A Mathematical Model of Network Communication

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Outline of Presentation

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• Communication Network
• Discrete Network Model
• Discrete Conservation of Packets Equations
• Continuum Network Model
• Example: One Dimensional Flow Model
• Current Work
Motivation

• Rapid Communication is essential in today’s world.
• Understand the dynamics of flow by creating a communication network model.
• Modeling flow on a communication network will allow us to:
  – Describe normal and congested flow on large communication networks.
  – Predict changes in flow pattern due to changes in the spatial density and per-link traffic.
Communication Networks

- A communication network is a global system of interconnected networks, both big and small.
- Packet switching network because all data traffic is broken down into data chunks called packets.
- Everything traveling on a communication network is called a packet.
Discrete Communication Network

- Use graph theory to describe the connectivity of a network, where a graph is composed of nodes and links.
- Information travels along links connecting nodes.
- The graph is undirected; information travels in both directions.
- The nodes are routers in this model
- Routers will act as both host and switch computers.
- Host computers is where information enters (source), and exits (destination) a network.
Route Matrix

• The Route matrix depicts the global state of a network.
• Describes how to direct packets from source to destination.
• Each entry describes the next appropriate router a packet will take along its path.
• Routes are pre-determined by an optimal path algorithm.
\[ r_{i,d} = x \]

- \( i \) = Current position
- \( d \) = Destination
- \( x \) = Next position
Queue Dynamics

- Each router contains a queue of buffered packets.
- FIFO unlimited memory buffer
  - Packets enter queue in one of two ways
    - Packets flowed from another router along a connecting link.
    - Generated at a router according to some packet generator rate appropriate to the router.
- Upon being generated, a packet is given a destination by its originator.
- Once a packet reaches its destination it is delivered and exits immediately.
Flow equation (Arrivals Age zero)

\[ b(j, d, 0, \tau + 1) = \sum_{a=0}^{\infty} \sum_{R_i,d}^{N} \delta_{j,R_i,d} \beta(i, d, a, \tau) + \nu(j, d, \tau) \]

- \( b(j, d, a, \tau) \), number of buffered packets at node \( j \), with destination \( d \), age \( a \) at time \( t \).

- \( \beta(i, d, a, \tau) \) is the sending rate of packets being sent from router \( i \) to router \( j \) toward its destination \( d \) at time \( \tau \).

- \( \nu(j, d, \tau) \) is the number of new packets entering the network at node \( j \).
Evolution Equation

• Describes how packets are aging in the buffer.

\[ b(j, d, a + 1, \tau + 1) = b(j, d, a, \tau) - \beta(j, d, a, \tau) \]

\( \beta(j, d, a, \tau) \) determines the number of packets that are sent outward from node \( j \) with destination \( d \).
Discrete Conservation of Packet Equation

• The number of buffered messages at router $j$ with destination $d$ at time $t$

\[
n(j, d, \tau) = \sum_{a=0}^{\infty} b(j, d, a, \tau)\]

\[
n(j, d, \tau + 1) - n(j, d, \tau) = \sum_{a=0}^{N} b(j, d, a, \tau + 1) - \sum_{a=0}^{N} b(j, d, a, \tau)\]
Discrete Conservation of Packet Equations

\[ n(j, d, \tau + 1) - n(j, d, t) = -\sum_{a=0}^{\infty} \beta(j, d, a, \tau) + \sum_{a=0}^{\infty} \sum_{i=1}^{N} \delta_{i, R_i, d} \beta(i, d, a, \tau) \]

\[ + \nu(j, d, \tau) - \sum_{a=0}^{\infty} \beta(j, j, a, \tau) \]
Continuum Network Model

- To view this model as a flow model, we'll discuss the collection of routers as opposed to one. The collection of routers create a Voronoi Diagram.
Voronoi Diagram

• The spatial location of each router is of importance.

• Each router serves a particular coverage area of users who are sending and receiving packets in an area

• Associate each router with a physical spatial location

• Each router shares physical boundaries with another set of routers so that packets moving through the network will pass through physical boundaries around routers.
Voronoi Diagram

\[ x_j = \text{Location of router } j \]
\[ y = \text{Location of destination } d \]

Each \( x_j \) generates a Voronoi polygon \( V(x_j) \), a tessellation in the plane. The Voronoi polygon \( V(x_j) \) is the set of points closer to the point (router) \( x_j \), than to all other routers in the diagram. A collection of Voronoi polygons is called a Voronoi Diagram.
Continuum Description

Let $V_p$ be a Voronoi Diagram and $\partial V_p$ be its boundary. The density of packets for Voronoi polygon $V(x_j)$

$$
\rho(x_j, y, t) = \frac{n(j, d, \tau)}{|V(x_j)|}
$$

The number of packets buffered in the Vorono Diagram $V_p$

$$
\sum_{j \in P} n(j, d, \tau)
$$
Continuum Description Cont.

The evolution equation for the density of packets in a Voronoi Diagram

\[ \int_{V_P} \frac{\rho(x, y, t + \Delta t) - \rho(x, y, t)}{\Delta t} \, dV = \sum_{j \in P} \frac{n(j, d, \tau + 1) - n(j, d, \tau)}{\Delta t} \]

Take the limit as \( \Delta t \) goes to zero, first term becomes

\[ \int_{V_P} \frac{\partial \rho}{\partial t} \, dV \]
Flux

The flow vector from router $x_j$ to router $x_i$ is

$$\frac{x_i - x_j}{|x_i - x_j|} \beta(j, d, a, \tau)$$

The outflow through boundary element $\partial V_{l,m}$

$$\sum_{a=0}^{\infty} \sum_{j \in P} \sum_{i \notin P} \delta_{(j,i)(l,m)} \delta_{i,R_{j,d}} \frac{1}{\Delta t} \frac{x_i - x_j}{|x_i - x_j|} \beta(j, d, a, \tau) \cdot n_{l,m} = \Phi^O(x_{l,m}, y, t) \cdot n_{l,m} |\partial V_{l,m}|$$

Similar derivation for the incoming flow terms

$$\sum_{a=0}^{\infty} \sum_{j \in P} \sum_{i \notin P} \delta_{(j,i)(l,m)} \delta_{j,R_{i,d}} \frac{1}{\Delta t} \frac{x_i - x_j}{|x_i - x_j|} \beta(i, d, a, \tau) \cdot n_{l,m} = \Phi^I(x_{l,m}, y, t) \cdot n_{l,m} |\partial V_{l,m}|$$
Flux Cont.

The total flux of packets entering and exiting a boundary $\partial V_{l,m}$

$$\Phi(x_{l,m}, y, t) \cdot n_{l,m} | \partial V_{l,m} | = \Phi^O(x_{l,m}, y, t) \cdot n_{l,m} | \partial V_{l,m} | - \Phi^I(x_{l,m}, y, t) \cdot n_{l,m} | \partial V_{l,m} |$$

Total flux of packets in Voronoi Diagram $V_p$ with destination $y$

$$\sum_{l \in \partial P \atop m \notin \partial P} \Phi(x_{l,m}, y, t) \cdot n_{l,m} | \partial V_{l,m} |$$

Written in the continuum limit and using the divergence theorem the total flux becomes

$$\sum_{l \in \partial P \atop m \notin \partial P} \Phi(x_{l,m}, y, t) \cdot n_{l,m} | \partial V_{l,m} | \rightarrow \int_{\partial V_p} \Phi(x, y, t) \cdot n \ ds$$
Source and Sink

The source and sink in their continuum limit

\[ \sum_{j \in P} \frac{1}{\Delta t} \nu(j, d, \tau) \rightarrow \int_{V_P} \gamma(x, y, t) \, dv \]

\[ \sum_{a=0}^{\infty} \sum_{j \in P} \frac{1}{\Delta t} \beta(j, d, a, \tau) \rightarrow \int_{V_P} \sigma(x, t) \, dV \]
Continuity Equation

Putting the continuum limits together we have the conservation of packets equation in a Voronoi Diagram $V_p$

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \Phi(x, y, t) = \gamma(x, y, t) - \sigma(x, t)
\]
One Dimensional Network Flow Model

• One dimensional Network flow model with x=0 and a destination x=y.
• Analyze inner nodes
• Continuity equation in one dimensions

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} = 0
\]

\[
\Phi(x, y, t) = \begin{cases} 
\rho(x, y, t) & \text{if } \rho(x, y, t) < \Phi_{\text{max}}(x, y, t) \\
\Phi_{\text{max}}(x, y, t) & \text{if } \rho(x, y, t) \geq \Phi_{\text{max}}(x, y, t)
\end{cases}
\]
Non-Saturated Flow

Flow Movement in Time
End of Flow
Example: Flow with Saturation

Initial Condition
End of Flow
Interruptions

• Disturbance in the flow
  – Limited bandwidth, link capacity drops to a lower value
  – Router (grid point) is down for some time.

• Example
  – Source node has constant flux of packets below link capacity.
  – One of the grid points has limited bandwidth for a while.
  – One of the inner grid points has a source term.
Interruptions
End of Flow
Current Work

- Currently route matrix is static.
- Make route matrix dynamic at every node (changes because of link weight, upstream traffic, etc.)

\[ R_{j,d} = i \]  Before

\[ R_{j,d}(t) \neq R_{j,d}(t + 1) \]  Now
Current Work

• Also leads to probabilities in which nodes packets will take next.
• Probability of going to node $i$ to node $j$ with destination $d$.

$$p(j, d, t)$$
Questions!!!!!!!!!!!!