Extremely Fast Numerical Integration of Ocean Surface Wave Dynamics

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LONG-TERM GOALS

1. Application of the ***inverse scattering transform*** **(IST)** to the ***time series analysis of laboratory and oceanic wave data***. The approach may be viewed as a **generalization of linear Fourier analysis** and is loosely referred to as "Nonlinear Fourier Analysis or **Generalized Fourier Transform**" (GFT).

2. A major focus has been the application of IST to the study of ***"rogue, freak or giant" ocean waves***. The emphasis has been on the study of the ***physical mechanisms leading to the generation of rogue waves in random sea states***.

3. A third long term goal is the ***development of fast algorithms for numerically integrating the space/time dynamics of deep-water wave trains***. While IST is limited to the numerical/analytic integration of the so-called "soliton equations," I have discovered how the GFT can be used to solved higher order equations for which study of the dynamics have previously been limited to numerical (as opposed to analytical) approaches. I discuss herein how the GFT can be used ***both*** for the ***analytical study and extremely fast numerical integration*** of the ***extended nonlinear Schroedinger equation for fully three dimensional wave motion***. I discuss a number of considerations for the development of ***extremely fast GFT algorithms***, which I refer to as the ***xGFT***.

OBJECTIVES

1. The objective of the present research program is the development of ***fast numerical multidimensional Fourier*** techniques applied to a wide range of wave modeling and analysis problems.

2. Important progress made in the past year has been the ***development of new algorithms for multidimensional Fourier analysis***. These algorithms are the key to future ***hyperfast applications of the method***.

APPROACH

The nonlinear modeling project that I discuss herein is quite new and, remarkably, has an overall structure as seen in Fig. 1. Basically this means that the *first step* consists of a *preprocessor* part which determines the *nonlinear time evolution* of the linear Fourier spectrum, i.e. it computes the linear Fourier spectrum at some large number (say $N \sim 1000$) of desired time steps (one normally takes a sufficient number of time steps to make a movie for studying the nonlinear wave behavior). The

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second step is to take the inverse FFT of the time varying Fourier transforms. Additionally the mathematics (soliton theory, Riemann theta functions and algebraic geometry) is not that ordinarily studied by physical oceanographers and this may be an impediment that could test the patience of those who might want to apply the method (see for example the book by [Baker, 1897]). As we shall see however, the speed gains are quite dramatic, i.e. two or three orders of magnitude faster than the direct numerical integration of nonlinear wave equations by the FFT. The preprocessor is unique for each ***integrable*** *wave equation* as discussed in detail below. Furthermore, for each ***nonintegrable*** *equation* a sub-processor must be added as shown in the blue box of Fig. 1.

We first consider the Kadomtsev-Petviashvili (KP) equation

$$\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} + \gamma\ \partial_x^{-1}\eta_{yy} = 0 \tag{1}$$

Here $\eta(x,y,t)$ is the wave amplitude as a function of the two spatial variables, x, y and time, *t*. The KP equation (1) is a natural two-space-dimension extension of the KdV equation. The periodic KP solutions include *directional spreading* in the wave field:

$$\eta(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\theta(x,y,t\,|\mathbf{B},\mathbf{K},\mathbf{\Omega},\mathbf{\Phi}) \tag{2}$$

Here the generalized Fourier series has the form given in (4) below, where the phase has the *two dimensional* expression:

$$\mathbf{X}(x,y,t) = \mathbf{K}x + \mathbf{L}y - \mathbf{\Omega}t - \mathbf{\Phi} \tag{3}$$

Here the spatial term $\mathbf{K}x$ has been joined by the lateral spatial term $\mathbf{L}y$, which allows wave spreading to be taken into account. The KP equation is the first nonlinear step toward a directional sea state; KP is however limited to small directional spreading. Improving the directional spreading characteristics of the KP equation requires the addition of physically important corrections to the equation: I give numerical examples below.

The ***generalized Fourier series***, $\theta(x,t\,|\tilde{\mathbf{B}},\mathbf{k},\boldsymbol{\omega},\boldsymbol{\phi})$, is given by the expression

$$\theta(x,y,t\,|\,\mathbf{B},\boldsymbol{\phi}) = \sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\cdots\sum_{m_N=-\infty}^{\infty} e^{i\sum_{n=1}^{N} m_n X_n - \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} m_m m_n B_{mn}} \tag{4}$$

where $X_n = k_n x + l_n y - \omega_n t + \phi_n$. The function $\theta(x,y,t\,|\mathbf{B},\boldsymbol{\phi})$ is also called a ***Riemann theta function*** or ***multidimensional Fourier series***. Here **B** is the Riemann matrix (the "spectrum" of the solution), the vectors **k**, **l** constitutes the usual wave numbers, the vector $\boldsymbol{\omega}$ contains the frequencies and the vector $\boldsymbol{\phi}$ forms the phases. The ***inverse problem*** associated with (2), (3) allows one to determine the Riemann matrix, wave numbers, frequencies and phases appropriate for solving the *Cauchy problem* for KdV: Given the spatial variation of the solutions at $t=0$, $\eta(x,y,0)$, compute the solution for all time, $\eta(x,y,t)$. This is a necessary step for the numerical simulations presented heein. The ***solitons, Stokes waves and sine waves lie on the diagonal of the Riemann matrix***; the ***off-diagonal terms contain the nonlinear interactions***.

Why is the above approach useful for hyperfast numerical simulations? Because the Riemann theta function can be programmed as a ***fast theta function transform*** (FTFT), just as the Fourier transform can be programmed as a ***fast Fourier transform*** (FFT). Therefore the numerical integration of KdV (1) can be evaluated at specific time points, necessary only for graphical purposes or for extracting useful properties of the sea surface. This contrasts to the FFT that must be evaluated at very small time steps when used for the numerical integration of a nonlinear partial differential equation.

The unidirectional, deep-water case is governed by the ***nonlinear Schroedinger equation***:

$$i(\psi_t + C_g \psi_x) + \mu \psi_{xx} + \nu |\psi|^2 \psi = 0 \tag{5}$$

where the coefficients are computed in the usual way in terms of the carrier wave number and frequency. The spectral solution to this equation is given by

$$\psi(x,t) = a \frac{\theta(x,t \mid \mathbf{B}, \delta^-)}{\theta(x,t \mid \mathbf{B}, \delta^+)} e^{2ia^2 t} \tag{6}$$

Thus the solution is the ratio of two multi-dimensional Fourier series with different phases, $\delta^{\pm}$.

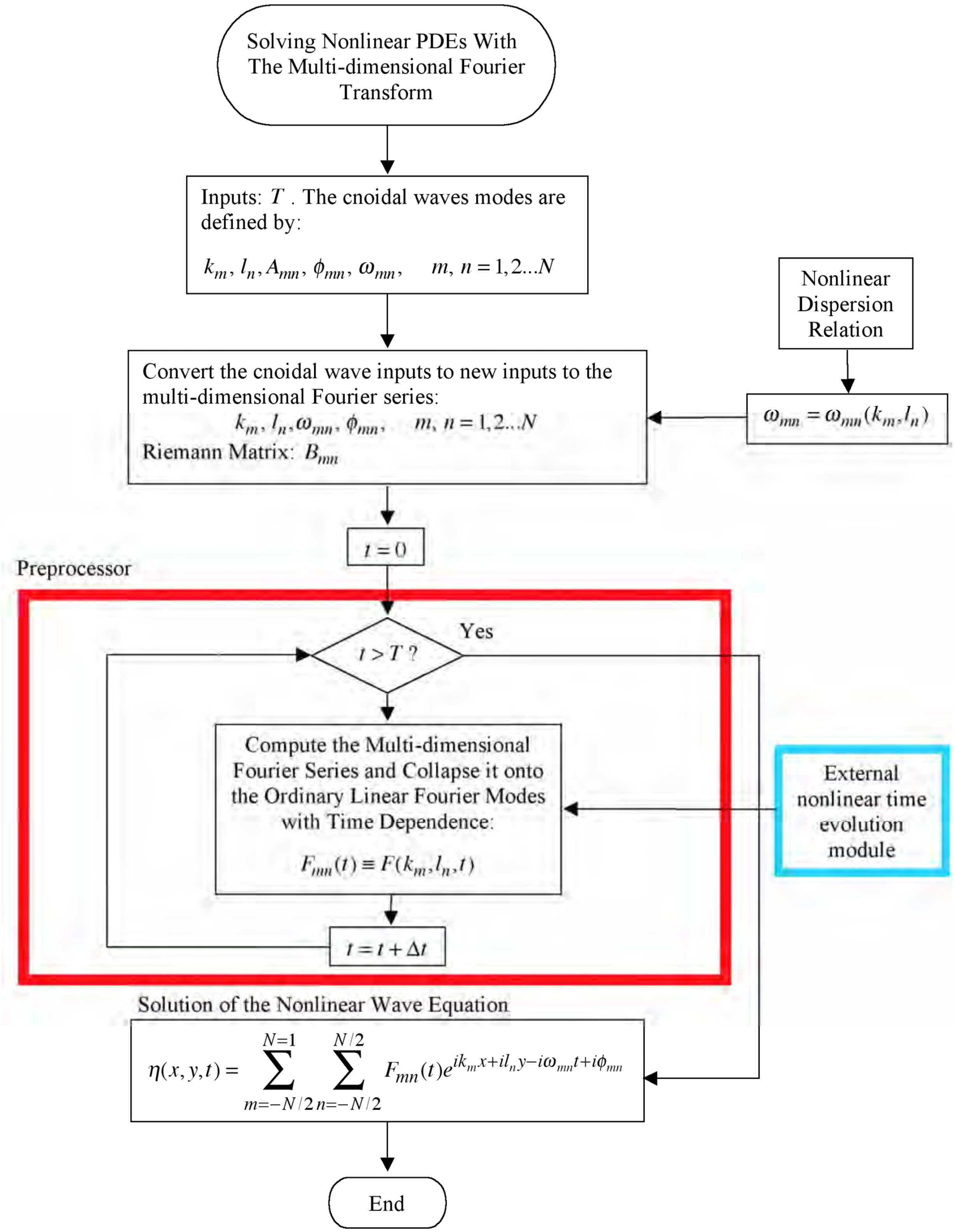

Fig. 1 Simple flow chart of the proposed numerical models for nonlinear partial differential equations with a well-defined dispersion relation.

For directionally spread waves in deep-water the nonlinear Schroedinger equation has the form:

$$i(\psi_t + C_g \psi_x) + \mu\psi_{xx} - 2\mu\psi_{yy} + \nu|\psi|^2 \psi = 0 \tag{7}$$

An important result, fundamental in the present research program, is that the ***above relatively simple cases form building blocks*** for the solution to more complex equations such as the ***Boussinesq equations*** (see Fig. 1) and the full ***Euler equations*** (see below). Another important aspect, to be addressed in the coming year, is that ***variable shaped boundaries***, ***variable bathymetry*** and ***wind*** and ***dissipation forces*** can be easily added to the above equations and to the Euler equations.

WORK COMPLETED

Up to this point I have not given any of the details of the nonlinear preprocessor in the new approach for obtaining numerical solutions to nonlinear wave equations. I will now do so, but without many of the mathematical details. ***First*** note that one begins with the Riemann theta functions and makes a transformation. One of the simplest types of transformation is given by (2). ***Second*** I will use the following result, i.e. that the theta function can be reduced to an ordinary linear Fourier series with time varying coefficients:

$$\theta(x,y,t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \theta_{mn}(t) e^{ik_m x + il_n y + i\phi_{mn}} \tag{8}$$

Here the time varying coefficients $\theta_{mn}(t)$ can be written analytically in terms of the Riemann matrix and phases. Note that the above expression is rather simple, i.e. it is just an ordinary linear Fourier series, which in numerical applications has $\sim N^2$ terms. The theta function itself is quite different in behavior for it has an exponential number of terms $\sim 10^N$, which can be a huge number. Consider for example a case where $N = 30$, i.e. 30 cnoidal waves in the spectrum. Then the number of terms in the theta function is far greater then the number of grains of sand on the earth, the number of galaxies in the universe, greater than Avogardro's number and greater then the number of seconds since the big bang! Clearly the preprocessor has got to be pretty efficient to reduce 10^N ($\sim 10^{30}$) to $\sim N^2$ (900)! I call the mathematical process to reduce the theta function onto the ordinary linear Fourier modes a "collapse" in analogy with quantum physics when one refers to the collapse of a wave function during the process of a physical measurement. Indeed we say that *we collapse the theta function onto the linear Fourier modes*. This process requires some knowledge of the algebraic geometry of theta functions [Baker, 1897] that I do not go into here.

I now consider some interesting numerical results. To be concrete I have chosen the KP equation, i.e. directionally spread waves in shallow water that are distributed over relatively small angles. An example of a directionally spread wave train is given in Fig. 2. The significant wave height is 1.5 m and the water depth is 8 m. There are 20 cnoidal waves in the spectrum with a maximum modulus of 0.84 (these are strongly nonlinear waves, but they are still less nonlinear than solitons which have a modulus of 1). Because the moduli of the spectral components is relatively large the waves are really Stokes-like in their shape, a result of the nonlinearities in the KP equation.

I now turn to the modified or ***extended KP equation***, which I refer to as xKP. This equation is found from the KP equation by extending it an additional two orders of approximation beyond the KP equation and for many ***purposes xKP behaves essentially like the Boussinesq equation***. To add this extended feature to the model in Fig. 1 we must of course add a ***time varying Riemann matrix and phases***. To this end the model of Fig. 1 must be modified by adding the dynamics of the Riemann matrix and phases to the blue box.

The numerical results for the Boussinesq model are shown in Figs. 3 and 4. One of the major new results which have come from these simulations is the appearance of a ***new kind of rogue wave*** which occurs only in shallow water wave dynamics and is not related to the Benjamin-Feir instability. Shallow water rogue waves of the type observed in the simulations occur due to strong nonlinear interactions between two or more cnoidal waves and in their simplest form are referred to as a *Mach stem*.

The other major result is the small amount of computer time required for these simulations. The waves are significantly nonlinear with the ratio of significant wave height to water depth given by $H_s / h = 1.5m / 8m = 0.1875$ and the maximum wave height to depth is $H_{\max} / h = 3.4m / 8m = 0.425$. The integration domain is 500 m by 500 m and has 128 x 128 spatial bins. A total of 500 time values were computed to make the film (I'll send you a .avi file if you request it). The total cpu time was 1 min 10 sec. This compares to a split-step FFT code which took 4.5 hours to compute the same problem. Thus the preprocessor multi-dimensional Fourier model which I have developed is about 230 times faster than the split-step FFT code. For a number of reasons which I will not mention here, I anticipate another order of magnitude improvement in computer time in the preprocessor part of the model over the next year or so.

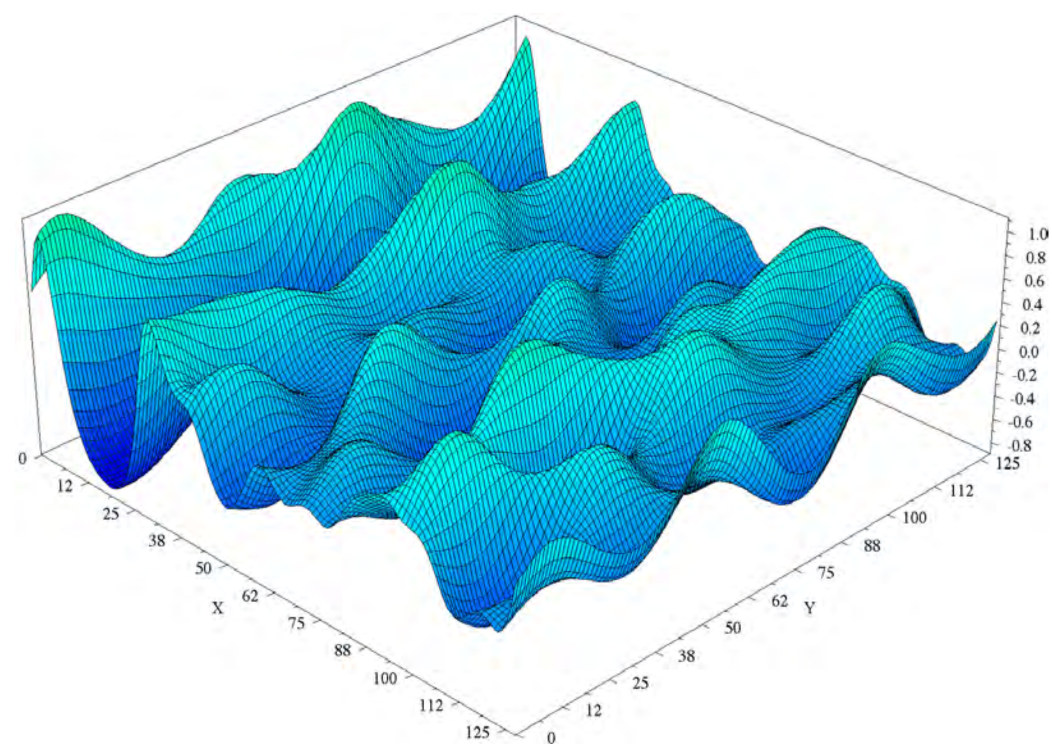

Fig. 2 Initial conditions for simulation of KP equation. The significant wave height is 1.5 m and the water depth is 8 m.

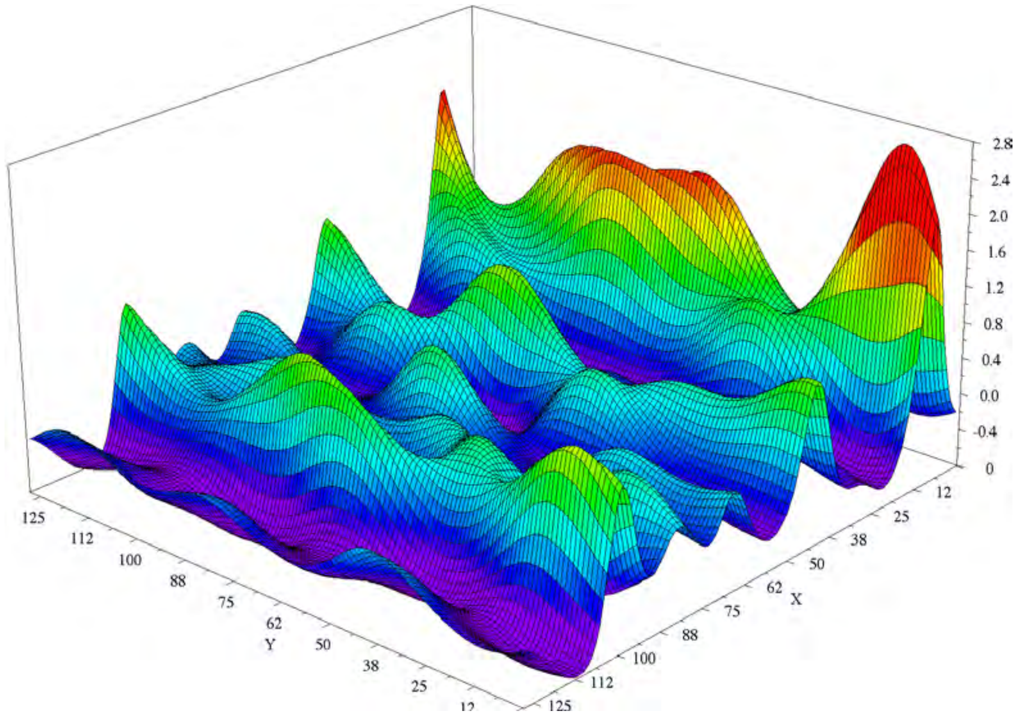

Fig. 3 A large rogue wave of height 3.4 m in the Boussinesq (xKP) simulation. The significant wave height is 1.5 m. The rogue waves have maxima shown in red.

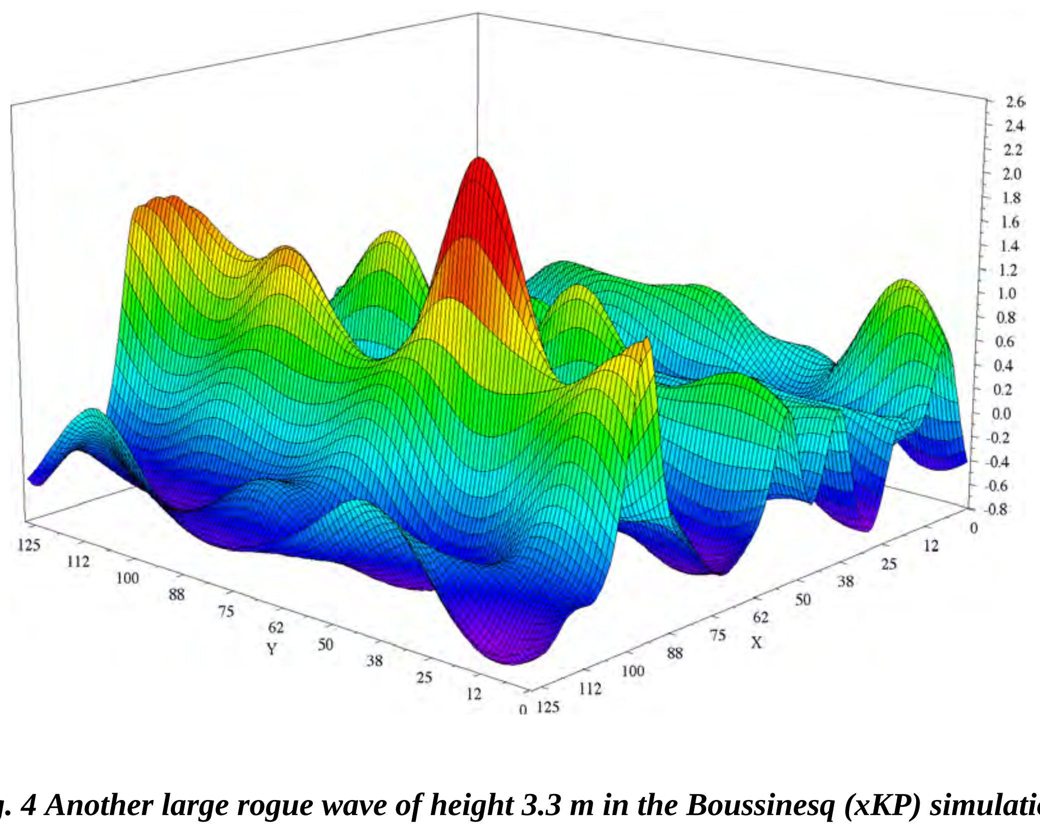

Fig. 4 Another large rogue wave of height 3.3 m in the Boussinesq (xKP) simulation. The significant wave height is 1.5 m.

RESULTS

The numerical model of the Euler equations uses particular transformations similar to (2) plus additional other mathematical machinery to simulate the wave motion simultaneously in terms of both the surface elevation and the velocity potential. There are three theta functions that have to be accounted for and collapsed onto the linear Fourier modes. One of the most important considerations is that the theta function now has a three dimensional form:

$$\theta(x,y,z,t) = \sum_{\mathbf{m}\in ¢} q_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{k}_x x + i\mathbf{m}\cdot\mathbf{k}_y y + i\mathbf{m}\cdot\mathbf{k}_z z - i\mathbf{m}\cdot\boldsymbol{\omega}\, t + i\mathbf{m}\cdot\boldsymbol{\phi}}$$

where the "nomes" are given by:

$$q_{\mathbf{m}} = e^{-\frac{1}{2}\mathbf{m}\cdot\mathbf{B}\mathbf{m}}$$

But in the formulation we need to evaluate this function at the free surface, $z = \eta(x,y,t)$. This gives for the ***theta function at the free surface***:

$$\theta(x,y,\eta,t) = \sum_{\mathbf{m}\in ¢} q_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{k}_x x + i\mathbf{m}\cdot\mathbf{k}_y y + i\mathbf{m}\cdot\mathbf{k}_z \eta(x,y,t) - i\mathbf{m}\cdot\boldsymbol{\omega}\, t + i\mathbf{m}\cdot\boldsymbol{\phi}}$$

It is tempting to write:

$$e^{i\mathbf{m}\cdot\mathbf{k}_z \eta(x,y,t)} = 1 + i\mathbf{m}\cdot\mathbf{k}_z \eta(x,y,t) - \frac{1}{2}(\mathbf{m}\cdot\mathbf{k}_z)^2 \eta^2(x,y,t) - \frac{i}{6}(\mathbf{m}\cdot\mathbf{k}_z)^3 \eta^3(x,y,t) + ...$$

which should converge provided that the wave steepness has the property

$$\mathbf{m}\cdot\mathbf{k}_z \eta(x,y,t) << 1$$

This can of course be substituted into the theta function to obtain

$$\theta(x,y,z,t) = \sum_{\mathbf{m}\in\mathbb{C}} q_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{k}_x x + i\mathbf{m}\cdot\mathbf{k}_y y + i\mathbf{m}\cdot\mathbf{k}_z \eta(x,y,t) - i\mathbf{m}\cdot\boldsymbol{\omega} t + i\mathbf{m}\cdot\boldsymbol{\phi}} +$$

$$+ i\eta(x,y,t) \sum_{\mathbf{m}\in\mathbb{C}} \mathbf{m}\cdot\mathbf{k}_z q_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{k}_x x + i\mathbf{m}\cdot\mathbf{k}_y y + i\mathbf{m}\cdot\mathbf{k}_z \eta(x,y,t) - i\mathbf{m}\cdot\boldsymbol{\omega} t + i\mathbf{m}\cdot\boldsymbol{\phi}}$$

$$- \frac{1}{2}\eta^2(x,y,t) \sum_{\mathbf{m}\in\mathbb{C}} (\mathbf{m}\cdot\mathbf{k}_z)^2 q_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{k}_x x + i\mathbf{m}\cdot\mathbf{k}_y y + i\mathbf{m}\cdot\mathbf{k}_z \eta(x,y,t) - i\mathbf{m}\cdot\boldsymbol{\omega} t + i\mathbf{m}\cdot\boldsymbol{\phi}}$$

$$- \frac{i}{6}\eta^3(x,y,t) \sum_{\mathbf{m}\in\mathbb{C}} (\mathbf{m}\cdot\mathbf{k}_z)^3 q_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{k}_x x + i\mathbf{m}\cdot\mathbf{k}_y y + i\mathbf{m}\cdot\mathbf{k}_z \eta(x,y,t) - i\mathbf{m}\cdot\boldsymbol{\omega} t + i\mathbf{m}\cdot\boldsymbol{\phi}} + \ldots$$

These are the fundamental types of multi-dimensional Fourier series which appear in the formulation.

A flowchart of the Euler equation simulation is shown in Fig. 5. In Fig. 6 we show the appearance of a rogue wave from one of the simulations in deep water.

IMPACT/APPLICATION

The impact of this research will occur in general for the nonlinear Fourier analysis of shallow and deep-water wave trains. Specific results will provide for a deeper understanding of nonlinear wave dynamics.

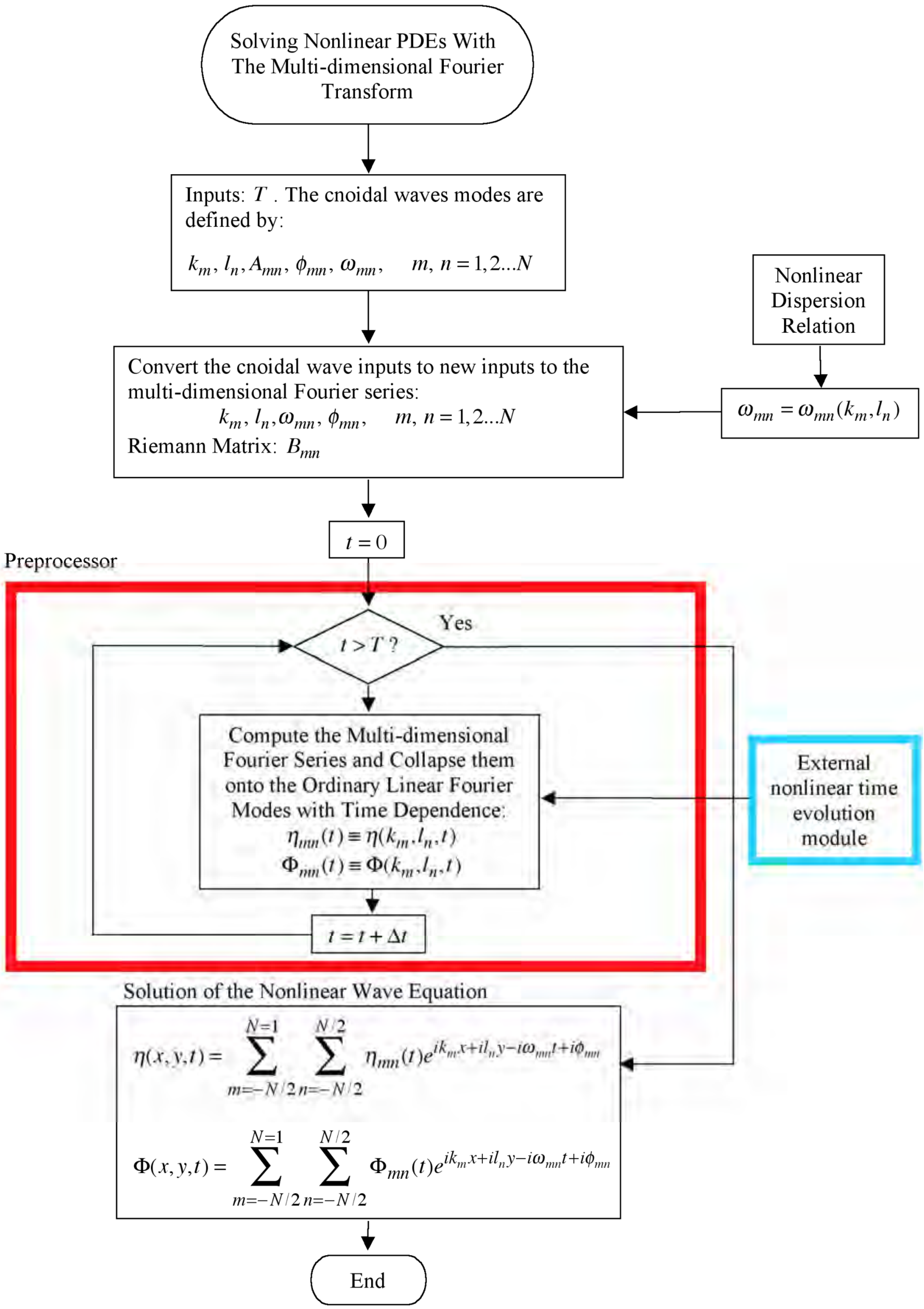

Fig. 5 Simple flow chart of the numerical model for the Euler Equations.

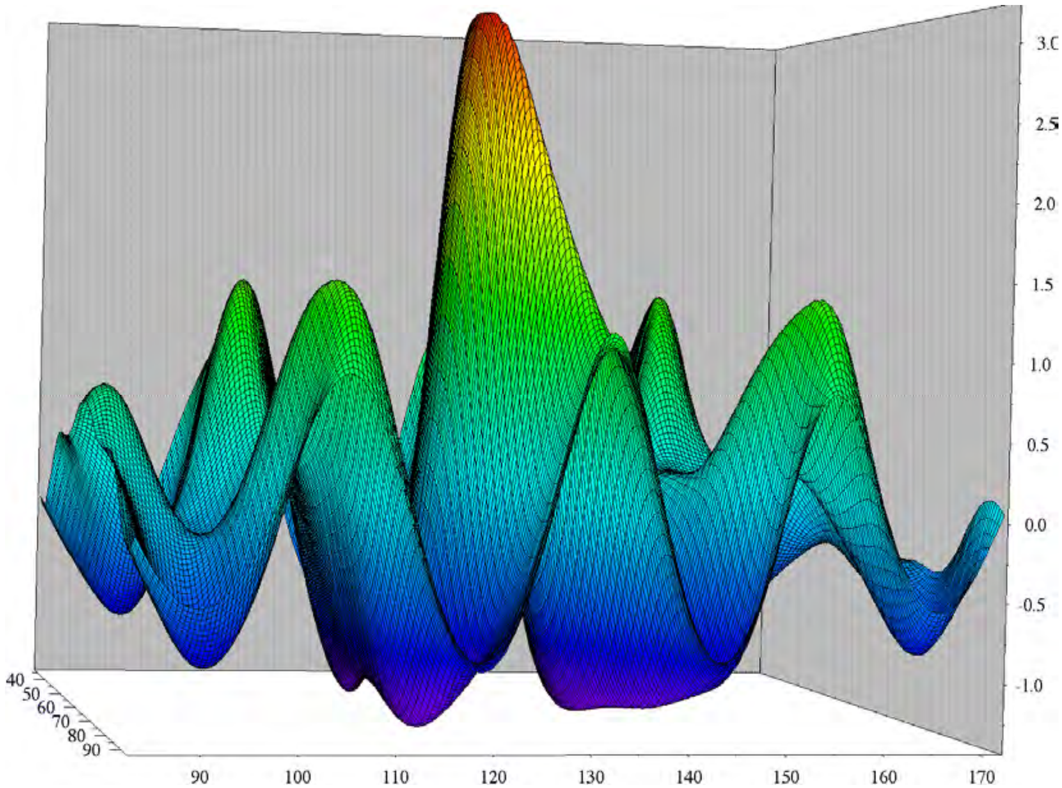

Fig. 6 A single-peak rogue wave with deep holes on the leading and trailing edges. The vertical scale should be multiplied by 10. The height of the wave is 46 m and has a wave length of 320 m.

TRANSITIONS

Transitions expected are related to the use of the codes as guidance to ships and unmanned, unteathered vehicles as the kind of environment in which one resides and for the real time sampling of the environment, including the acoustic environment.

RELATED PROJECTS

An intimate relationship between our results and other projects exists because the sea surface provides a major forcing input to many kinds of offshore activities, including the dynamics of floating and drilling vessels, barges, risers and tethered vehicles. The present work leads to a nonlinear representation of the sea surface forcing and vessel response for shallow water waves.

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