A Detailed Solution to the Hyperbolic Lines of Position Formulation of the Shooter Detection Problem

by David L. Doria

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14. ABSTRACT

In this report we detail a classical method of shooter detection, Hyperbolic Lines of Position, to determine the position of a gunshot when the muzzle blast is heard by a distributed microphone array. The core of this solution is computing the intersection of multiple hyperbolae. The traditional approach to intersecting two hyperbolae involves a change of coordinates to align one hyperbola with the Cartesian axes. However, when multiple hyperbolae must be considered simultaneously, we have found it more convenient to perform the intersection calculations in the original coordinate frame.

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1. Introduction

When a sound is heard by a single, omnidirectional microphone, it is impossible to tell from which direction or from how far away the sound originated. That is, if this sound was a gunshot, the shooter could be anywhere. This situation is illustrated in figure 1. However, when several microphones are present, we are sometimes able to estimate the location of the source of the sound. To simplify the problem, we make several assumptions. First, we assume that the observers have a direct line of sight to the shooter. That is, there are no occluding objects (building, trees, vehicles, etc.) between the observer positions and the shooter position. Second, we assume that the curvature of the earth is negligible (a safe assumption at ballistic threat ranges). Third, we assume that the air temperature is constant over both time and position so that the sound of the gunshot travels at a constant speed. Finally, we treat the situation as a two-dimensional (2-D) problem, that is, we assume the shooter and all of the observers lie on the same plane.

In section 2.1, we discuss the geometry of the situation at a conceptual level, and then in section 2.2 we introduce the corresponding mathematics and discuss the details of the solution.
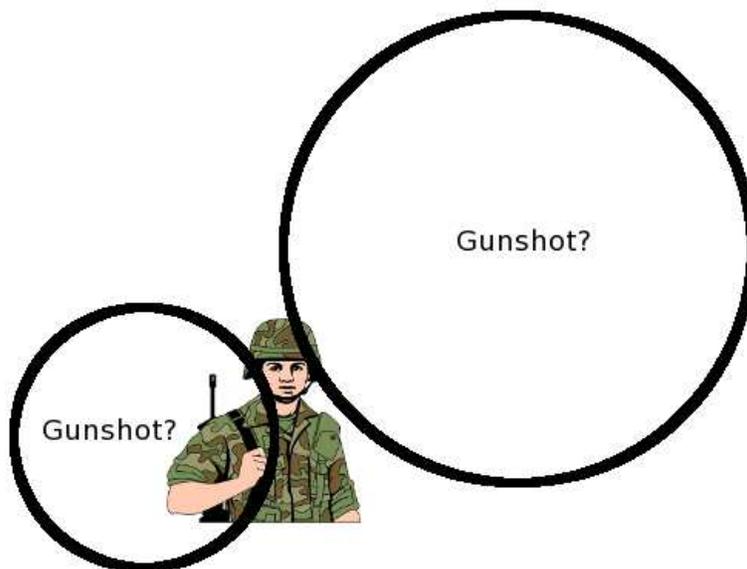
Figure 1. With only one microphone, it is not possible to determine
the distance or direction of the origin of a sound. The
black circles indicate the spherical propagation of the
sound from two possible sources of a gunshot.

2. Hyperbolic Lines of Position Solution

With two observers of the same sound, we can constrain the potential origin of the sound, S, to lie on a hyperbola. The geometry of the situation is shown in figure 2.
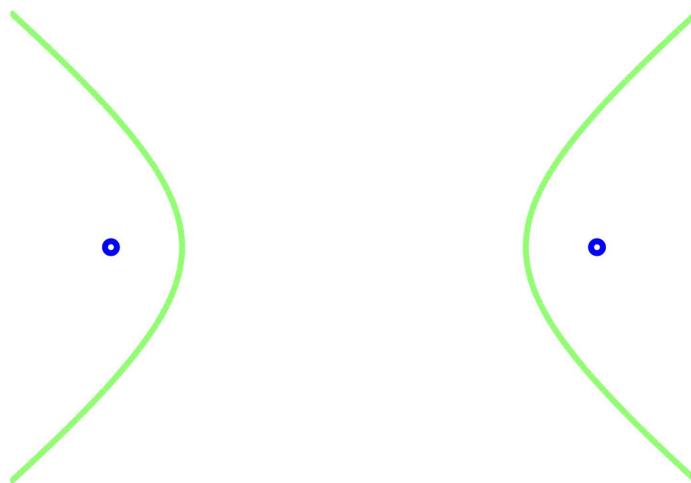
Figure 2. With two listeners (blue circles), we can only narrow down the possible shooter position to lie on a hyperbola (both branches of the hyperbola are shown as the green curve).

It is only with at least three observers that we can determine the actual position of the shooter. From each pair (there are $\binom{N}{2}$ pairs) of listeners, we can produce a hyperbola constraining the position of the shooter. In figure 3, we show the three pairs and corresponding hyperbolic constraints in a three-listener scenario.

Additionally, since we know the order in which the listeners received the sound (as explained in section 2.1), we can restrict the shooter position to one branch of each hyperbola, as shown in figure 4.

The only point that satisfies these three constraints simultaneously is the intersection of these hyperbolas, which is the position of the shooter, as shown in figure 5. We note that in an error-free system, the intersection of each pair of hyperbolas is exactly the same point. However, in a system with errors (in either receive time, clock synchronization, or global positioning system [GPS] localization), these points may not be coincident and an averaging procedure must be used to determine the shooter location, as shown in figure 6.
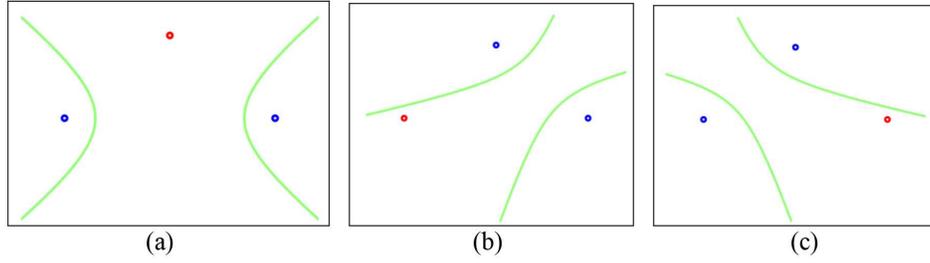
Figure 3. The hyperbolas constraining the shooter position generated by each pair of listeners. In each subfigure, the two listeners that generated the hyperbola are shown in blue.
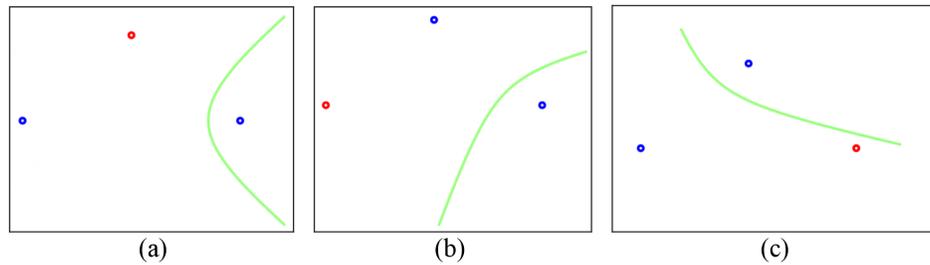
Figure 4. The branch of each hyperbola constraining the shooter position generated by each pair of two listeners. In each subfigure, the two listeners that generated the hyperbola are shown in blue.

2.1   Determining a Hyperbola From Two Observers

We now proceed to give a mathematical explanation of how to compute these hyperbolae and their intersections. We denote the ith observer position as O_i, and the time that the sound arrives at O_i by t_i. By computing the difference between the time arrival of the sound at the two receivers as $t_\Delta = |t_i - t_j|$, we can determine that the shooter must be at a position such that the difference in distance between the shooter and observer i and the shooter and observer j is $d = t_\Delta v$, where v is the speed of sound in air at the given temperature, typically 343.2 m/s (at 20 °C). The description of this situation is exactly the definition of a hyperbola—the locus of points whose difference in distance from a pair of points (the foci) is a constant. Here, the foci are the two observers, and the constant difference is d. We can write this as

$$\bigl||O_i - S| - |O_j - S|\bigr| = d. \tag{1}$$
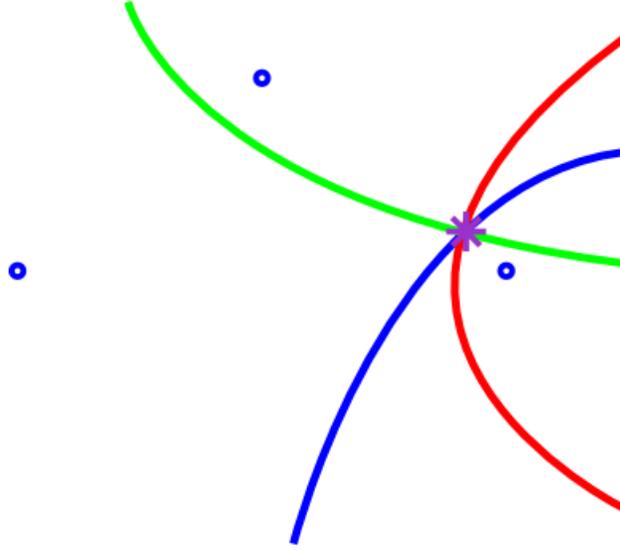
Figure 5. The intersection of the hyperbolas generated by each pair of listeners is the position of the shooter (shown as a purple asterisk).

(a)   (b)
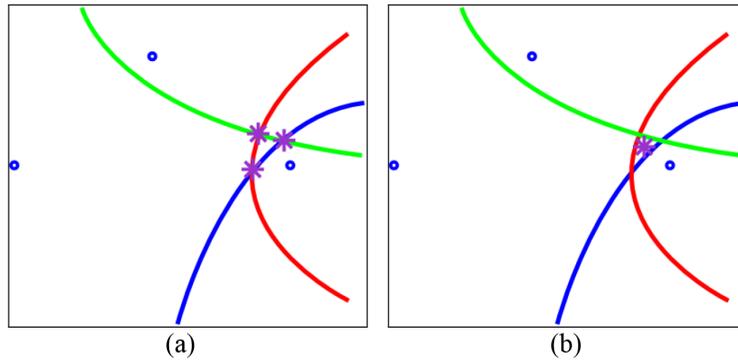
Figure 6. The branch of each hyperbola constraining the shooter position generated by each pair of two listeners. In each subfigure, the two listeners that generated the hyperbola are shown in blue.

The Euclidean distance between two points p_1 and p_2 is

$$\text{distance} = \sqrt{(p_{1x} - p_{2x})^2 + (p_{1y} - p_{2y})^2}. \tag{2}$$

Therefore, we can write our situation as

$$\left| \sqrt{(O_{ix} - S_x)^2 + (O_{iy} - S_y)^2} - \sqrt{(O_{jx} - S_x)^2 + (O_{jy} - S_y)^2} \right| = d. \tag{3}$$

4

If we order O_i and O_j such that

$$\sqrt{(O_{ix} - S_x)^2 + (O_{iy} - S_y)^2} > \sqrt{(O_{jx} - S_x)^2 + (O_{jy} - S_y)^2} \tag{4}$$

we can remove the absolute value to obtain

$$\sqrt{(O_{ix} - S_x)^2 + (O_{iy} - S_y)^2} - \sqrt{(O_{jx} - S_x)^2 + (O_{jy} - S_y)^2} = d. \tag{5}$$

There is tremendous manipulation necessary to write this equation in general quadratic form $(Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0)$. The steps are not all intuitive, so we detail the procedure and rationale step-by-step. First, add $\sqrt{(b_x - x)^2 + (b_y - y)^2}$ to both sides of the equation so that when we square both sides we will not eventually introduce quartic terms:

$$\sqrt{(a_x - x)^2 + (a_y - y)^2} = \sqrt{(b_x - x)^2 + (b_y - y)^2} + d. \tag{6}$$

Square both sides:

$$(a_x - x)^2 + (a_y - y)^2 = (b_x - x)^2 + (b_y - y)^2 + 2d\sqrt{(b_x - x)^2 + (b_y - y)^2} + d^2. \tag{7}$$

Expand:

$$a_x^2 - 2xa_x + x^2 + a_y^2 - 2a_yy + y^2 = b_x^2 - 2xb_x + x^2 + b_y^2 - 2b_yy + y^2 + 2d\sqrt{(b_x - x)^2 + (b_y - y)^2} + d^2. \tag{8}$$

Cancel x^2 and y^2 terms that occur on both sides of the equation:

$$a_x^2 - 2xa_x + a_y^2 - 2a_yy = b_x^2 - 2xb_x + b_y^2 - 2b_yy + 2d\sqrt{(b_x - x)^2 + (b_y - y)^2} + d^2. \tag{9}$$

Prepare to square both sides again by isolating the remaining square root:

$$a_x^2 - 2xa_x + a_y^2 - 2a_yy - b_x^2 + 2xb_x - b_y^2 + 2b_yy - d^2 = 2d\sqrt{(b_x - x)^2 + (b_y - y)^2}. \tag{10}$$

Group terms on the left side into the form $Jx + Ky + L$ to make squaring easier:

$$\left((-2a_x + 2b_x)x + (-2a_y + 2b_y)y + (a_x^2 + a_y^2 - b_x^2 - b_y^2 - d^2)\right) = 2d\sqrt{(b_x - x)^2 + (b_y - y)^2} \tag{11}$$

For notational convenience, assign $J = -2a_x + 2b_x$, $k = -2a_y + 2b_y$, $L = a_x^2 + a_y^2 - b_x^2 - b_y^2 - d^2$ so this can be written as

$$Jx + Ky + L = 2d\sqrt{(b_x - x)^2 + (b_y - y)^2} \qquad (12)$$

Square both sides:

$$(Jx + Ky + L)^2 = 4d^2\left((b_x - x)^2 + (b_y - y)^2\right) \qquad (13)$$

Expand:

$$J^2x^2 + JxKy + JxL + KyJx + K^2y^2 + KyL + LJx + LKy + L^2 = 4d^2\left(b_x^2 - 2xb_x + x^2 + b_y^2 - 2yb_y + y^2\right) \qquad (14)$$

Combine terms on the left and expand the expression on the right:

$$J^2x^2 + K^2y^2 + 2JxKy + 2JxL + 2KyL + L^2 = 4d^2b_x^2 - 8d^2xb_x + 4d^2x^2 + 4d^2b_y^2 - 8d^2yb_y + 4d^2y^2 \qquad (15)$$

Combine all terms $(x^2, xy, y^2, x, y, constant)$:

$$(J^2 - 4d^2)x^2 + (2JK)xy + (K^2 - 4d^2)y^2 + (2JL + 8d^2b_x)x + (2KL + 8d^2b_y)y + (L^2 - 4d^2b_x^2 - 4d^2b_y^2) = 0 \qquad (16)$$

Substitute the defined values of J, K, and L:

$$\left((-2a_x + 2b_x)^2 - 4d^2\right)x^2 + \qquad (17)$$

$$(2(-2a_x + 2b_x)(-2a_y + 2b_y))\,xy + \qquad (18)$$

$$\left((-2a_y + 2b_y)^2 - 4d^2\right)y^2 + \qquad (19)$$

$$\left(2(-2a_x + 2b_x)(a_x^2 + a_y^2 - b_x^2 - b_y^2 - d^2) + 8d^2b_x\right)x + \qquad (20)$$

$$\left(2(-2a_y + 2b_y)(a_x^2 + a_y^2 - b_x^2 - b_y^2 - d^2) + 8d^2b_y\right)y + \qquad (21)$$

$$\left((a_x^2 + a_y^2 - b_x^2 - b_y^2 - d^2)^2 - 4d^2b_x^2 - 4d^2b_y^2\right) = 0 \qquad (22)$$

From here, we can see that in the general bivariate quadratic polynomial form of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, the coefficients are

$$A = (-2a_x + 2b_x)^2 - 4d^2 \tag{23}$$

$$B = 2(-2a_x + 2b_x)(-2a_y + 2b_y) \tag{24}$$

$$C = (-2a_y + 2b_y)^2 - 4d^2 \tag{25}$$

$$D = 2(-2a_x + 2b_x)(a_x^2 + a_y^2 - b_x^2 - b_y^2 - d^2) + 8d^2 b_x \tag{26}$$

$$E = 2(-2a_y + 2b_y)(a_x^2 + a_y^2 - b_x^2 - b_y^2 - d^2) + 8d^2 b_y \tag{27}$$

$$F = (a_x^2 + a_y^2 - b_x^2 - b_y^2 - d^2)^2 - 4d^2 b_x^2 - 4d^2 b_y^2 \tag{28}$$

2.2 Determining The Shooter Location

For every pair of observers (there are $\binom{N_O}{2}$, where N_O is the number of observers) we can derive a hyperbola as we did in section 2.1. For now we focus on the hyperbola Q_1 formed from the information at O_0 and O_1, and the hyperbola Q_2 formed from the information at O_1 and O_2. (Note: Additional hyperbolae can be formed using other pairs of observers in an identical fashion. In the next section, we show how to combine the information from these multiple pairs of hyperbolas.) In the noise-free case, the point of intersection of Q_1 and Q_2 is the position of the shooter. We discuss the case with noise in section 2.3.

We can write the equation of the hyperbola as

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = \begin{pmatrix} x & y & 1 \end{pmatrix} Q_i \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \tag{29}$$

where

$$Q_i = \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix} \tag{30}$$

Generally, there are four intersection points of two hyperbolas Q_1 and Q_2. It is difficult to directly find these intersection points from the general forms of the hyperbolas. However, there is an entire family of conics that pass through the same intersection points, namely, any linear combination of the two equations

$$\mu Q_1 + \lambda Q_2. \tag{31}$$

In this set of conics (more details of this family of conics can be found in reference 1, there are up to three degenerate conics, which are simply a pair of straight lines rather than the typical curves of a hyperbola. We take advantage of this by finding these degenerate equations and using them to perform the intersection computations, as intersecting a linear function with a general quadratic function is straightforward. We can identify a degenerate conic because the determinant of its coefficient matrix is zero. Therefore, to find the degenerate conics of the family we are interested in, we set

$$det\left(\mu Q_1 + \lambda Q_2\right) = 0. \tag{32}$$

Without loss of generality, we can set $\mu = 1$ so our problem becomes solving

$$det\left(Q_1 + \lambda Q_2\right) = 0 \tag{33}$$

for λ. The determinant expression is a cubic function in λ. That is,

$$\det(Q_1 + \lambda Q_2) = \sum_{k=0}^{3} a_k \lambda^k \tag{34}$$

If we write the matrix

$$Q_1 + \lambda Q_2 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{35}$$

we can write

$$det(Q_1 + \lambda Q_2) = aei + bfg + cdh - ceg - bdi - afh \tag{36}$$

Each of $a, b, c, d, e, f, g, h, i$ have the form $a_1 + \lambda a_2$. For example,
$aei = (a_1 + \lambda a_2)(e_1 + \lambda e_2)(i_1 + \lambda i_2)$. Expanding this, we get

$$aei = a_1 e_1 i_1 + a_1 e_1 i_2 \lambda + a_1 e_2 i_1 \lambda + a_1 e_2 i_2 \lambda^2 + \tag{37}$$
$$a_2 e_1 i_1 \lambda + a_2 e_1 i_2 \lambda^2 + a_2 e_2 i_1 \lambda^2 + a_2 e_2 i_2 \lambda^3 \tag{38}$$
$$= a_1 e_1 i_1 + (a_1 e_1 i_2 + a_1 e_2 i_1 + a_2 e_1 i_1)\lambda + \tag{39}$$
$$(a_1 e_2 i_2 + a_2 e_1 i_2 + a_2 e_2 i_1)\lambda^2 + a_2 e_2 i_2 \lambda^3 \tag{40}$$

Following this pattern, we can compute the coefficients a_k of the cubic equation resulting from equation 34 as follows. We use parenthesis to indicate the terms that come from each term in the

determinant expansion.

$$a_0 = (a_1 e_1 i_1) + (b_1 f_1 g_1) + (c_1 d_1 h_1) - (c_1 e_1 g_1) - (b_1 d_1 i_1) - (a_1 f_1 h_1) \tag{41}$$

$$a_1 = (a_1 e_1 i_2 + a_1 e_2 i_1 + a_2 e_1 i_1) + (b_1 f_1 g_2 + b_1 f_2 g_1 + b_2 f_1 g_1) + \tag{42}$$

$$(c_1 d_1 h_2 + c_1 d_2 h_1 + c_2 d_1 h_1) - (c_1 e_1 g_2 + c_1 e_2 g_1 + c_2 e_1 g_1) - \tag{43}$$

$$(b_1 d_1 i_2 + b_1 d_2 i_1 + b_2 d_1 i_1) - (a_1 f_1 h_2 + a_1 f_2 h_1 + a_2 f_1 h_1) \tag{44}$$

$$a_2 = (a_1 e_2 i_2 + a_2 e_1 i_2 + a_2 e_2 i_1) + (b_1 f_2 g_2 + b_2 f_1 g_2 + b_2 f_2 g_1) + \tag{45}$$

$$(c_1 d_2 h_2 + c_2 d_1 h_2 + c_2 d_2 h_1) - (c_1 e_2 g_2 + c_2 e_1 g_2 + c_2 e_2 g_1) - \tag{46}$$

$$(b_1 d_2 i_2 + b_2 d_1 i_2 + b_2 d_2 i_1) - (a_1 f_2 h_2 + a_2 f_1 h_2 + a_2 f_2 h_1) \tag{47}$$

$$a_3 = (a_2 e_2 i_2) + (b_2 f_2 g_2) + (c_2 d_2 h_2) - (c_2 e_2 g_2) - (b_2 d_2 i_2) - (a_2 f_2 h_2) \tag{48}$$

We then compute the roots (r_1, r_2, and r_3) of this cubic equation using the procedure described in section 5.6 of reference 2.

Selecting the root that is most distant from the other two (to avoid ill-conditioning [1]), we can now write the coefficient matrix for the degenerate conic as

$$Q_D = Q_1 + r_1 Q_2 \tag{49}$$

The two lines that constitute the degenerate conic can be written as $a_i x + b_i y + c_i = 0$, where $i = 1, 2$. Denote the vector of these coefficients as

$$u_i = \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}. \tag{50}$$

The entries of u_i can be found using the eigenvectors of the non-zero eigenvalues of Q_D. Specifically,

$$u_1 = \sqrt{|\lambda_1|} v_1 + \sqrt{|\lambda_2|} v_2 \tag{51}$$

and

$$u_2 = \sqrt{|\lambda_1|} v_1 - \sqrt{|\lambda_2|} v_2. \tag{52}$$

Once we have u_1 and u_2, we can intersect these lines with both of the original conics independently to obtain up to four points of intersection (potentially two intersections with each

conic). Solving $ax + by + c = 0$ for x, we obtain

$$x = \frac{-by - c}{a} \tag{53}$$

Substituting this into the general form of the conic equation, we obtain

$$A\left(\frac{-by - c}{a}\right)^2 + B\left(\frac{-by - c}{a}\right)y + Cy^2 + D\left(\frac{-by - c}{a}\right) + Ey + F = 0 \tag{54}$$

Collecting y^2, y, and constant terms, we can write this equation as

$$\left(\frac{Ab^2}{a^2} - \frac{bB}{a} + c\right)y^2 + \left(\frac{2Abc}{a^2} - \frac{Bc}{a} - \frac{Db}{a} + E\right)y + \left(\frac{Ac^2}{a^2} - \frac{Dc}{a} + F\right) = 0. \tag{55}$$

Substituting this into the expression for y from the conic, we have

$$y = \frac{-(B\frac{-by-c}{a} + E) \pm \sqrt{(B\frac{-by-c}{a} + E)^2 - 4(C)(A(\frac{-by-c}{a})^2 + D\frac{-by-c}{a} + F)}}{2C}. \tag{56}$$

These points (there are up to two because of the $\pm$ in the expression for y) (x, y) are intersections of the hyperbolae. The (up to) other two points can be found by intersecting the degenerate conic with the other hyperbola.

2.3 Selecting the Intersection that is the Shooter Position

There are many intersections of the hyperbolas generated by all pairs of listeners. We can discard half of these intersections by checking their relative distances to the listeners, as they are not consistent with the receive times that we observed (the receive time differences are negative). For example, if $t_1 > t_2$ but $|detected - S_1| < |detected - S_2|$, this detected position is inconsistent.

Furthermore, we can discard additional intersections if they are not intersections of all of the pairs of hyperbolas. Since there is typically error introduced in the simulation, as well as an always-present numerical computation error, this discarding is done using an epsilon test. An intersection is removed if its distance to all hyperbolas is not less than ϵ.

Computing the distance from a point to a hyperbola is not a trivial task. This distance is defined as the distance from the point to the closest orthogonal projection on the hyperbola. To find these orthogonal projections, we again use the machinery of degenerate conics. We refer the reader to reference 3 for a complete explanation of this procedure. Denote a point not on the hyperbola as

a. In our case, a is each of the intersections, successively. We compute a matrix B as

$$B = \begin{pmatrix} -C_{12} & C_{11} & -a_y C_{11} + a_x C_{12} \\ -C_{22} & C_{12} & -a_y C_{12} + a_x C_{22} \\ -C_{23} & C_{13} & -a_y C_{13} + a_x C_{23} \end{pmatrix} \tag{57}$$

and from it

$$D = \frac{B + B^T}{2}. \tag{58}$$

Now we find the intersections of the conics given by matrices C and D as we have done in the previous section. Out of these intersections (at most four), we compute the closest one to a, which we call p. This is the closest orthogonal projection of a onto the hyperbola. We then simply compute the distance from a to p and this is the distance from a to the hyperbola.

Unfortunately, with only three listeners, we are not guaranteed to arrive at a unique solution. That is, there may be more than one point that is an intersection of all hyperbolas and is physically consistent with the situation. However, with at least four listeners, where no three, are collinear, this ambiguity is removed.

Finally, assuming we are in an unambiguous case, we have identified a collection of points that are all supposed to be coincident (in the error-free case). We can simply compute the spatial average of these points to arrive at a final estimation of the shooter position.

3. Conclusions

In this report we have detailed a solution to the hyperbolic lines of position problem that does not require any coordinate transformations. This allows us to compute the location of a shooter given that the shot is heard by at least three observers. Though we have made assumptions about the environment and restricted the discussion to the two dimensional case, the techniques presented provide a direction for work on these problems in the future.

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