SOLUTION SETS FOR GAMES ON THE SQUARE

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Summary: Some necessary and sufficient* conditions that a pair of non-void weak closed convex sets of strategies form the solution set of a game with continuous payoff on the square are given.

SOLUTION SETS FOR GAMES ON THE SQUARE

I. Glicksberg and O. Gross

Let K denote the set of all optimal strategies for one player, L the corresponding set for his opponent in a game. We shall refer to K x L, the set of all pairs (f,g), f ∈ K, g ∈ L, as the solution set of the game. Any non-void weak* closed convex set K is the set of all optimal strategies for one player in some game with continuous payoff, as was shown in [1], but of course not all pairs K,L of such sets will yield solution sets. By means of constructions similar to those used in [1] we shall determine which pairs do occur in terms of the spectra, 1) σ-K, σ-L of these sets and the number of independent containing hyperplanes.

1. Preliminaries. As was shown in [1], any non-void weak* (w*) closed convex set K of strategies is the intersection

1) $\sigma K = \bigcup_{f \in K} \sigma(f)$, which is easily seen to be a closed set.
of a sequence of half spaces, which we may express by

\[ K = \{ f \mid (\psi_n, f) = \int \psi_n(x) df(x) \geq 0, \ n = 1, \ldots \} \]

where \( \{\psi_n\} \) is a sequence of continuous functions and we may assume, for each \( n \), \( (\psi_n, f) = 0 \) for some \( f \) in \( K \). Certain of these \( \psi \)'s will yield \( (\psi, f) = 0 \) for all \( f \) in \( K \), and these we shall denote by \( p \)'s. Thus we shall write

\[ K = S(\{\psi_m; p_n\})^2 \]

to express the fact that \( K = \{ f \mid (\psi_m, f) \neq 0 = (p_n, f) \} \) as well as the fact that \( (\psi_m, K) \) is a non-degenerate interval.

The functions \( p_m \) thus define hyperplanes containing \( K \) while the \( \psi_m \) do not. If the set \( K \) is the intersection of a set of hyperplanes, one may show exactly as in the proof of (1) that it is the intersection of a sequence of these and one may write \( K = S(p_m) \).

What we shall be concerned with in large part in the following constructions will be the hyperplanes containing \( K \). It is immediately evident that if we select from the functions \( \{p_n\} \) a maximal subsequence \( \{p'_n\} \) which is linearly independent on \( \circ K \) then the relations \( (p_n, f) = 0 \) are consequences of the relations \( (p'_n, f) = 0 \) for \( f \) for which

\[ 2) \text{ For the opponent we shall write } L = S(\psi_m; q_n) \text{ where we take } (\psi_m, g) \neq 0. \]
$\sigma(f) \subset \sigma K$. Consequently if we set $p^*(x) = \text{dist}(x, \sigma K)$, then $(p^*, f) = 0$, $(p_n^*, f) = 0$ all $n \iff (p_n, f) = 0$ all $n$; thus in most of what follows we shall assume the $\{p_n\}$ to be linearly independent, and actually orthonormal:

Suppose we define a measure on $\sigma K$ in the following way: select a sequence \( \{x_n\} \) dense in $\sigma K$ and place weight $2^{-n}$ at $x_n$. Then clearly we may apply the Gram–Schmidt process to the $\{p_n\}$ to obtain an orthonormal sequence $\{p_n'\}$ of the same length (we take $\{x_n\}$ dense to insure that only the function $0$ has the integral of its square zero). Just as clear is the fact that $(p_n, f) = 0$ for all $n$ is equivalent to $(p_n', f) = 0$ for all $n$.

2. Constructions. We shall now construct payoffs which will have three types of solution sets. That these are the only types which occur will be shown later.

Case I: Suppose $\sigma K = [0, 1] = \sigma L$ and $K$ and $L$ are the intersections of the same number of independent hyperplanes. The orthonormal sequences $\{p_n\}$ and $\{q_n\}$ defining $K$ and $L$ are thus of the same length, and if we set

$$M(x, y) = \sum a_n p_n(x) q_n(y),$$

3) We shall say that the hyperplanes $H_n$ defined by $H_n = \{f | (p_n, f) = 0\}$ are independent hyperplanes containing $K$ if the $p_n$ are linearly independent on $\sigma K$, $K \subset H_n$. 
where the $a_n$ are chosen to insure uniform convergence of the series, then for $f$ in $K$ and $g$ in $L$,

$$\int M df = \sum a_n(p_n,f)q_n(y) = 0 = \sum a_n p_n(x)(q_n,g) = \int M dg;$$
on the other hand, if $f$ is optimal

$$\int M df = \sum a_n(p_n,f)q_n(y) = 0,$$
and in view of the orthogonality of the $q_n$, if $f$ is in $K$.
Similarly every optimal $g$ is in $L$, and $K \times L$ is the solution set.

**Case II:** Suppose $\sigma K = [0,1] \neq \sigma L$, and $K = S(\phi_m; p_n)$, $L = S(q_n)$ where there are at least as many independent hyperplanes containing $K$ as there are containing $L$ (thus we may assume that a maximal linearly independent set of $p_n$'s is at least as long as the set of $q_n$'s linearly independent on $\sigma L$). Since $\sigma L$ is not the full unit interval we may select an open interval $I$ which has one end point $y_0$ in $\sigma L$. Select a disjoint sequence $\{I_n\}$ of open subintervals of $I$ for which $\text{dist} (y_0, I_n) \rightarrow 0$, and an open subinterval $I_n^*$ of each $I_n$ whose closure lies entirely in $I_n$. Let $k_n$ be a continuous non-negative function which vanishes outside $I_n$ but is non-zero inside $I_n$, and which assumes the value 1 at a point $y_n$ of $I_n^*$. Define a continuous function $m_n$ which vanishes at $y_n$ and outside $I_n^*$, but takes on the values $\pm 1$. 
If we then set \( q(y) = \text{dist}(y, \sigma L \cup \bigcup I^*_n) \), then for every \( y \) not in \( \sigma L \) one of the non-negative functions \( q, k_n \) is non-zero at \( y \).

We now define our payoff as follows: we divide the sequence \( \{p_n\} \) into \( \{p'_n\} \), orthonormal and of the same length as the \( \{q_n\} \), and \( \{p'_n\} \). If either of the sequences \( \{p'_n\} \) or \( \{\psi_n\} \) are finite we use repetitions to form a sequence, and if there are no \( \psi_n \)'s say, we take \( \psi_n = 1 \) for all \( n \).

We set (for \( b_n > 0 \), chosen to insure uniform convergence)

\[
M(x, y) = \sum a_n p_n(x) q_n(y) + \sum b_n \left[ k_n(y)\psi_{N_n} (x) + nm_n(y)p^*_n(x) \right] + q(y)
\]

where \( \{N_n\} \) is an enumeration of the integers in which each integer occurs infinitely often. For \( f \) in \( K \) and \( g \) in \( L \)

\[
\int Mdf = \sum b_n k_n(y)(\psi_{N_n}, f) + q(y) \geq 0 = \int Mdg,
\]

so that both are optimal.

Suppose \( f \) is optimal; then for \( y \) in \( \sigma L \),

\[
\sum a_n(p_n, f)q_n(y) = 0
\]

whence \( (p_n, f) = 0 \), and thus

\[
0 \leq \sum b_n \left[ k_n(y)(\psi_{N_n}, f) + nm_n(y)p^*_n, f \right] + q(y),
\]

and at setting \( y = y_n \), \( b_n(\psi_{N_n}, f) \geq 0 \), so that \( (\psi_n, f) \geq 0 \) for all \( n \). For \( y \) in \( I^*_n \) we have

\[
0 \leq b_n \left[ k_n(y)(\psi_{N_n}, f) + nm_n(y)p^*_n, f \right]
\]
whence $0 \neq \left( \varphi_{N_n}, f \right) + m_n(\psi)(p_{n}^1, f)$, and since $m_n$ assumes the values $\pm 1$,

$\left( \varphi_{N_n}, f \right) = \pm n(p_{n}^1, f),$

hence

$\left( \varphi_{N_n}, f \right) \neq n|\left( p_{n}^1, f \right)|.$

Since $N_n$ takes on the value $n_0$ infinitely often,

$\left( \varphi_{n_0}, f \right) \neq n_0|\left( p_{n_0}^1, f \right)|$ for arbitrarily large $n$, and

$(p_{n_0}^1, f) = 0$ for each $n_0$. Thus $f$ is in $K$.

If $g$ is optimal, then for any $f$ in $K$,

$$Q = \int \int Mdfdg = \sum b_n(k_n, g)(\varphi_{N_n}, f) + (q, g).$$

But each term of this sum is non-negative ($\left( k_n, g \right) \neq 0$ since $k_n \neq 0$) so that surely $(q, g) = 0$. If $(k_n, g) > 0$ for some $n$ then since there is an $f$ in $K$ for which $(\varphi_{n}, f) > 0$, we would have a contradiction. Thus

$$(q, g) = 0, (k_n, g) = 0, \text{ and } (m_n, g) = 0$$

since $(k_n, g) = 0$ implies $g$ places no weight on $I_n$. Thus

$\varphi(g) \subset \varphi L$, and since we now may write

$$0 = \sum a_n p_n(x)(q_n, g), \quad x \text{ in } \varphi K,$$

and $(q_n, g) = 0$, $g$ is in $L$. 

Case III: $\sigma K \neq [0,1] \neq \sigma L$. Here we may take any $K$ and $L$ without further restriction, so that $K = S(\varphi_m; p_n)$ and $L = S(\psi_m; q_n)$ (if $\varphi_m, \psi_m \neq 0$ here, however, in our definitions). We construct functions $h_n$ similar to the $k_n$ of case II, and $l_n$ similar to the $m_n$, on an interval abutting $\sigma K$. We set

$$M(x,y) = \sum_a h_n(x) \varphi_n(y) + n h_n(x) q_n(y) + k_n(y) \psi_n(x) + n m_n(y) p_n(x).$$

Arguments entirely similar to those used in case II show $K \times L$ to be the solution set.

3. Generality. In case I ($\sigma K = [0,1] = \sigma L$) we restricted our attention to the case in which $K$ and $L$ were intersections of the same number of independent hyperplanes. Suppose now that a game with payoff $M$ has as its solution set $K \times L$ where $\sigma K$ and $\sigma L$ are the full intervals. $K$ is determined as the set of all $f$ for which

$$\int M(x,y) df(x) = 0$$

(for convenience we take the value to be zero), and thus is the intersection of hyperplanes given by the functions \{ $M(\cdot, y)$ \}, and similarly $L$ is the intersection of the hyperplanes determined by the functions \{ $M(x, \cdot)$ \}.

If a maximal linearly independent set \{ $M(x_i, \cdot)$ \} of the first set, say, is finite, $i = 1, \ldots, n$, then the same is true of the second, indeed there are just as many. For, as is
well known, n functions $F_1, \ldots, F_n$ are linearly independent on a set $X$ if and only if there exist $x_1, \ldots, x_n$ in $X$ for which

$$\det (F_i(x_j)) \neq 0;$$

consequently we have $y_1, \ldots, y_n$ for which

$$(2) \quad \det (M(x_1, y_j)) \neq 0,$$

so that the functions $\{M(\cdot, y_j)\}_{j=1, \ldots, n}$ are linearly independent. Of course if $\{M(\cdot, y_j)\}_{j=1, \ldots, n+1}$ were linearly independent by the same argument we should have an $x_{n+1}$ for which $\{M(x_1, \cdot)\}_{i=1, \ldots, n+1}$ were, which contradicts our assumption, and there are exactly $n$. Thus the type of solution sets considered in case I are the only type which can occur. (One might note that here finite set of independent containing hyperplanes can only occur in a polynomial-like game, since for every $x$ we have coefficients $a_i(x)$ for which

$$M(x, y) = \sum a_i(x)M(x_1, y),$$

and (2) shows the functions $a_i$ to be continuous.)

In case II, ($\sigma K = [0,1] \neq \sigma L$) we considered only those $K$ and $L$ for which we had as many independent hyperplanes containing $K$ as there are containing $L$. But if $M$ is the payoff of a game with solution set $K \times L$, $\sigma K = [0,1] \neq \sigma L$, then as before since $L$ is determined by

$$\int M(x, y)dg(y) = 0, \quad \text{all } x,$$
L is just the intersection of hyperplanes. If there are only \( n \) independent containing hyperplanes, then, as we shall see in a moment, these must be given by the functions
\[
\{M(x_i, \cdot)\}_{i=1, \ldots, n}
\]
linearly independent on \( \sigma L \), for some set \( x_1, \ldots, x_n \); consequently there exist \( y_1, \ldots, y_n \) in \( \sigma L \) for which (2) holds, and \( \{M(\cdot, y_j)\}_{j=1, \ldots, n} \) are linearly independent.

Since \( \int M(x, y) d\gamma(x) = 0 \) for \( y \) in \( \sigma L \), these functions define \( n \) independent hyperplanes containing \( K \).

To see that the \( n \) independent hyperplanes containing \( L \) arise from functions \( M(x_i, \cdot) \) we note that for each \( x \), \( M(x, \cdot) \) defines a containing hyperplane since \( x \) is in \( \sigma K = [0, 1] \).

Consequently there can be only \( m \) points, \( m \neq n, x_1, \ldots, x_m \) for which \( \{M(x_i, \cdot)\} \) are linearly independent, so that clearly
\[
L = \{g| (M(x_i, \cdot), g) = 0, i = 1, \ldots, m\}
\]

If \( m < n \), we can find a function \( q_0 \) for which, denoting \( M(x_i, \cdot) \) by \( q_i \), the set \( q_0, \ldots, q_m \) is linearly independent on \( \sigma L \) and \( (q_0, g) = 0 \) for all \( g \) in \( L \). But then the mapping
\[
T: g \mapsto ((q_0, g), \ldots, (q_m, g))
\]
of the set \( S \) of all strategies into \( m + 1 \) space, takes \( S \) into a convex subset containing \( (0, \ldots, 0) \) (since \( L \) is non-void). But \( T(S) \) intersects the line \( (t, 0, \ldots, 0) \) in only one point (since \( (q_0, g) = 0 \) for \( g \) in \( L \))—thus \( (0, 0, \ldots, 0) \) is a boundary point and we have a supporting hyperplane at this point given by constants (not all zero) \( a_0, \ldots, a_m \). Thus
\[
\sum_{i=0}^{\infty} a_i (q_i, g) \geq 0
\]
for all g in S, hence $\sum a_iq_i(y) \geq 0$ for y in $\sigma$-$L$. If inequality holds for any y it holds in some neighborhood, and this is, of course, of positive measure with respect to some g in L (from the definition of $\sigma$-$L$), whence $\sum a_i(q_i,g) > 0$ for some g in L — a contradiction. Thus $\sum a_iq_i = 0$ on $\sigma$-$L$, which contradicts the linear independence on $\sigma$-$L$, and we must have $m = n$.

Thus the theme of things is as follows: The necessary and sufficient condition that K x L be the solution set for a game with continuous payoff on the square (where K and L are non-void $\omega^*$ closed convex sets of strategies) is that one of the following hold:

(a) $\sigma$-$K = [0,1] = \sigma$-$L$ and K and L are the intersection of the same number (finite if and only if the game is polynomial-like) of independent containing hyperplanes

(b) $\sigma$-$K = [0,1] \neq \sigma$-$L$, L is the intersection of hyperplanes and K has as many independent containing hyperplanes as L

(c) $\sigma$-$K \neq [0,1] \neq \sigma$-$L$.

The constructions we have used can be duplicated in the case of a game with continuous payoff played on a pair of infinite compact metric spaces; the character of solution sets, however, involves slightly different conditions:
\( \sigma K = [0,1] = \sigma L \) must be replaced by \( \sigma K, \sigma L \) open, 
\( \sigma K = [0,1] \neq \sigma L \) by \( \sigma K \) open, \( \sigma L \) not open, 
\( \sigma K \neq [0,1] \neq \sigma L \) by \( \sigma K \) and \( \sigma L \) not open. In the case of 
a unique optimal strategy forming \( K \) and another forming \( L \) 
we are thus guaranteed a game having \( K \times L \) as the solution 
set, which generalizes the result of [2].

As a final remark, we note that solution sets for 
symmetric games on the square (where \( M(x,y) = -M(y,x) \)) can be 
easily described. For such games the value is always zero 
and any optimal strategy for one player is optimal for his 
opponent, so that a solution set is of the form \( K \times K \). The 
necessary and sufficient condition that \( K \times K \) be the solution 
set of a symmetric game is that either 

(a) \( \sigma K = [0,1] \) and \( K \) is the intersection of an even 

we take \( \omega \) as even) number of independent hyper-
planes, or 
(b) \( \sigma K \neq [0,1] \).

For if \( \sigma K = [0,1] \) and \( K \) is the intersection of an 
even number of independent hyperplanes given by functions 
\( \{p_n\} \) (which we may take orthonormal), then, dividing these 
into two sets \( \{p_n\}, \{p'_n\} \) of equal cardinality, we may set 

\[
M(x,y) = \sum a_n [p_n(x)p'_n(y) - p'_n(x)p_n(y)],
\]

which is easily seen to have \( K \times K \) as its solution, and is 
symmetric. On the other hand, if \( K \times K \) is the solution set
of a game with payoff \( M \) and \( \sigma K = [0, 1] \), then \( K \) is, of course, the intersection of a set of hyperplanes. If only a finite number of these are independent, then, as before, \( M \) is polynomial-like, that is,

\[
M(x, y) = \sum_{n=1}^{k} \phi_n(x) \psi_n(y),
\]

where \( \{\phi_n\} \) and \( \{\psi_n\} \) are linearly independent sets of functions. Since \( M \) is symmetric

\[
M(x, y) = -M(y, x) = -\sum_{n=1}^{k} \phi_n(y) \psi_n(x),
\]

so

\[
M(x, y) = \frac{1}{2} \sum_{n=1}^{k} [\phi_n(x) \psi_n(y) - \phi_n(y) \psi_n(x)].
\]

If the functions \( \{\phi_n, \psi_n\} \) are not a linearly independent set, replacement of a dependent \( \phi \) or \( \psi \) again yields a sum of the same type, and we finally obtain a similar expression for \( M \) in which the set \( \{\phi_n, \psi_n\} \) is linearly independent; however, there are an even number of terms in the resulting sums, and thus there must be an even number of independent hyperplanes determining \( K \).

In case (b), \( K = S(\psi_m, p_n) \), and we may set

\[
M(x, y) = \sum_{n} a_n \left[ k_n(y) \psi_n(x) + n \psi_n(y)p_n(x) - k_n(x) \psi_n(y) - n \psi_n(x)p_n(y) \right]
\]

to obtain a symmetric game in which \( K \times K \) is the solution set.
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1. I. Glicksberg and O. Gross, Optimal Sets for Games over the Square, RM-889.
2. I. Glicksberg and O. Gross, Continuous Games with Given Unique Solutions, RM-620.