{2}} = \frac{\mu^4}{(\mu^2 + \sigma^2)^{\frac{3}{2}}}$$

The equation for  $D=1$  allows us to calculate the abscissa of the mode of the corresponding lognormal cost-risk function, given the function's mean and the variance. Using the equations for  $b$  and  $c$ , we get:

$$D = 1 = ace^{\frac{1}{4b}} \sqrt{\frac{\pi}{b}}$$

$$a = \frac{e^{-\frac{1}{4b}}}{c} \sqrt{\frac{b}{\pi}} = \frac{e^{-\frac{1}{4} \left( 2 \ln \left( \frac{\sigma^2}{\mu^2} + 1 \right) \right)}}{\left( \frac{\mu^4}{(\mu^2 + \sigma^2)^{\frac{3}{2}}} \right)} \sqrt{\frac{1}{2\pi \ln \left( \frac{\sigma^2}{\mu^2} + 1 \right)}} = \frac{\sqrt{\frac{\sigma^2}{\mu^2} + 1}}{\left( \frac{\mu^4}{(\mu^2 + \sigma^2)^{\frac{3}{2}}} \right) \sqrt{2\pi \ln \left( \frac{\sigma^2}{\mu^2} + 1 \right)}}$$

$$a = \frac{\frac{(\mu^2 + \sigma^2)^{\frac{3}{2}}}{\mu^4}}{\frac{(\mu^2 + \sigma^2)^{\frac{1}{2}}}{\mu} \sqrt{2\pi \ln\left(\frac{\sigma^2}{\mu^2} + 1\right)}} = \frac{\mu^2 + \sigma^2}{\mu^3 \sqrt{2\pi \ln\left(\frac{\mu^2 + \sigma^2}{\mu^2}\right)}}$$

So, for a corresponding lognormal PDF, the values of the coefficients are shown in Table 2:

**Table 2: Equations for the Coefficients of a Corresponding Lognormal PDF**

<b>Coefficient</b>	<b>Equation for the Coefficient</b>
<b>a</b>	$\frac{\mu^2 + \sigma^2}{\mu^3 \sqrt{2\pi \ln\left(\frac{\mu^2 + \sigma^2}{\mu^2}\right)}}$
<b>b</b>	$\frac{1}{2\ln\left(\frac{\sigma^2}{\mu^2} + 1\right)}$
<b>c</b>	$\frac{\mu^4}{(\mu^2 + \sigma^2)^{\frac{3}{2}}}$

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## Addendum 2: Mean & Variance of the Convolution of Functions of Uncorrelated Random Variables

### Determining the Mean of the Convolution of Functions of Uncorrelated Random Variables

If all the functions in a convolution are functions of uncorrelated random variables, we can derive an exact relationship between the mean value of a convolution and the mean values of the individual functions being convolved. Let  $\mathbf{f}(\mathbf{x})$  be the convolution of two functions  $\mathbf{a}(\mathbf{x})$  and  $\mathbf{b}(\mathbf{x})$ , i.e.,  $\mathbf{f}(\mathbf{x})=\mathbf{a}(\mathbf{x})*\mathbf{b}(\mathbf{x})$ , and let these two functions have mean values  $\mu_a$  and  $\mu_b$ , i.e.:

$$\mu_a = \frac{\int_{-\infty}^{\infty} xa(x) dx}{\int_{-\infty}^{\infty} a(x) dx}$$

$$\mu_b = \frac{\int_{-\infty}^{\infty} xb(x) dx}{\int_{-\infty}^{\infty} b(x) dx}$$

Since the integrals in the denominators are simply the areas  $\mathbf{A}$  under the functions  $\mathbf{a}(\mathbf{x})$  and  $\mathbf{b}(\mathbf{x})$ , we have:

$$A_a = \int_{-\infty}^{\infty} a(x) dx$$

$$A_b = \int_{-\infty}^{\infty} b(x) dx$$

giving us:

$$\mu_a = \frac{\int_{-\infty}^{\infty} xa(x) dx}{A_a}$$

$$\mu_a A_a = \int_{-\infty}^{\infty} xa(x) dx$$

and:

$$\mu_b = \frac{\int_{-\infty}^{\infty} xb(x) dx}{A_b}$$

$$\mu_b A_b = \int_{-\infty}^{\infty} xb(x) dx$$

The mean value  $\mu_f$  of the convolution  $\mathbf{f}(\mathbf{x})$  is:

$$\mu_f = \frac{\int_{-\infty}^{\infty} xf(x) dx}{\int_{-\infty}^{\infty} f(x) dx}$$

Since  $\mathbf{f}(\mathbf{x})$  is the convolution of  $\mathbf{a}(\mathbf{x})$  and  $\mathbf{b}(\mathbf{x})$ , then the area under  $\mathbf{f}(\mathbf{x})$  is the product of the areas under  $\mathbf{a}(\mathbf{x})$  and  $\mathbf{b}(\mathbf{x})$ , i.e.,  $A_f=A_aA_b$ , so we then have:

$$\mu_f = \frac{\int_{-\infty}^{\infty} xf(x) dx}{A_f}$$

$$\mu_f A_f = \int_{-\infty}^{\infty} xf(x) dx$$

and:

$$\mu_f A_f = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} a(x - \alpha) b(\alpha) d\alpha dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x a(x - \alpha) b(\alpha) dx d\alpha$$

By using  $\beta = x - \alpha$  ( $\rightarrow x = \alpha + \beta$ ) and, if we take the derivative of  $\beta$  with respect to  $x$  and express the result in differential form, we get  $d\beta = dx$ . We then have:

$$\mu_f A_f = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha + \beta) a(\beta) b(\alpha) d\beta d\alpha = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha a(\beta) b(\alpha) + \beta a(\beta) b(\alpha)) d\beta d\alpha$$

$$\mu_f A_f = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha a(\beta) b(\alpha) d\beta d\alpha + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta a(\beta) b(\alpha) d\beta d\alpha$$

If the variables  $\alpha$  and  $\beta$  are uncorrelated random variables, then the functions  $a(x)$  and  $b(x)$  are separable, giving us:

$$\mu_f A_f = \int_{-\infty}^{\infty} a(\beta) \left( \int_{-\infty}^{\infty} \alpha b(\alpha) d\alpha \right) d\beta + \int_{-\infty}^{\infty} b(\alpha) \left( \int_{-\infty}^{\infty} \beta a(\beta) d\beta \right) d\alpha$$

$$\mu_f A_f = \int_{-\infty}^{\infty} a(\beta) d\beta \left( \int_{-\infty}^{\infty} \alpha b(\alpha) d\alpha \right) + \int_{-\infty}^{\infty} b(\alpha) d\alpha \left( \int_{-\infty}^{\infty} \beta a(\beta) d\beta \right)$$

The quantities in the parentheses are the mean values, and the quantities in front of the parentheses are the areas  $A_a$  and  $A_b$ . So, the terms being added are mean values and their associated areas,  $\mu_a A_a$  and  $\mu_b A_b$ , giving us:

$$\mu_f A_f = \mu_a A_a A_b + \mu_b A_a A_b = (\mu_a + \mu_b) A_a A_b$$

But, since  $A_f = A_a A_b$ , and we have:

$$\mu_f = \mu_a + \mu_b$$

So, for functions of uncorrelated random variables, the mean of a convolution is the sum of the means of each of the functions in the convolution. If the functions  $a(x)$  and  $b(x)$  are themselves convolutions of functions of uncorrelated random variables, then we can express each of their means as the sum of the means of the functions in their convolutions and we can then express the mean of  $f(x)$  as the sum of both sets of means. We can extend this indefinitely for any set of  $n$  uncorrelated functions to get:

$$\mu_f = \mu_1 + \mu_2 + \mu_3 + \dots + \mu_n = \sum_{i=1}^n \mu_i$$

where  $n$  is any positive integer.

## Determining the Variance of the Convolution of Functions of Uncorrelated Random Variables

For functions of uncorrelated random variables, we can also derive an exact relationship between the variance of a convolution and the variances of the individual functions being convolved. Let  $f(x)$  be the convolution of two functions  $a(x)$  and  $b(x)$ , i.e.,  $f(x) = a(x) * b(x)$ , and let these functions have mean values  $\sigma_f^2$ ,  $\sigma_a^2$  and  $\sigma_b^2$ , respectively. The variance  $\sigma_f^2$  of the function  $f(x)$  is:

$$\sigma_f^2 = \frac{\int_{-\infty}^{\infty} (x - \mu_f)^2 f(x) dx}{\int_{-\infty}^{\infty} f(x) dx} = \frac{\int_{-\infty}^{\infty} (x - \mu_f)^2 f(x) dx}{A_f}$$

$$A_f \sigma_f^2 = \int_{-\infty}^{\infty} (x - \mu_f)^2 f(x) dx = \int_{-\infty}^{\infty} (x - \mu_f)^2 \int_{-\infty}^{\infty} a(x - \alpha) b(\alpha) d\alpha dx$$

$$A_f \sigma_f^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_f)^2 a(x - \alpha) b(\alpha) dx d\alpha$$

By using  $\mathbf{\beta} = \mathbf{x} - \mathbf{\alpha}$  ( $\rightarrow \mathbf{x} = \mathbf{\alpha} + \mathbf{\beta}$ ) and, if we take the derivative of  $\mathbf{\beta}$  with respect to  $\mathbf{x}$  and express the result in differential form, we get  $d\mathbf{\beta} = d\mathbf{x}$ . We then have:

$$A_f \sigma_f^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha + \beta - \mu_a - \mu_b)^2 a(\beta) b(\alpha) d\beta d\alpha$$

$$A_f \sigma_f^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha^2 + \beta^2 + \mu_a^2 + \mu_b^2 + 2\alpha\beta - 2\alpha\mu_a - 2\alpha\mu_b - 2\beta\mu_a - 2\beta\mu_b + 2\mu_a\mu_b) a(\beta) b(\alpha) d\beta d\alpha$$

$$A_f \sigma_f^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ((\beta^2 - 2\beta\mu_a + \mu_a^2) + (\alpha^2 - 2\alpha\mu_b + \mu_b^2) + 2(\alpha\beta - \alpha\mu_a - \beta\mu_b + \mu_a\mu_b)) a(\beta) b(\alpha) d\beta d\alpha$$

$$A_f \sigma_f^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\beta - \mu_a)^2 a(\beta) b(\alpha) d\beta d\alpha + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha - \mu_b)^2 a(\beta) b(\alpha) d\beta d\alpha + 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha\beta - \alpha\mu_a - \beta\mu_b + \mu_a\mu_b) a(\beta) b(\alpha) d\beta d\alpha$$

$$A_f \sigma_f^2 = \int_{-\infty}^{\infty} (\beta - \mu_a)^2 a(\beta) d\beta \int_{-\infty}^{\infty} b(\alpha) d\alpha + \int_{-\infty}^{\infty} (\alpha - \mu_b)^2 b(\alpha) d\alpha \int_{-\infty}^{\infty} a(\beta) d\beta + 2 \int_{-\infty}^{\infty} \alpha b(\alpha) d\alpha \int_{-\infty}^{\infty} \beta a(\beta) d\beta + 2\mu_a \int_{-\infty}^{\infty} \alpha b(\alpha) d\alpha \int_{-\infty}^{\infty} a(\beta) d\beta + 2\mu_b \int_{-\infty}^{\infty} b(\alpha) d\alpha \int_{-\infty}^{\infty} \beta a(\beta) d\beta + 2\mu_a\mu_b \int_{-\infty}^{\infty} b(\alpha) d\alpha \int_{-\infty}^{\infty} a(\beta) d\beta$$

$$A_f \sigma_f^2 = A_a \sigma_a^2 A_b + A_b \sigma_b^2 A_a + 2(A_a \mu_a \mu_b A_b - \mu_a \mu_b A_b A_a - \mu_b \mu_a A_a A_b + \mu_a \mu_b A_a A_b)$$

$$A_f \sigma_f^2 = \sigma_a^2 A_a A_b + \sigma_b^2 A_a A_b + 2(0) = \sigma_a^2 A_a A_b + \sigma_b^2 A_a A_b$$

Since  $\mathbf{f}(\mathbf{x})$  is the convolution of  $\mathbf{a}(\mathbf{x})$  and  $\mathbf{b}(\mathbf{x})$ , then the area under  $\mathbf{f}(\mathbf{x})$  is the product of the areas under  $\mathbf{a}(\mathbf{x})$  and  $\mathbf{b}(\mathbf{x})$ , i.e.,  $A_f = A_a A_b$ , so we then have:

$$A_a A_b \sigma_f^2 = \sigma_a^2 A_a A_b + \sigma_b^2 A_a A_b$$

$$\sigma_f^2 = \sigma_a^2 + \sigma_b^2$$

So, for functions of uncorrelated random variables, the variance of a convolution is the sum of the variances of each of the functions in the convolution. If the functions  $\mathbf{a}(\mathbf{x})$  and  $\mathbf{b}(\mathbf{x})$  are themselves convolutions of functions of uncorrelated random variables, then we can express each of their variances as the sum of the variances of the functions in their convolutions and we can then express the variance of  $\mathbf{f}(\mathbf{x})$  as the sum of both sets of variances. We can extend this indefinitely for any set of  $\mathbf{n}$  uncorrelated functions to get:

$$\sigma_f^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \dots + \sigma_n^2 = \sum_{i=1}^n \sigma_i^2$$

where  $\mathbf{n}$  is any positive integer.

## Determining the Mean and Variance of the Convolution of Discrete Functions

In the special cases of discrete functions, where the functions are a set of one or more delta functions, we can use the convolution properties of delta functions to see how they affect the mean and the variance. If  $\mathbf{f}(\mathbf{x})$  is the convolution of any real function  $\mathbf{a}(\mathbf{x})$  with a delta function  $\mathbf{b}(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}_0)$ , we get:

$$f(x) = a(x) * b(x) = \int_{-\infty}^{\infty} a(\alpha)b(x - \alpha)d\alpha = \int_{-\infty}^{\infty} a(\alpha)\delta(x - \alpha - x_0)d\alpha$$

Let  $\beta=x-x_0$ :

$$f(x) = \int_{-\infty}^{\infty} a(\alpha)\delta(x - \alpha - x_0)d\alpha = \int_{-\infty}^{\infty} a(\alpha)\delta(\beta - \alpha)d\alpha = a(\beta) = a(x - x_0)$$

So,  $\mathbf{a}(\mathbf{x})$  is simply shifted to the right by the amount  $\mathbf{x}_0$ . This will also shift the mean to the right by the amount  $\mathbf{x}_0$  and leave the variance of  $\mathbf{a}(\mathbf{x})$  unchanged. So, if  $\mathbf{f}(\mathbf{x})$  is the convolution of any real function  $\mathbf{a}(\mathbf{x})$  with a set of  $\mathbf{n}$  delta functions:

$$b(x) = \sum_{i=1}^n c_i \delta(x - x_i)$$

where  $\mathbf{n}$  is any positive integer, the  $\mathbf{c}_i$  are constant coefficients and the  $\mathbf{x}_i$  are fixed real numbers. We then get:

$$f(x) = a(x) * b(x) = \int_{-\infty}^{\infty} a(\alpha)b(x - \alpha)d\alpha = \int_{-\infty}^{\infty} a(\alpha) \left( \sum_{i=1}^n c_i \delta(x - \alpha - x_i) \right) d\alpha$$

$$f(x) = \sum_{i=1}^n c_i \int_{-\infty}^{\infty} a(\alpha)\delta(x - \alpha - x_i)d\alpha$$

Let  $\beta_i=x-x_i$ :

$$f(x) = \sum_{i=1}^n c_i \int_{-\infty}^{\infty} a(\alpha)\delta(x - \alpha - x_i)d\alpha = \sum_{i=1}^n c_i \int_{-\infty}^{\infty} a(\alpha)\delta(\beta_i - \alpha)d\alpha = \sum_{i=1}^n c_i a(\beta_i) = \sum_{i=1}^n c_i a(x - x_i)$$

So,  $\mathbf{f}(\mathbf{x})$  is the sum of  $\mathbf{n}$  copies of  $\mathbf{a}(\mathbf{x})$  with each  $\mathbf{i}^{\text{th}}$  copy shifted to the right by the amount  $\mathbf{x}_i$ . The mean  $\mu$  of  $\mathbf{f}(\mathbf{x})$  is:

$$\mu = \frac{\int_{-\infty}^{\infty} x f(x) dx}{\int_{-\infty}^{\infty} f(x) dx} = \frac{\int_{-\infty}^{\infty} x \sum_{i=1}^n c_i a(x - x_i) dx}{\int_{-\infty}^{\infty} \sum_{i=1}^n c_i a(x - x_i) dx} = \frac{\sum_{i=1}^n c_i \int_{-\infty}^{\infty} x a(x - x_i) dx}{\sum_{i=1}^n c_i \int_{-\infty}^{\infty} a(x - x_i) dx}$$

The denominator is the area  $\mathbf{A}_f$  under the function  $\mathbf{f}(\mathbf{x})$ .  $\mathbf{A}_f$  is the weighted sum of the areas  $\mathbf{A}_a$  under the  $\mathbf{n}$  shifted copies of the function  $\mathbf{a}(\mathbf{x})$ :

$$A_f = \sum_{i=1}^n c_i A_a = A_a \sum_{i=1}^n c_i$$

Using this, we get:

$$A_f \mu = A_a \sum_{i=1}^n c_i \mu = \sum_{i=1}^n c_i \int_{-\infty}^{\infty} x a(x - x_i) dx = \sum_{i=1}^n c_i A_a \mu_i = A_a \sum_{i=1}^n c_i \mu_i$$

$$\mu = \frac{A_a}{A_f} \sum_{i=1}^n c_i \mu_i$$

So the mean  $\mu$  of  $\mathbf{f}(\mathbf{x})$  is the weighted average of the means of the  $\mathbf{n}$  shifted copies of  $\mathbf{a}(\mathbf{x})$ .

The variance  $\sigma^2$  of  $\mathbf{f}(\mathbf{x})$  is:

$$\sigma^2 = \frac{\int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx}{\int_{-\infty}^{\infty} f(x) dx} = \frac{\int_{-\infty}^{\infty} (x - \mu)^2 (\sum_{i=1}^n c_i a(x - x_i)) dx}{A_f} = \frac{\sum_{i=1}^n c_i \int_{-\infty}^{\infty} (x - \mu)^2 a(x - x_i) dx}{A_f}$$

$$\sigma^2 = \frac{\sum_{i=1}^n c_i A_a \sigma_i^2}{A_f} = \frac{A_a}{A_f} \sum_{i=1}^n c_i \sigma_i^2$$

So the variance of  $\mathbf{f}(\mathbf{x})$  is the weighted average of the variances of the  $\mathbf{n}$  shifted copies of  $\mathbf{a}(\mathbf{x})$ .

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