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A Convective Coordinate Approach to Continuum Mechanics with Application to Electrodynamics

by Daniel S. Weile, David A. Hopkins, George A. Gazonas, and Brian M. Powers

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Weapons and Materials Research Directorate, ARL

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14. ABSTRACT

The description of nonmechanical physical processes is simplified by using convective coordinates in place of reference coordinates for the mathematical characterization of physical boundaries. While the reference coordinates of standard continuum mechanics remain fixed in a single (possibly fictitious) configuration, convected coordinates move along with the particles of the simulation describing an evolving geometry as time progresses. This allows nonmechanical physical processes (such as electromagnetic fluxes) to be tracked easily even when introducing discontinuities in the continuum. Moreover, because the convective description refers to the evolving physics in real time, it may be used to evaluate the fidelity of physical models in different reference frames. To demonstrate the utility of these coordinates and investigate an important application, Maxwell's equations of electrodynamics are formulated in the convective system.

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1. Introduction

The mechanics of a continuum are commonly described by relating two sets of coordinates: a set of *reference coordinates* that serves to label the particles in an arbitrary (perhaps initial or even fictitious) configuration, and a set of *spatial coordinates* that fixes locations in space (that is, in the laboratory) (*1, 2*). The importance of the reference coordinates in continuum mechanics is that they serve to *name* each individual particle so it can be tracked in the spatial coordinates. This is the sense in which the reference coordinates may even be fictitious—since they merely serve to label each particle; reference coordinates themselves need not have any particular geometric meaning so long as they fulfill this requirement. In this report, we seek to clarify the relationship between the two standard coordinate sets and a third set of coordinates, generally called *convective* coordinates which are related in subtle ways to both the spatial and reference coordinates. Convective coordinates are particularly important with respect to nonmechanical physics occurring against the background of a deforming body, and are almost essential within the framework of the theory of relativity, as that theory does not recognize the universality of simultaneity.

More specifically, in this report, we clarify the relationships between these three coordinate systems, and use the convective coordinate system to formulate the classical theory of electromagnetism in the presence of material deformation. This reformulation is useful in numerical work because the physically required continuity of electromagnetic field descriptions depends intimately on the geometry of the boundary, which is generally simple to describe only in reference or convective coordinates. By reformulating the Maxwell equations in convective coordinates, we also demonstrate that the standard vector formulation of them is not covariant in classical physics (*3*). In other words, we demonstrate that a proper formulation of Maxwell's equations, implying the invariance of physical law for all observers, cannot be achieved within the confines of classical physics. This difficulty is a primary source of the conflicting formulation of the Maxwell equations in continuum mechanics literature (*3–6*).

2. Background and Curvilinear Coordinates

To understand the subtle differences in the various coordinate systems presented in this work, definitions need to be made clearly and with a modicum of rigor. Moreover, the formulations presented in the following concentrate on the use of general curvilinear coordinates, a subject likely unfamiliar to many readers. Therefore, in the interest of precision and clarity, we describe curvilinear tensor theory in the next two sections. This section concentrates on the algebra of curvilinear systems, their metric structure, and the vector bases used to define field quantities. The next section describes the formulation of differential operators for the differentiation of field quantities defined in such spaces.

2.1 The Background Cartesian System

Before defining the physical coordinate systems at the heart of this report, we first describe an absolute, mathematical coordinate system disconnected from any material continuum and fixed firmly as a background description of space. This background space is assumed Cartesian and fixed for all time, and is denoted by $\mathfrak{s}$. The space $\mathfrak{s}$ is constructed given a point O (the *origin*) and three orthonormal vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. An arbitrary point P has *coordinates* (x^1, x^2, x^3) if the vector from O to P is given by the *position vector*

$$\mathbf{x} = x^i \mathbf{u}_i. \tag{1}$$

Superscripts are used here for indexing for reasons that will become clear later. In any case, here we use the Einstein convention that an index repeated in a superscript and a subscript is to be summed over its range.

In addition to serving as the basis for geometrical description, $\mathfrak{s}$ can be used to describe physical scalar and vector fields. For instance, if we have a (static) pressure associated with each point in space, we may write

$$p = p\big(x^i\big)\,, \tag{2}$$

where the x^i refer to points of space as defined by equation 1. Similarly, the electric field at the point with coordinates x^i can be written in the form

$$\mathbf{e} = e^i\big(x^j\big)\,\mathbf{u}_i, \tag{3}$$

where now both the coordinates and the basis vectors of underlying background description are employed.

2.2 Curvilinear Coordinates and Metrics

Our initial description is based on Cartesian coordinates because such a description allows us to define coordinates through equation 1 and thus directly connect coordinates with position vectors.

On the other hand, any set of three numbers $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ suffices to locate points in space if every such triplet is mapped uniquely to a point in space; that is, if the mapping

$$\tilde{x}^{\tilde{\imath}} = \tilde{\xi}^{\tilde{\imath}}\left(x^i\right) \qquad \tilde{\imath}, i \in \{1,2,3\} \tag{4}$$

has the properties that each coordinate set $\tilde{x}^{\tilde{\imath}}$ is the image of a exactly one coordinate set x^i. (From this point on, the range on indices, which for this work is always $\{1,2,3\}$, is assumed and suppressed.) That is, we are assuming the mapping $\tilde{\xi}^{\tilde{\imath}}$ is *bijective*, i.e., there is a unique set of functions ξ^i such that

$$x^i = \xi^i\left(\tilde{x}^{\tilde{\imath}}\right). \tag{5}$$

These new $\tilde{x}^{\tilde{\imath}}$ coordinates function as well for point location as did the old x^i coordinates because of this one-to-one mapping: Given the $\tilde{x}^{\tilde{\imath}}$, the x^i are determined uniquely, and associated with the point described by equation 1. We therefore refer to this new curvilinear coordinate systems as $\tilde{\mathfrak{s}}$, and label all vectors in $\tilde{\mathfrak{s}}$ just like their counterparts in $\mathfrak{s}$, but with a tilde. The only weakness of using $\tilde{\mathfrak{s}}$ to describe space is that nothing as simple as equation 1 relates the point location to the coordinates.

Given this failure of equation 1 and the unclear relationship between the $\tilde{x}^{\tilde{\imath}}$ and the $\mathbf{u}_i$, it makes sense to define new basis vectors to describe vector fields in $\tilde{\mathfrak{s}}$ (if not position vectors). Inspired by the original Cartesian basis vectors, we might define our new basis to be in the direction of increase of a single coordinate, computed holding the other coordinates constant. Specifically, define

$$\tilde{\mathbf{u}}_{\tilde{\imath}} \doteq \frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}} \mathbf{u}_i. \tag{6}$$

These vectors are not necessarily of unit length or orthogonal, but they do point in the direction desired. The basis is illustrated at a fixed point in figure 1a. These vectors, and any other tensorial quantities indexed by a subscript and therefore subject to variable changes in the manner of equation 6, are called *covariant*.

The Cartesian basis vectors are not only well suited to their coordinate system because they point in the direction of increase of a coordinate holding all other coordinates constant; they are also orthogonal to the constant coordinate value surfaces constructed by holding a single variable constant and letting all other coordinates vary. The $\tilde{\mathbf{u}}_{\tilde{\imath}}$ do not have this property. In view of this, we can define a new set of basis vectors

$$\tilde{\mathbf{u}}^{\tilde{\imath}} \doteq \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} \mathbf{u}^i, \tag{7}$$

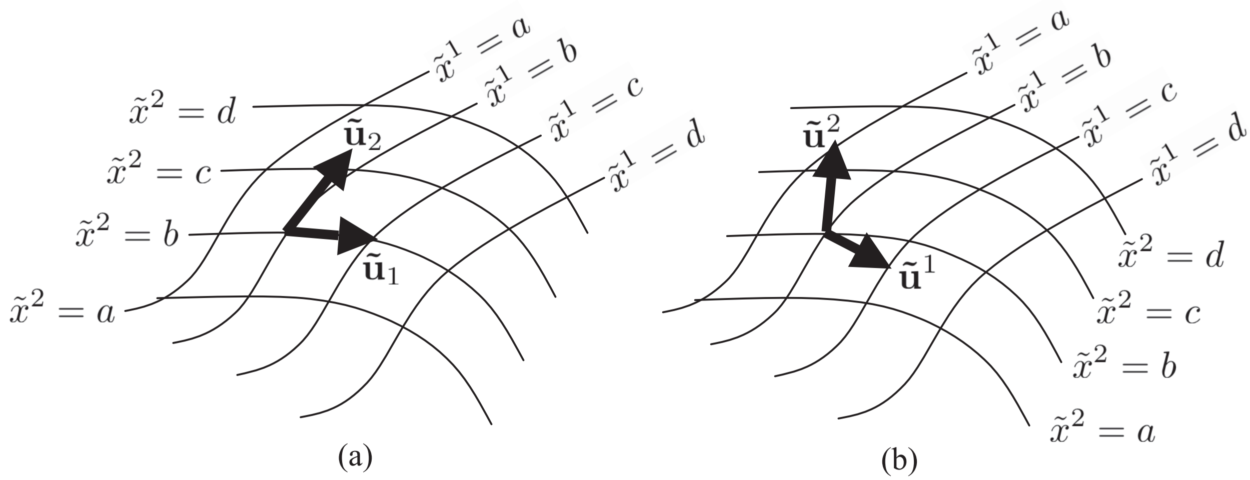

Figure 1. (a) Covariant basis vectors $\tilde{\mathbf{u}}_{\tilde{\imath}}$, and (b) contravariant basis vectors $\tilde{\mathbf{u}}^{\tilde{\imath}}$ at the point (b, b). The grid is drawn assuming $a < b < c < d$.

where for notational convenience we define

$$\mathbf{u}^i \doteq \delta^{ij}\mathbf{u}_j, \tag{8}$$

and as usual δ^{ij} (with indices as superscripts or subscripts or both) is the Kronecker delta. These vectors are orthogonal to the constant coordinate value surfaces (i.e., $\tilde{\mathbf{u}}^{\tilde{\imath}}$ is orthogonal to the $\tilde{x}^{\tilde{\imath}} =$ constant surface) since they are the gradients of the curvilinear coordinate values with respect to the underlying Cartesian system. Indeed, the *biorthogonality* of the covariant and contravariant components follows from the chain rule:

$$\tilde{\mathbf{u}}^{\tilde{\imath}} \cdot \tilde{\mathbf{u}}_{\tilde{\jmath}} = \left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}\mathbf{u}^i\right) \cdot \left(\frac{\partial \tilde{x}^{\tilde{\jmath}}}{\partial x^j}\mathbf{u}_j\right) = \delta^i_j \frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}\frac{\partial \tilde{x}^{\tilde{\jmath}}}{\partial x^j} = \frac{\partial x^j}{\partial \tilde{x}^{\tilde{\imath}}}\frac{\partial \tilde{x}^{\tilde{\jmath}}}{\partial x^j} = \delta^{\tilde{\imath}}_{\tilde{\jmath}} \tag{9}$$

They are illustrated in figure 1b, and are called *contravariant* because they change in a manner opposite that of the covariant basis as expressed by equation 7.

Finally, we note that we can define higher-order tensors that obey similar rules, and that they can be covariant or contravariant in any of their indices. In this work, no tensors of higher than second order appears. As an example, a twice-contravariant set of second-order tensor coefficients a^{ij} transforms according to the formula

$$\tilde{a}^{\tilde{\imath}\tilde{\jmath}} = \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i}\frac{\partial \tilde{x}^{\tilde{\jmath}}}{\partial x^j}a^{ij}, \tag{10}$$

precisely because the vector *outer product* $\mathbf{u}_i\mathbf{u}_j$ is twice-covariant:

$$\tilde{\mathbf{u}}_{\tilde{\imath}}\tilde{\mathbf{u}}_{\tilde{\jmath}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}\frac{\partial x^j}{\partial \tilde{x}^{\tilde{\jmath}}}\mathbf{u}_i\mathbf{u}_j. \tag{11}$$

Indeed, these two equations taken together ensure the product of such quantities is *invariant*, since

$$\begin{aligned}\tilde{a}^{\tilde{\imath}\tilde{\jmath}}\tilde{\mathbf{u}}_{\tilde{\imath}}\tilde{\mathbf{u}}_{\tilde{\jmath}} &= \left(\frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^m}\frac{\partial \tilde{x}^{\tilde{\jmath}}}{\partial x^m}a^{mn}\right)\left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}\frac{\partial x^j}{\partial \tilde{x}^{\tilde{\jmath}}}\mathbf{u}_i\mathbf{u}_j\right)\\ &= \delta^i_m\delta^j_n a^{mn}\mathbf{u}_i\mathbf{u}_j\\ &= a^{ij}\mathbf{u}_i\mathbf{u}_j.\end{aligned} \tag{12}$$

This invariance under the change of coordinates is a hallmark of a correctly formulated physical theory since it ensures that the meaning of physical quantities is independent of their mathematical description. Similarly, a mixed second-order tensor a^i_j changes coordinates in the by now expected fashion

$$\tilde{a}^{\tilde{\imath}}_{\tilde{\jmath}} = \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i}\frac{\partial x^j}{\partial \tilde{x}^{\tilde{\jmath}}}a^i_j, \tag{13}$$

because the product $\tilde{\mathbf{u}}_{\tilde{\imath}}\tilde{\mathbf{u}}^{\tilde{\jmath}}$ changes in the opposite fashion.

Because of these notational observations, it is easy to compute the inner product of two vectors expanded with respect to the two different bases. If we write

$$\mathbf{a} = \tilde{a}^{\tilde{\imath}}\tilde{\mathbf{u}}_{\tilde{\imath}} = \tilde{a}_{\tilde{\imath}}\tilde{\mathbf{u}}^{\tilde{\imath}}, \tag{14}$$

and

$$\mathbf{b} = \tilde{b}^{\tilde{\imath}}\tilde{\mathbf{u}}_{\tilde{\imath}} = \tilde{b}_{\tilde{\imath}}\tilde{\mathbf{u}}^{\tilde{\imath}}, \tag{15}$$

then

$$\mathbf{a}\cdot\mathbf{b} = \tilde{a}^{\tilde{\imath}}\tilde{b}_{\tilde{\imath}} = \tilde{a}_{\tilde{\imath}}\tilde{b}^{\tilde{\imath}}. \tag{16}$$

(As an aside, note that the boldface vector notation does not distinguish between covariant and contravariant coordinate systems. This is because vectors themselves, as independent, physically meaningful entities, are by definition independent of coordinate system. Basis vectors, which seem to be an exception to this rule, are in fact not exceptions: For them, the use (or nonuse) of a tilde indicates for which space they form a covariant or contravariant basis, rather than having anything to do with their expansion in any system.)

To ease the computation of scalar products between pairs of covariant or contravariant vectors, a metric tensor is introduced. Defining the metric tensor

$$\tilde{g}_{\tilde{\imath}\tilde{\jmath}} \doteq \tilde{\mathbf{u}}_{\tilde{\imath}} \cdot \tilde{\mathbf{u}}_{\tilde{\jmath}}, \tag{17}$$

the inner product of $\mathbf{a}$ and $\mathbf{b}$ can be computed from contravariant components alone:

$$\mathbf{a} \cdot \mathbf{b} = \tilde{g}_{ij} a^{\tilde{\imath}} b^{\tilde{\jmath}}. \tag{18}$$

In a similar manner, the twice contravariant metric tensor is defined:

$$\tilde{g}^{ij} \doteq \tilde{\mathbf{u}}^{\tilde{\imath}} \cdot \tilde{\mathbf{u}}^{\tilde{\jmath}}. \tag{19}$$

These two tensors are trivially symmetric and are inverses of one another by the chain rule:

$$\tilde{g}_{\tilde{\imath}\tilde{\jmath}} \tilde{g}^{\tilde{\jmath}\tilde{k}} = \delta_{\tilde{\imath}}^{\tilde{k}}. \tag{20}$$

The introduction of the metric tensor creates the possibility of working entirely with vector coefficients, and assuming and suppressing the basis vectors that are always defined by equations 6 and 7. (Indeed, some books (*7*) never even mention these vectors, assuming them superfluous.) For this approach to be useful, the metric tensor should be computable from the functions ξ and $\tilde{\xi}$, and indeed it is. The standard covariant metric tensor may be computed from

$$\tilde{g}_{\tilde{\imath}\tilde{\jmath}} = \tilde{\mathbf{u}}_{\tilde{\imath}} \cdot \tilde{\mathbf{u}}_{\tilde{\jmath}} = \left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}} \mathbf{u}_i \right) \cdot \left(\frac{\partial x^j}{\partial \tilde{x}^{\tilde{\jmath}}} \mathbf{u}_j \right) = \delta_{ij} \frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}} \frac{\partial x^j}{\partial \tilde{x}^{\tilde{\jmath}}} \tag{21}$$

recalling in the final step that $\mathfrak{s}$ is orthonormal so $\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$. This computation has the added benefit of demonstrating that $\tilde{g}_{\tilde{\imath}\tilde{\jmath}}$ is a tensor, since the metric tensor in $\mathfrak{s}$ is trivially given by $g_{ij} \equiv \delta_{ij}$. In precisely the same way, the inverse of the metric tensor $\tilde{g}^{\tilde{\imath}\tilde{\jmath}}$ can be directly computed according to the formula

$$\tilde{g}^{\tilde{\imath}\tilde{\jmath}} = \delta^{ij} \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} \frac{\partial \tilde{x}^{\tilde{\jmath}}}{\partial x^j}, \tag{22}$$

and is a twice contravariant tensor.

Finally, computing the dot product of the vector $\mathbf{a}$ defined above with each of the basis vectors in turn demonstrates that the covariant and contravariant components of a vector are related through the metric tensor:

$$\tilde{a}_{\tilde{\imath}} = \tilde{g}_{\tilde{\imath}\tilde{\jmath}} \tilde{a}^{\tilde{\jmath}}, \tag{23}$$

and

$$\tilde{a}^{\tilde{\imath}} = \tilde{g}^{\tilde{\imath}\tilde{\jmath}}\tilde{a}_{\tilde{\jmath}}. \tag{24}$$

These operations are known as the *lowering* and *raising* of an index, respectively.

2.3 Determinants and Cross Products

In addition to the computation of dot products (and thereby lengths), work in physics requires the computation of cross products and determinants. These computations depend intimately upon the definition of the Levi-Civita system

$$\epsilon_{ijk} \doteq \begin{cases} 0 & \text{if } i=j, j=k, \text{ or } i=k, \\ 1 & \text{if } ijk \text{ is an even permutation of 123}, \\ -1 & \text{if } ijk \text{ is an odd permutation of 123}. \end{cases} \tag{25}$$

This definition shows clearly that all completely antisymmetric third-order systems are scalar multiples of the Levi-Civita system, a fact we will need later. We can also define the contravariant Levi-Civita system in the obvious way, specifically

$$\epsilon^{ijk} = \delta^{ir}\delta^{js}\delta^{kt}\epsilon_{rst} \tag{26}$$

Determinants and Jacobians can be easily computed in terms of this tensor. The Jacobian of the mapping from the x^i to the $\tilde{x}^{\tilde{\imath}}$ is given by the formula

$$\det\left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}\right) \doteq \epsilon_{ijk}\frac{\partial x^i}{\partial \tilde{x}^1}\frac{\partial x^j}{\partial \tilde{x}^2}\frac{\partial x^k}{\partial \tilde{x}^3}. \tag{27}$$

This observation can be used to clarify the tensorial nature of the Levi-Civita system. Specifically, we can consider the system

$$\Delta_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \doteq \epsilon_{ijk}\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}\frac{\partial x^j}{\partial \tilde{x}^{\tilde{\jmath}}}\frac{\partial x^k}{\partial \tilde{x}^{\tilde{k}}}. \tag{28}$$

This system is completely antisymmetric because, for instance,

$$\Delta_{\tilde{\jmath}\tilde{\imath}\tilde{k}} = \epsilon_{ijk}\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\jmath}}}\frac{\partial x^j}{\partial \tilde{x}^{\tilde{\imath}}}\frac{\partial x^k}{\partial \tilde{x}^{\tilde{k}}} = -\epsilon_{jik}\frac{\partial x^j}{\partial \tilde{x}^{\tilde{\imath}}}\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\jmath}}}\frac{\partial x^k}{\partial \tilde{x}^{\tilde{k}}} = -\Delta_{\tilde{\imath}\tilde{\jmath}\tilde{k}}. \tag{29}$$

Therefore, because all antisymmetric systems of third order are uniquely determined up to a multiplicative constant, we find that

$$\epsilon_{ijk}\frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\frac{\partial x^j}{\partial \tilde{x}^{\tilde{j}}}\frac{\partial x^k}{\partial \tilde{x}^{\tilde{k}}} = \det\left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\right)\epsilon_{\tilde{i}\tilde{j}\tilde{k}}, \tag{30}$$

and, by the same token,

$$\epsilon^{ijk}\frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\frac{\partial \tilde{x}^{\tilde{j}}}{\partial x^j}\frac{\partial \tilde{x}^{\tilde{k}}}{\partial x^k} = \det\left(\frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\right)\epsilon^{\tilde{i}\tilde{j}\tilde{k}}. \tag{31}$$

These two equations imply that the Levi-Civita system is not a tensor in the usual sense, but in a new sense in which the usual tensorial transformation is accompanied by multiplication with a power of the Jacobian determinant. Such tensors are called *relative tensors*, or, if the power of the determinant is ± 1, *tensor densities* (*7*). (Note that no tilde is ever put on ϵ since it has the same values in all systems and so needs no such distinction.) In this sense, we may refer to it as the “Levi-Civita tensor.”

Because the determinant of the variable transformation tensor occurs so frequently in the following, we can simplify the remaining exposition with better notation. If we let the determinant of a twice-covariant tensor be represented by its unsubscripted symbol, we can write

$$\begin{aligned}\tilde{g} \doteq \det\left(\tilde{g}_{\tilde{i}\tilde{j}}\right) &= \det\left[\left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\mathbf{u}_i\right)\cdot\left(\frac{\partial x^j}{\partial \tilde{x}^{\tilde{j}}}\mathbf{u}_j\right)\right] \\ &= \det\left[\delta_{ij}\left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\right)\left(\frac{\partial x^j}{\partial \tilde{x}^{\tilde{j}}}\right)\right] \\ &= \left[\det\left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\right)\right]^2,\end{aligned} \tag{32}$$

or, more simply,

$$\sqrt{\tilde{g}} = \det\left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\right). \tag{33}$$

Note again that this definition pertains only to the determinants of twice-covariant tensors, and in particular to the twice-covariant metric tensor.

This clean expression for the Jacobian can be used to simplify the expression of cross products in the curvilinear system. If the original Cartesian space is right-handed, as we shall always assume, the cross product $\mathbf{c} = c^i\mathbf{u}_i = c_i\mathbf{u}^i$ of the vectors $\mathbf{a}$ and $\mathbf{b}$ of equations 14 and 15 has contravariant components given by

$$c^i = \epsilon^{ijk}a_j b_k \tag{34}$$

and covariant components given by

$$c_i = \epsilon_{ijk}a^j b^k. \tag{35}$$

Applying all the change of basis rules described above to these formulas, we find that the coefficients of the cross product are given in the curvilinear system by

$$\tilde{c}^{\tilde{\imath}} = \frac{1}{\sqrt{\tilde{g}}}\epsilon^{\tilde{\imath}\tilde{\jmath}\tilde{k}}\tilde{a}_{\tilde{\jmath}}\tilde{b}_{\tilde{k}}, \tag{36}$$

and

$$\tilde{c}_{\tilde{\imath}} = \sqrt{\tilde{g}}\,\epsilon_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\tilde{a}^{\tilde{\jmath}}\tilde{b}^{\tilde{k}}. \tag{37}$$

3. Differential Operators in Curvilinear Spaces

Having completed the description of the algebra underlying the description of fields in curvilinear coordinate systems in the last section, we turn to the differentiation of such fields in this section. The main idea is that tensors should be differentiated in a physically meaningful way; that is, that the results of such differentiations should be tensors. A so-called covariant derivative ensures the physicality of results. It is defined immediately below.

3.1 The Covariant Derivative

The last section described the algebra involved in changing coordinates between a fixed Cartesian system $\mathfrak{s}$ and a curvilinear system $\tilde{\mathfrak{s}}$. In this section, we explore the analytical properties of coordinate system changes. In particular, given a vector $\mathbf{a}$, we wish to determine how its derivatives with respect to space can be computed such that their meaning is not bound to any particular coordinate system.

To this end, consider the change undergone by a vector $\mathbf{a}$ over a differential distance $\mathrm{d}\mathbf{x}$. In the $\tilde{\mathfrak{s}}$ system, the change in $\mathbf{a}$ can be expressed in terms of a dot product as

$$\mathrm{d}\mathbf{a} = \frac{\partial \mathbf{a}}{\partial \tilde{x}^{\tilde{\imath}}}\mathrm{d}\tilde{x}^{\tilde{\imath}}. \tag{38}$$

To actually apply this formula, a more specific formula for the components of the partial derivative indicated above is needed. To this end, we define the *covariant derivatives* of the components as

$$\frac{\mathrm{D}\tilde{a}_{\tilde{\jmath}}}{\mathrm{D}\tilde{x}^{\tilde{\imath}}} \doteq \frac{\partial \mathbf{a}}{\partial \tilde{x}^{\tilde{\imath}}} \cdot \tilde{\mathbf{u}}_{\tilde{\jmath}}, \tag{39}$$

and

$$\frac{\mathrm{D}\tilde{a}^{\tilde{\jmath}}}{\mathrm{D}\tilde{x}^{\tilde{\imath}}} \doteq \frac{\partial \mathbf{a}}{\partial \tilde{x}^{\tilde{\imath}}} \cdot \tilde{\mathbf{u}}^{\tilde{\jmath}}. \tag{40}$$

The word “covariant” used here in concert with the derivative definition is not meant to indicate the nature of an index (though covariant differentiation does result in the addition of a covariant index to a tensor). Rather, the word implies that the result is tensorial; that is, it is covariant in that it changes form appropriately under a change of coordinates.

To find practical formulas for the covariant derivative components, we begin with equation 39 and apply the product rule of differentiation to find

$$\frac{\partial \mathbf{a}}{\partial \tilde{x}^{\tilde{\imath}}} = \frac{\partial \left(\tilde{a}_{\tilde{k}} \tilde{\mathbf{u}}^{\tilde{k}}\right)}{\partial \tilde{x}^{\tilde{\imath}}} = \frac{\partial \tilde{a}_{\tilde{k}}}{\partial \tilde{x}^{\tilde{\imath}}} \tilde{\mathbf{u}}^{\tilde{k}} + \tilde{a}_{\tilde{k}} \frac{\partial \tilde{\mathbf{u}}^{\tilde{k}}}{\partial \tilde{x}^{\tilde{\imath}}}. \tag{41}$$

Computing the inner product of this equation with the basis vector $\tilde{\mathbf{u}}_{\tilde{\jmath}}$, and defining the *Christoffel symbol of the second kind* by

$$\left\{ \begin{array}{c} \tilde{k} \\ \tilde{\jmath}\,\tilde{\imath} \end{array} \right\} \doteq -\tilde{\mathbf{u}}_{\tilde{\jmath}} \cdot \frac{\partial \tilde{\mathbf{u}}^{\tilde{k}}}{\partial \tilde{x}^{\tilde{\imath}}} = \tilde{\mathbf{u}}^{\tilde{k}} \cdot \frac{\partial \tilde{\mathbf{u}}_{\tilde{\jmath}}}{\partial \tilde{x}^{\tilde{\imath}}}, \tag{42}$$

we find that (*7*, *8*)

$$\frac{\mathrm{D}\tilde{a}_{\tilde{\jmath}}}{\mathrm{D}\tilde{x}^{\tilde{\imath}}} = \frac{\partial \tilde{a}_{\tilde{\jmath}}}{\partial \tilde{x}^{\tilde{\imath}}} - \left\{ \begin{array}{c} \tilde{k} \\ \tilde{\jmath}\,\tilde{\imath} \end{array} \right\} \tilde{a}_{\tilde{k}}. \tag{43}$$

Other authors use the notation

$$\Gamma^{\tilde{k}}_{\tilde{\jmath}\tilde{\imath}} = \left\{ \begin{array}{c} \tilde{k} \\ \tilde{\jmath}\,\tilde{\imath} \end{array} \right\} \tag{44}$$

for the Christoffel symbols, but this notation is not employed here. In particular, the alternative notation makes the Christoffel symbols look like third-order tensors, which they are not.

The covariant derivative of the contravariant components of a vector can similarly be shown to be (*7*, *8*)

$$\frac{\mathrm{D}\tilde{a}^{\tilde{\jmath}}}{\mathrm{D}\tilde{x}^{\tilde{\imath}}} = \frac{\partial \tilde{a}^{\tilde{\jmath}}}{\partial \tilde{x}^{\tilde{\imath}}} + \left\{ \begin{array}{c} \tilde{\jmath} \\ \tilde{k}\,\tilde{\imath} \end{array} \right\} \tilde{a}^{\tilde{k}}. \tag{45}$$

The definition can even be extended to covariant, contravariant, and mixed tensors of second order yielding (*7, 8*)

$$\frac{\mathrm{D}\tilde{a}_{\tilde{\imath}\tilde{\jmath}}}{\mathrm{D}\tilde{x}^{\tilde{k}}} = \frac{\partial \tilde{a}_{\tilde{\imath}\tilde{\jmath}}}{\partial \tilde{x}^{\tilde{k}}} - \left\{ \begin{array}{c} \tilde{m} \\ \tilde{\imath}\;\tilde{k} \end{array} \right\} \tilde{a}_{\tilde{m}\tilde{\jmath}} - \left\{ \begin{array}{c} \tilde{m} \\ \tilde{k}\;\tilde{\jmath} \end{array} \right\} \tilde{a}_{\tilde{\imath}\tilde{m}}, \tag{46}$$

$$\frac{\mathrm{D}\tilde{a}^{\tilde{\imath}\tilde{\jmath}}}{\mathrm{D}\tilde{x}^{\tilde{k}}} = \frac{\partial \tilde{a}^{\tilde{\imath}\tilde{\jmath}}}{\partial \tilde{x}^{\tilde{k}}} + \left\{ \begin{array}{c} \tilde{\imath} \\ \tilde{m}\;\tilde{k} \end{array} \right\} \tilde{a}^{\tilde{m}\tilde{\jmath}} + \left\{ \begin{array}{c} \tilde{k} \\ \tilde{m}\;\tilde{\jmath} \end{array} \right\} \tilde{a}^{\tilde{\imath}\tilde{m}}, \tag{47}$$

$$\frac{\mathrm{D}\tilde{a}^{\tilde{\imath}}_{\tilde{\jmath}}}{\mathrm{D}\tilde{x}^{\tilde{k}}} = \frac{\partial \tilde{a}^{\tilde{\imath}}_{\tilde{\jmath}}}{\partial \tilde{x}^{\tilde{k}}} + \left\{ \begin{array}{c} \tilde{\imath} \\ \tilde{m}\;\tilde{k} \end{array} \right\} \tilde{a}^{\tilde{m}}_{\tilde{\jmath}} - \left\{ \begin{array}{c} \tilde{m} \\ \tilde{k}\;\tilde{\jmath} \end{array} \right\} \tilde{a}^{\tilde{\imath}}_{\tilde{m}}, \tag{48}$$

as the appropriate formulas for the coefficient derivatives.

3.2 The Christoffel Symbols

The Christoffel symbols used in the definition of the covariant derivative and defined in equation 42 obey some important identities that find uses in this work. These identities are for the most part proven by complicated (but purely formal) manipulations, and their proofs can be found in almost any work on tensor analysis, such as references 7 or 8.

First, it can be demonstrated that the symbols are symmetric in their lower indices; that is, that

$$\left\{ \begin{array}{c} \tilde{\imath} \\ \tilde{\jmath}\;\tilde{k} \end{array} \right\} \equiv \left\{ \begin{array}{c} \tilde{\imath} \\ \tilde{k}\;\tilde{\jmath} \end{array} \right\}, \tag{49}$$

so that in particular

$$\epsilon^{\tilde{\imath}\tilde{\jmath}\tilde{k}} \left\{ \begin{array}{c} \tilde{\imath} \\ \tilde{\jmath}\;\tilde{k} \end{array} \right\} \equiv 0. \tag{50}$$

This symmetry can be used to derive their expansion in terms of the metric tensor, which after some tedious manipulations can be shown to be (*7, 8*)

$$\left\{ \begin{array}{c} \tilde{\imath} \\ \tilde{\jmath}\;\tilde{k} \end{array} \right\} = \frac{1}{2}\tilde{g}^{\tilde{\imath}\tilde{m}} \left(\frac{\partial \tilde{g}_{\tilde{m}\tilde{\jmath}}}{\partial \tilde{x}^{\tilde{k}}} + \frac{\partial \tilde{g}_{\tilde{m}\tilde{k}}}{\partial \tilde{x}^{\tilde{\jmath}}} - \frac{\partial \tilde{g}_{\tilde{\jmath}\tilde{k}}}{\partial \tilde{x}^{\tilde{m}}} \right). \tag{51}$$

Two more facts about Christoffel symbols are important in the remainder of the work. The first is Ricci's lemma, which states that the covariant derivative of the metric tensor vanishes. The proof is simple: In $\mathfrak{s}$, $g_{ij} = \delta_{ij}$, so its covariant derivative vanishes trivially. Since the metric tensor is itself a tensor, its covariant derivative is also a tensor. Therefore, its vanishing in one system proves its vanishing in all systems. Ricci's lemma implies that the metric tensor can be treated as a constant for the purposes of covariant differentiation.

Finally, intricate (but ultimately straightforward) manipulations of the Christoffel symbol definition show that (*7, 8*)

$$\left\{ \begin{array}{c} \tilde{\imath} \\ \tilde{\imath}\ \tilde{\jmath} \end{array} \right\} = \frac{1}{\sqrt{\tilde{g}}} \frac{\partial \sqrt{\tilde{g}}}{\partial \tilde{x}^{\tilde{\jmath}}}. \tag{52}$$

3.3 Differential Operators

With all of this detail in hand, the standard differential operators of multivariate calculus can now be introduced. In $\mathfrak{s}$, the gradient of a scalar $\phi(x^i)$ has components equal to the partial derivatives with respect to the underlying coordinate dimensions:

$$\nabla \phi \doteq \frac{\partial \phi}{\partial x^i} \mathbf{u}^i. \tag{53}$$

Note that the components of the gradient are covariant, as they are associated with the contravariant basis vectors ensuring the consistency of the Einstein summation convention. Assuming that ϕ has a physical meaning independent of the coordinate system in which it is defined (that is, assuming ϕ a true physical scalar), its value in $\tilde{\mathfrak{s}}$ must obey

$$\tilde{\phi}\left(\tilde{x}^{\tilde{\imath}}\right) \doteq \phi\left(x^i\right). \tag{54}$$

To find the gradient value in $\tilde{\mathfrak{s}}$, we invoke the formula for contravariant change of basis,

$$\frac{\partial \phi}{\partial x^i} \left(\mathbf{u}^i\right) = \frac{\partial \phi}{\partial x^i} \left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}} \tilde{\mathbf{u}}^{\tilde{\imath}} \right), \tag{55}$$

so that, by the chain rule, the gradient is given by

$$\nabla \phi = \frac{\partial \tilde{\phi}}{\partial \tilde{x}^{\tilde{\imath}}} \tilde{\mathbf{u}}^{\tilde{\imath}}. \tag{56}$$

The divergence of a vector $\mathbf{a}$ is given by the formula

$$\nabla \cdot \mathbf{a} \doteq \frac{\mathrm{D} a^i}{\mathrm{D} x^i} \tag{57}$$

in $\mathfrak{s}$. Covariant differentiation is used here to ensure that the result is a true scalar. (It is trivially identical to partial differentiation here anyway given the Cartesian assumption on $\mathfrak{s}$.) Now, because covariant derivatives change coordinates according to the laws of tensor algebra, we can immediately write

$$\nabla \cdot \mathbf{a} = \frac{\mathrm{D}\tilde{a}^{\tilde{\imath}}}{\mathrm{D}\tilde{x}^{\tilde{\imath}}}. \tag{58}$$

Using equations 45 and 52, we can expand this expression to find

$$\frac{\mathrm{D}\tilde{a}^{\tilde{\imath}}}{\mathrm{D}\tilde{x}^{\tilde{\imath}}} = \frac{\partial \tilde{a}^{\tilde{\imath}}}{\partial \tilde{x}^{\tilde{\imath}}} + \left\{ \begin{array}{c} \tilde{\imath} \\ \tilde{\imath}\;\tilde{\jmath} \end{array} \right\} \tilde{a}^{\tilde{\jmath}} = \frac{\partial \tilde{a}^{\tilde{\imath}}}{\partial \tilde{x}^{\tilde{\imath}}} + \frac{1}{\sqrt{\tilde{g}}} \frac{\partial \sqrt{\tilde{g}}}{\partial \tilde{x}^{\tilde{\jmath}}} \tilde{a}^{\tilde{\jmath}}, \tag{59}$$

or finally, by separating a factor of $\tilde{g}^{-\frac{1}{2}}$ and recognizing the derivative of a product,

$$\nabla \cdot \mathbf{a} = \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{\imath}}} \left(\sqrt{\tilde{g}}\, \tilde{a}^{\tilde{\imath}} \right). \tag{60}$$

Lastly, there is the curl. In $\mathfrak{s}$, the curl is defined as

$$\nabla \times \mathbf{a} \doteq \epsilon^{ijk} \frac{\mathrm{D}a_k}{\mathrm{D}x^j} \mathbf{u}_i. \tag{61}$$

This formula can be brought into $\tilde{\mathfrak{s}}$ using equation 31 and the usual covariant variable change formulas to find

$$\begin{aligned} \epsilon^{ijk} \frac{\mathrm{D}a_k}{\mathrm{D}x^j} \mathbf{u}_i &= \left(\frac{1}{\sqrt{\tilde{g}}} \frac{\partial x^i}{\partial \tilde{x}^{\tilde{m}}} \frac{\partial x^j}{\partial \tilde{x}^{\tilde{n}}} \frac{\partial x^k}{\partial \tilde{x}^{\tilde{p}}} \epsilon^{\tilde{m}\tilde{n}\tilde{p}} \right) \left(\frac{\partial \tilde{x}^{\tilde{\jmath}}}{\partial x^j} \frac{\partial \tilde{x}^{\tilde{k}}}{\partial x^k} \frac{\mathrm{D}\tilde{a}_{\tilde{k}}}{\mathrm{D}\tilde{x}^{\tilde{\jmath}}} \right) \left(\frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} \tilde{\mathbf{u}}_{\tilde{\imath}} \right) \\ &= \frac{1}{\sqrt{\tilde{g}}} \epsilon^{\tilde{m}\tilde{n}\tilde{p}} \delta^{\tilde{m}}_{\tilde{\imath}} \delta^{\tilde{n}}_{\tilde{\jmath}} \delta^{\tilde{p}}_{\tilde{k}} \frac{\mathrm{D}\tilde{a}_{\tilde{k}}}{\mathrm{D}\tilde{x}^{\tilde{\jmath}}} \tilde{\mathbf{u}}_{\tilde{\imath}} \\ &= \frac{1}{\sqrt{\tilde{g}}} \epsilon^{\tilde{\imath}\tilde{\jmath}\tilde{k}} \frac{\mathrm{D}\tilde{a}_{\tilde{k}}}{\mathrm{D}\tilde{x}^{\tilde{\jmath}}} \tilde{\mathbf{u}}_{\tilde{\imath}}. \end{aligned} \tag{62}$$

In view of equation 50, however,

$$\epsilon^{\tilde{\imath}\tilde{\jmath}\tilde{k}} \frac{\mathrm{D}\tilde{a}_{\tilde{k}}}{\mathrm{D}\tilde{x}^{\tilde{\jmath}}} \equiv \epsilon^{\tilde{\imath}\tilde{\jmath}\tilde{k}} \frac{\partial \tilde{a}_{\tilde{k}}}{\partial \tilde{x}^{\tilde{\jmath}}}, \tag{63}$$

so that the curl may finally be written without the covariant derivative notation as

$$\nabla \times \mathbf{a} = \frac{1}{\sqrt{\tilde{g}}} \epsilon^{\tilde{\imath}\tilde{\jmath}\tilde{k}} \frac{\partial \tilde{a}_{\tilde{k}}}{\partial \tilde{x}^{\tilde{\jmath}}} \tilde{\mathbf{u}}_{\tilde{\imath}}. \tag{64}$$

4. The Coordinate Systems of Continuum Mechanics

4.1 Reference Coordinates

To begin applying the mathematics of the foregoing sections to continuum mechanics, coordinate systems must be introduced to describe continua and their motions and deformations. The first set of coordinates introduced in this regard is the *reference coordinate set*, which serves to name the particles. Indeed, in more general, mathematical renderings of continuum mechanics, the reference "coordinates" hardly even need to be coordinates or even numerical, so long as each particle is identified as a member of a set (*9*). We therefore assume that a material body is composed of particles, which, at some time, could have been identified by their location X^I in a Cartesian coordinate system $\mathfrak{S}$. By the seeming circumlocution "at some time, could have been," we mean to imply that while the particles may never have been in the reference configuration X^I, the mapping from that configuration to any future configuration x^i is

- Injective, so that each particle in the current configuration corresponds to only one reference particle,
- Surjective, so that each particle in the current configuration has a name, and
- Characterized by a mapping with a positive Jacobian determinant, so that the chirality of the mapping is preserved throughout space and time.

In most work in elasticity, the X^I are imagined to represent the rest state of the matter, devoid of elastic potential energy. In other work, the reference coordinates may be the initial state of the continuum, or they may even be ignored altogether, except to assume their existence. (This last approach is often seen in fluid mechanics.)

Finally, in this work, we always assume that $\mathfrak{S}$ is Cartesian, as mentioned above but this need not be the case in more general applications. For some problems involving lower dimensional formulations of the theory or special symmetry, it might be useful to introduce curvilinear material coordinates through the mapping $\tilde{X}^{\tilde{I}} = \tilde{\Xi}^{\tilde{I}}(X^I)$. By the assumptions listed above and the discussion of the previous two sections, these new coordinates would name the particles as well as the original coordinates, and could therefore be used as the basis of a theory with no essential complications.

4.2 Spatial Coordinates

The current state of the particles of the body named by the reference coordinates is related by the *spatial coordinates* x^i. The location of a given particle X^I at a given time t is completely specified by the *motion* of the particle

$$x^i = \xi^i\left(X^I, t\right) . \tag{65}$$

This relation completely specifies particle kinematics and is the basis for the theory of continuum mechanics.

At any fixed time $t = t_0$, this relationship must be bijective with a positive Jacobian determinant as mentioned in the previous section. Because of this, we may also write

$$X^I = \Xi^I\left(x^i, t\right) . \tag{66}$$

This equation should not be interpreted as meaning that the X^I depend on time; they don't. For an arbitrary fixed value of time $t = t_0$, what equations 65 and 66 mean is that ξ and Ξ are inverse functions, i.e., that

$$x^i = \xi^i\left(X^I, t_0\right) = \xi^i\left(\Xi^I\left(x^i, t_0\right), t_0\right) , \tag{67}$$

or, equivalently,

$$X^I = \Xi^I\left(x^i, t_0\right) = \Xi^I\left(\xi^i\left(X^I, t_0\right), t_0\right) , \tag{68}$$

for all t_0.

Because the particles move, the relationship between their current location and their reference designation does evolve, and that is captured by equation 66. When combined with the curvilinear coordinate information provided in the previous sections, this observation can be used as a basis for the definition of the coordinate system at the center of the work, which is provided in the next subsection.

4.3 Convective Coordinates

The description of boundaries, inclusions, and other geometrically relevant physical features of a body are often more simply described in the reference system $\mathfrak{S}$ than in the spatial system $\mathfrak{s}$. After all, the reference system is merely a matter of particle nomenclature, whereas the particle trajectories in $\mathfrak{s}$ are governed by the applied excitations and the laws of physics. This can be illustrated through the example of an elastic bar in the shape of a rectangular prism whose axes are aligned with the coordinate axes, subjected to a large force. Assuming that the reference coordinates are taken to be the initial coordinates of the bar at rest at $t = 0$, this situation is depicted in figure 2a. In $\mathfrak{S}$, the boundaries of the bar correspond to equations of the form $X^I = K^I$, where K^I is some fixed constant for each boundary in each dimension. Of course, at $t = 0$, the boundaries in $\mathfrak{s}$ are just as simple to describe, since at this instant they coincide.

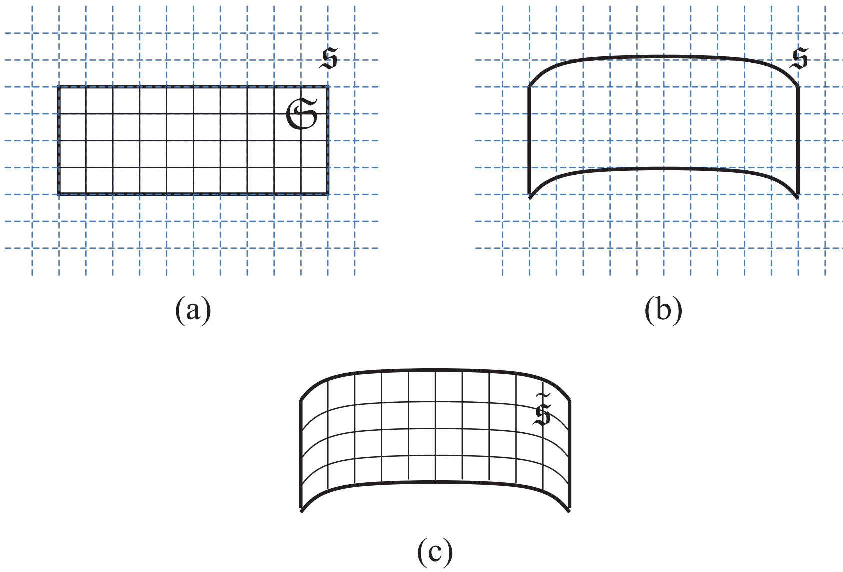

Figure 2. (a) The initial state of the bar labeled in the spatial (dashed, $\mathfrak{s}$) and reference (solid, $\mathfrak{S}$) coordinate systems, (b) the deformed bar labeled in spatial coordinates ($\mathfrak{s}$), and (c) the deformed bar labeled in convective coordinates ($\tilde{\mathfrak{s}}$). The reference coordinates never change, remaining associated with (a) even as the deformation occurs .

At a later time, under the action of outside forces, the bar will have deformed. Suppose it takes the shape depicted in figure 2b. The spatial coordinates are depicted in this picture, and the boundaries (obviously) do not align with them. By definition, the reference coordinates never change, and are still depicted in figure 2a.

To arrive at a coordinate system in which features simply described in reference coordinates retain their simple description as the body evolves and deforms, we turn to a set of spatial curvilinear coordinates commonly called *convective coordinates*. Since they are curvilinear spatial coordinates, we denote the convective variables by $\tilde{x}^{\tilde{i}}$ and the convective system by $\tilde{\mathfrak{s}}$. The mapping that gives rise to these coordinates can be written, at each instant t as

$$\tilde{x}^{\tilde{i}} = \tilde{\xi}^{\tilde{i}}(x^i, t) \doteq \delta^{\tilde{i}}_I \Xi^I(x^i, t) \,, \tag{69}$$

where the function Ξ is defined in equation 66. (Note that these coordinates evolve as a function of the "parameter" t.) The virtue of this choice is that at any arbitrary time t_0, the particle with material coordinates X^I in $\mathfrak{S}$ has coordinates $\tilde{x}^{\tilde{\imath}} = X^{\tilde{\imath}}$ in $\tilde{s}$. After all, by equation 68,

$$\tilde{x}^{\tilde{\imath}} = \delta_I^{\tilde{\imath}} \Xi^I \left(x^i, t_0\right) = \delta_I^{\tilde{\imath}} \Xi^I \left(\xi^i \left(X^I, t_0\right), t_0\right) = \delta_I^{\tilde{\imath}} X^I. \tag{70}$$

In particular, the algebraic description of interesting object features never changes. This is depicted in figure 2c, which labels the deformed bar of figure 2b with the convective coordinates of system $\tilde{\mathfrak{s}}$.

The system $\tilde{\mathfrak{s}}$ does both of the following:

- Describes the system as it evolves, even if the "continuum" depicted breaks apart during its evolution, and
- Tracks the trajectory of fixed particles during that evolution.

This can be very useful for the discussion of nonmechanical physics unfolding against the background of the continuum. In particular, because simultaneity is not absolute in the theory of relativity (*10*, *11*), convective descriptions can become essential. In the next section, operations necessary for the formulation of continuum kinematics in convective coordinates are described.

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5. The Operations of Continuum Mechanics in Convective Coordinates

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5.1 Temporal Differentiation

Temporal differentiation in $\tilde{\mathfrak{s}}$ is not very different than in $\mathfrak{S}$ because at all times $\tilde{x}^{\tilde{\imath}} \equiv X^{\tilde{\imath}}$. Therefore, many of the formulas presented here have direct reference counterparts discussed at length in standard texts on continuum mechanics such as references 1, 2, and 9.

Perhaps the most important temporal derivative definition is also the most basic: the particle velocity, which is computed by holding the material coordinates (the X^I) of the particle fixed, and differentiating the spatial coordinates with respect to time:

$$v^i \doteq \frac{\mathrm{d}x^i}{\mathrm{d}t}. \tag{71}$$

The use of the "total derivative" notation $\mathrm{d}/\mathrm{d}t$ here is meant to indicate that the X^I remain fixed but the x^i vary during the differentiation. The resulting vector v^i is the velocity of a fixed particle through space.

Another type of velocity that can be measured is the speed with which particles flow past a given point in space. To compute this, we need to fix the point in space (the x^i), and differentiate the $\tilde{x}^{\tilde{i}}$ with respect to time. We can thus define

$$\tilde{\psi}^{\tilde{i}} \doteq \frac{\partial \tilde{x}^{\tilde{i}}}{\partial t}, \tag{72}$$

where the "partial derivative" notation $\partial/\partial t$ indicates that the x^i remain fixed and the X^I are allowed to vary as particles pass. The definitions of the temporal differentiation operators above should make it clear that

$$\frac{\partial x^i}{\partial t} \equiv 0, \tag{73}$$

and

$$\frac{\mathrm{d}\tilde{x}^{\tilde{i}}}{\mathrm{d}t} \equiv 0. \tag{74}$$

Finally, functions are often presented in terms of their dependence on the spatial variables, but need to be differentiated holding the particle constant. Given a function $f = f(x^i, t)$, its *material derivative* is computed as

$$\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i}. \tag{75}$$

5.2 Deformations, Transformations, and Metrics

Consider two points in $\mathfrak{s}$ separated by an infinitesimal displacement $\mathrm{d}x^i$. This displacement is related to an infinitesimal displacement $\mathrm{d}X^I$ in $\mathfrak{S}$ by the equation

$$\mathrm{d}x^i = \frac{\partial x^i}{\partial X^I}\mathrm{d}X^I, \tag{76}$$

and to an infinitesimal variable transformation $\mathrm{d}\tilde{x}^{\tilde{i}}$ in $\tilde{\mathfrak{s}}$ by

$$\mathrm{d}x^i = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\mathrm{d}\tilde{x}^{\tilde{i}}. \tag{77}$$

Equations of this form, of course, hold for any change of variables; after all, this is just a straightforward application of the chain rule of calculus. Moreover, because $X^I \equiv \tilde{x}^I$ at all times, the set of numbers invoked on the right-hand sides of equations 76 and 77 are the same.

Nonetheless, their meanings are different because they refer to spaces with different metric structures. The squared magnitude of $\mathrm{d}X^I$ is given by

$$\mathrm{d}S^2 = \delta_{IJ}\mathrm{d}X^I\mathrm{d}X^J, \tag{78}$$

whereas the squared magnitude of $\mathrm{d}\tilde{x}^{\tilde{\imath}}$ is

$$\mathrm{d}\tilde{s}^2 = \tilde{g}_{\tilde{\imath}\tilde{\jmath}}\mathrm{d}\tilde{x}^{\tilde{\imath}}\mathrm{d}\tilde{x}^{\tilde{\jmath}}. \tag{79}$$

The reason for this "discrepancy" is that two different lengths are involved: $\mathrm{d}S^2$ is the square length of the *reference* segment that corresponds to $\mathrm{d}\mathbf{x}$ in the spatial domain, whereas $\mathrm{d}\tilde{s}^2 = \mathrm{d}s^2$ because they are the squared length of the same segment $\mathrm{d}\mathbf{x}$ in different coordinate systems. Of course, by equation 76 we may write

$$\mathrm{d}s^2 = C_{IJ}\mathrm{d}X^I\mathrm{d}X^J \tag{80}$$

where the Green deformation tensor (*1*) or right Cauchy-Green tensor (*12*) is defined by

$$C_{IJ} = \delta_{ij}\frac{\partial x^i}{\partial X^I}\frac{\partial x^j}{\partial X^J} \tag{81}$$

Recalling that $\tilde{x}^I \equiv X^I$ at all times, and recalling the definition of the metric tensor in terms of the variable transformations given in equation 21, we find that

$$\tilde{g}_{\tilde{\imath}\tilde{\jmath}} \equiv \delta^I_{\tilde{\imath}}\delta^J_{\tilde{\jmath}}C_{IJ}, \tag{82}$$

that is, that the right Cauchy-Green tensor and the convective metric tensor are numerically indistinguishable.

From these observations, the computation of (oriented) areas and lengths using reference, convective, and spatial coordinates is simple. A differential oriented area in the reference configuration $\mathfrak{S}$ is given by

$$dA_I = \epsilon_{IJK}\mathrm{d}X^J\mathrm{d}X^K. \tag{83}$$

In the current configuration, this oriented area is transformed into a new vector with components given by

$$da_i = \epsilon_{ijk}\mathrm{d}x^j\mathrm{d}x^k \tag{84}$$

in $\mathfrak{s}$, or by

$$d\tilde{a}_{\tilde{\imath}} = \sqrt{\tilde{g}}\,\epsilon_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\mathrm{d}\tilde{x}^{\tilde{\jmath}}\mathrm{d}\tilde{x}^{\tilde{k}} \tag{85}$$

in $\tilde{\mathfrak{s}}$. The coefficients of the oriented areas $\mathrm{d}a_i$ and $\mathrm{d}\tilde{a}_{\tilde{\imath}}$ obey a simple tensor relationship; they represent the same tensor quantity in two different coordinate systems. The relationship between these coefficients and the $\mathrm{d}A_I$ can be computed from the definition, resulting finally in

$$\mathrm{d}a_i = \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i}\mathrm{d}\tilde{a}_{\tilde{\imath}} = \sqrt{C}\,\frac{\partial X^I}{\partial x^i}\mathrm{d}A_I. \tag{86}$$

Volumes are computed in much the same way. Reference volumes in $\mathfrak{S}$ are computed by the Euclidean scalar triple product

$$\mathrm{d}V = \mathrm{d}X^1\mathbf{U}_1 \cdot \left(\mathrm{d}X^2\mathbf{U}_2 \times \mathrm{d}X^3\mathbf{U}_3\right) = \mathrm{d}X^1\mathrm{d}X^2\mathrm{d}X^3. \tag{87}$$

Spatial volumes in $\mathfrak{s}$ are computed likewise, resulting in

$$\mathrm{d}v = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3. \tag{88}$$

The curvilinear definition of the vector product needs to be invoked to compute volumes in $\tilde{\mathfrak{s}}$, resulting in

$$\mathrm{d}\tilde{v} = \mathrm{d}\tilde{x}^1\tilde{\mathbf{u}}_1 \cdot \left(\mathrm{d}\tilde{x}^2\tilde{\mathbf{u}}_2 \times \mathrm{d}\tilde{x}^3\tilde{\mathbf{u}}_3\right) = \sqrt{\tilde{g}}\,\mathrm{d}\tilde{x}^1\mathrm{d}\tilde{x}^2\mathrm{d}\tilde{x}^3. \tag{89}$$

These three different expressions for two different differential volumes are related by

$$\mathrm{d}v = \mathrm{d}\tilde{v} = \sqrt{C}\,\mathrm{d}V, \tag{90}$$

since at all times $C = \tilde{g}$ and $dX^I = \delta^I_{\tilde{\imath}}\mathrm{d}\tilde{x}^{\tilde{\imath}}$.

5.3 Three More Basic Relationships

Before presenting the Maxwell equations in convective coordinates, three more basic results of geometry and continuum mechanics are needed. In the standard continuum mechanics literature, these relationships are often written in reference coordinates, but here we translate them (where needed) into convective coordinates.

5.3.1 Particle Velocity and Identity Flux

The first basic equation derived here relates the particle velocity components v^i of equation 71 to the identity flux components $\tilde{\psi}^{\tilde{\imath}}$ of equation 72 (*3*). By equation 74, $\mathrm{d}\tilde{x}^{\tilde{\imath}}/\mathrm{d}t = 0$, so that choosing $f = \tilde{x}^{\tilde{\imath}}$ in equation 75 gives

$$\tilde{\psi}^{\tilde{\imath}} = -\frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i}v^i. \tag{91}$$

5.3.2 The Spatial Equation of Continuity

The second equation presented here is the spatial equation of continuity (*1–3, 9*). This relationship is predicated on the ability to compute the temporal derivative to the evolving metric $\sqrt{\tilde{g}}$, which, in turn, involves differentiating the transformation tensor with respect to time. In particular, the temporal derivative of the transformation tensor is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}\right) = \frac{\partial v^i}{\partial \tilde{x}^{\tilde{\imath}}} = \frac{\partial v^i}{\partial x^j}\frac{\partial x^j}{\partial \tilde{x}^{\tilde{\imath}}}. \tag{92}$$

Given the definition of the transformation determinant, the product rule of calculus, and the vanishing of determinants with equal rows, this relationship may be used to show that

$$\frac{\mathrm{d}\sqrt{\tilde{g}}}{\mathrm{d}t} = \frac{\mathrm{d}v^i}{\mathrm{d}x^i}\sqrt{\tilde{g}}. \tag{93}$$

Given any (scalar, vector, or tensor) quantity $\mathcal{F}$, invoking the above along with the product rule gives rise to the *spatial equation of continuity*:

$$\frac{1}{\sqrt{\tilde{g}}}\frac{\mathrm{d}\left(\sqrt{\tilde{g}}\mathcal{F}\right)}{\mathrm{d}t} = \frac{\partial \mathcal{F}}{\partial t} + \frac{\partial (v^i\mathcal{F})}{\partial x^i}. \tag{94}$$

5.3.3 Convected Time Derivative

Finally, we have occasion to compute the material derivative of a flux. Given a vector g^i we seek a vector $\mathring{g}^i$ called the *convected time derivative* (*3, 9*) of g^i such that

$$\int\limits_{\Gamma} \mathring{g}^i \, da_i = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Gamma} g^i \, \mathrm{d}a_i. \tag{95}$$

Here, $\Gamma \subset \mathbb{R}^3$ is an open surface in space; that is, it is isomorphic to a finite part of a plane. We may compute the necessary quantity in several steps. First, by invoking equations 30 and 77, the integrand may be brought into convective coordinates and the derivative moved inside the integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Gamma} g^i \, \mathrm{d}a_i = \int\limits_{\Gamma} \frac{\mathrm{d}}{\mathrm{d}t}\left[g^i\sqrt{\tilde{g}}\frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i}\right]\frac{1}{\sqrt{\tilde{g}}}\frac{\partial x^j}{\partial \tilde{\imath}}\,\epsilon_{jk\ell}\mathrm{d}x^k\mathrm{d}x^\ell. \tag{96}$$

Thus, using the product differentiation formula and equation 94,

$$\mathring{g}^i = \frac{\partial g^i}{\partial t} + \frac{\partial}{\partial x^k}\left(v^k g^i\right) + g^j\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^j}\right). \tag{97}$$

Using the fact that δ^i_j is constant in time,

$$\frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^j}\right) = -\frac{\partial v^i}{\partial x^j}, \tag{98}$$

follows immediately. Therefore, inserting this relation into equation 97, we find

$$\mathring{g}^i = \frac{\partial g^i}{\partial t} + \frac{\partial (v^k g^i)}{\partial x^k} - \frac{\partial v^i}{\partial x^j} g^j, \tag{99}$$

which, upon relabeling indices and applying the product rule becomes the desired relationship

$$\mathring{g}^i = \frac{\partial g^i}{\partial t} + \epsilon^{ijk}\frac{\partial}{\partial x^j}\left[\epsilon_{k\ell m} v^m g^\ell\right] - \frac{\partial g^j}{\partial x^j} v^i. \tag{100}$$

This is the final form of the convected time derivative.

6. Maxwell's Equations in Convective Coordinates

6.1 Maxwell's Equations in Vacuum in Spatial Coordinates

Before transforming the Maxwell equations to a convective frame, they must be stated in the usual spatial frame. In SI units, the basic quantities of electromagnetic theory are *total charge* q measured in coulombs per meter cubed (C/m^3), *total current density* j^i measured in amperes per meter squared (A/m^2), the *electric field* e_i measured in volts per meter (V/m), and the *magnetic flux density* b^i measured in tesla (T). We have not demonstrated that any of these quantities is tensorial; therefore, we make these definitions at the outset and do not assume they have tensorial properties. In particular, we do not change bases by raising or lowering indices. (These definitions are the ones that make sense later in that they preserve the form of the macroscopic Maxwell equations in all systems. Therefore, these definitions have been made with considerable hindsight.) In vacuum, the Maxwell equations are

$$\frac{\partial}{\partial x^i}\left(\delta^{ij} e_j\right) = \frac{q}{\epsilon_0}, \tag{101}$$

$$\frac{\partial b^i}{\partial x^i} = 0, \tag{102}$$

$$\epsilon^{ijk}\frac{\partial e_k}{\partial x^j} = -\frac{\partial b^i}{\partial t}, \tag{103}$$

$$\epsilon^{ijk}\frac{\partial}{\partial x^j}\left(\delta_{k\ell} b^\ell\right) = \mu_0\epsilon_0\frac{\partial}{\partial t}\left(\delta^{ij} e_j\right) + \mu_0 j^i. \tag{104}$$

The quantity $\mu_0 = 4\pi \times 10^{-7}$ henries per meter (H/m) is called the *permeability of free space*, and the quantity $\epsilon_0 = (\mu_0 c^2)^{-1}$ farads per meter (F/m) is called the *permittivity of free space*, where $c = 299792458$ m/s is the *speed of light in vacuum*. The electric field and magnetic flux density described above are basic in the sense that the Lorentz force equation

$$f_i = q\left(e_i + \epsilon_{ijk} v^j b^k\right) \tag{105}$$

relates the mechanical force density f_i (N/m^3) to these two fields, the charge density q, and the speed at which the charge is moving v^j. Of course, the current density is related to the charge density by the usual formula $j^i = q v^i$. Equation 101 is known as Gauß's law for the electric field; equation 102 is known as Gauß's law for the magnetic field; equation 103 is known as Faraday's law, and equation 104 is known as the Ampère-Maxwell law. Charge conservation follows immediately upon taking the divergence of equation 104 and inserting equation 101, resulting in the equation of continuity

$$\frac{\partial j^i}{\partial x^i} + \frac{\partial q}{\partial t} = 0. \tag{106}$$

6.2 Maxwell's Equations in Ponderable Materials in Spatial Coordinates

Ponderable, that is massive, materials affect electromagnetic wave propagation because they are constructed out of charged particles like protons and electrons. Equations 101–104 are not particularly convenient in practice in such media because the charge and current they reference are the total charge and current; that is, they make no distinction between charges and currents impressed externally to create fields and those arising merely because fields acting upon charges bound in matter are acted upon by other fields. (Another way of saying this is that they are in *microscopic* form; they consider all matter from a corpuscular, rather than continuum, viewpoint.) Bound charge densities arise either through the creation of new dipoles or the alignment of existing dipoles in response to an external field. The density of these dipoles can be reckoned as a vector dipole moment per unit volume p^i (C/m^2) related to the bound charge density by

$$q_{\mathrm{B}} \doteq \frac{\partial p^i}{\partial x^i}, \tag{107}$$

and this dipole density can be related, in turn, to the total field (*13*). Defining the free charge as the difference between the total charge and the bound charge,

$$q_{\mathrm{F}} \doteq q - q_{\mathrm{B}}, \tag{108}$$

the Gauß law for the electric field becomes

$$\epsilon_0 \frac{\partial}{\partial x^i}\left(\delta^{ij} e_j\right) = q_{\mathrm{F}} - \frac{\partial p^i}{\partial x^i}. \tag{109}$$

This is generally simplified by defining the *electric displacement density*

$$d^i \doteq \epsilon_0 \left(\delta^{ij} e_j\right) + p^i, \tag{110}$$

(in C/m^2) and writing the Gauß law as

$$\frac{\partial d^i}{\partial x^i} = q_{\mathrm{F}}. \tag{111}$$

Similarly, the total current density may be broken into its constitutive parts:

$$j^i \doteq j^i_{\mathrm{F}} + \frac{\partial p^i}{\partial t} + \epsilon^{ijk} \frac{\partial}{\partial x^j}\left(m_k + \epsilon_{kmn} p^m v^n\right). \tag{112}$$

Here j^i_{F} is the free current (impressed and conducted) density, the term $\partial p^i / \partial t$ is the current caused by local increase or decrease of polarization charges, and the term related to $\epsilon_{kmn} p^m v^n$ has to do with the convected polarization charges. Finally, the m_k is the magnetization per unit volume (A/m^3), resulting from the formation of magnetic dipoles in reaction to the field (*3, 13*). Inserting this definition into equation 104 gives rise to the macroscopic Ampère-Maxwell law

$$\epsilon^{ijk} \frac{\partial h_k}{\partial x^j} = j^i_{\mathrm{F}} + \frac{\partial d^i}{\partial t}, \tag{113}$$

where the *magnetic field* h_k (in A/m) is defined as

$$h_i \doteq \frac{\left(\delta_{ij} b^j\right)}{\mu_0} - m_i - \epsilon_{ijk} p^j v^k. \tag{114}$$

6.3 Maxwell's Macroscopic Equations in Convective Coordinates

In principle, translating the Maxwell equations into convective coordinate form should be easy. After all, if electromagnetics is a proper physical theory, its quantities should change coordinate systems as tensors, and formulas can be written immediately. Unfortunately, that basic physical observations prohibit this is clear upon momentary reflection: An observer holding a charge sees an electric field, but another observer moving with respect to the first sees a current and therefore a magnetic field. This implies immediately that the electric and magnetic fields must be part of the same physical entity, as indeed they are in the theory of relativity (*10, 11, 13*). Despite this, nonunique theories may be constructed via other means; this is the approach taken here to illustrate the convective form of reference 3. This work aimed to preserve the macroscopic Maxwell equations. Though other approaches are possible, we do not consider them here.

To convert equation 113 to convective coordinates, we integrate both sides over an arbitrary surface and apply Stokes's theorem to find

$$\int_{\Gamma} \epsilon^{ijk} \frac{\partial h_k}{\partial x^j} \mathrm{d}a_i = \oint_{\partial\Gamma} h_i \mathrm{d}x^i. \tag{115}$$

Here, the covariant components of the spatial directed differential area are given by equation 84, and the right-hand side indicates a line integral around the closed path $\partial\Gamma$ surrounding Γ in spatial coordinates. (The path $\partial\Gamma$ has direction related to that of the normal to Γ by the right-hand rule. The circle notation on the path integral indicates that the path is necessarily closed.) We can simplify the right-hand side of equation 113 using the convected time derivative defined in equation 100. Specifically, using equation 100 to substitute for the partial derivative of the electric displacement, and recognizing the meaning of that derivative, equation 113 can be written in the form

$$\oint_{\partial\Gamma} \left(h_i - \epsilon_{ijk} v^j d^k\right) \mathrm{d}x^i = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma} d^i \, \mathrm{d}a_i + \int_{\Gamma} \left(j_{\mathrm{F}}^i - v^i \frac{\partial d^j}{\partial x^j}\right) \mathrm{d}a_i. \tag{116}$$

Because the time derivative in this equation is the material time derivative, the integrals can be converted to convective coordinates using the techniques from the previous sections. Recognizing the free charge, this procedure results in the final integral form of the Ampère-Maxwell law in convective coordinates,

$$\oint_{\partial\Gamma} \left(h_i - \epsilon_{ijk} v^j d^k\right) \frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}} \mathrm{d}\tilde{x}^{\tilde{\imath}} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma} d^i \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} \mathrm{d}\tilde{a}_{\tilde{\imath}} + \int_{\Gamma} \left(j_{\mathrm{F}}^i - q_{\mathrm{F}} v^i\right) \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} \mathrm{d}\tilde{a}_{\tilde{\imath}}, \tag{117}$$

and Stokes's theorem gives its final differential form:

$$\epsilon_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \frac{\partial}{\partial \tilde{x}^{\tilde{\jmath}}} \left[\left(h_i - \epsilon_{ijk} v^j d^k\right) \frac{\partial x^i}{\partial \tilde{x}^{\tilde{k}}}\right] = \frac{\mathrm{d}}{\mathrm{d}t} \left(d^i \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i}\right) + \left(j_{\mathrm{F}}^i - q_{\mathrm{F}} v^i\right) \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i}. \tag{118}$$

Because we want the form of this equation preserved under the coordinate change, we insist that this equation be interpreted as the convective Ampère-Maxwell law

$$\frac{1}{\sqrt{\tilde{g}}} \epsilon^{\tilde{\imath}\tilde{\jmath}\tilde{k}} \frac{\partial \tilde{h}_{\tilde{k}}}{\partial \tilde{x}^{\tilde{\jmath}}} = \frac{\mathrm{d}\tilde{d}^{\tilde{\imath}}}{\mathrm{d}t} + \tilde{\jmath}_{\mathrm{C}}, \tag{119}$$

so that we define the convective magnetic field by

$$\tilde{h}_{\tilde{\imath}} \doteq \left(h_i - \epsilon_{ijk} v^j d^k\right) \frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}, \tag{120}$$

the convective displacement density by

$$\tilde{d}^{\tilde{\imath}} \doteq \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} d^i, \tag{121}$$

and the conducted convected current by

$$\tilde{\jmath}_{\mathrm{C}} \doteq \left(j^i_{\mathrm{F}} - q_{\mathrm{F}} v^i\right) \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i}. \tag{122}$$

(This last is called *conducted current* and not free current because the only charge moving relative to the convected frame must be that conducted through it.) Faraday's law is converted in the same way and becomes

$$\frac{1}{\sqrt{\tilde{g}}} \epsilon^{\tilde{\imath}\tilde{\jmath}\tilde{k}} \frac{\partial \tilde{e}_{\tilde{k}}}{\partial \tilde{x}^{\tilde{\jmath}}} = -\frac{\mathrm{d}\tilde{b}^{\tilde{\imath}}}{\mathrm{d}t}, \tag{123}$$

where

$$\tilde{e}_{\tilde{\imath}} \doteq \left(e_i + \epsilon_{ijk} v^j b^k\right) \frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}}, \tag{124}$$

and

$$\tilde{b}^{\tilde{\imath}} \doteq \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} b^i. \tag{125}$$

The divergence equations can be converted even more simply. Integrating both sides of equation 111 over an arbitrary volume and applying the divergence theorem immediately gives the convective, integral form of the Gauß law for the electric field:

$$\oint_{\partial\Omega} d^i \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} \mathrm{d}\tilde{a}_{\tilde{\imath}} = \int_{\Omega} q_{\mathrm{F}} \mathrm{d}\tilde{v}. \tag{126}$$

Here, $\Omega \subset \mathbb{R}^3$ is a volume and $\partial\Omega$ is the surface enclosing it. The circle around the integral symbol used on the left-hand side of this equation implies a surface that is necessarily closed. If we define

$$\tilde{q}_{\mathrm{F}} \doteq q_{\mathrm{F}}, \tag{127}$$

and apply equation 121 and the divergence theorem again, the convective differential form of the electric Gauß law obtains:

$$\frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{\imath}}} \left(\sqrt{\tilde{g}}\, \tilde{d}^{\tilde{\imath}}\right) = \tilde{q}_{\mathrm{F}}. \tag{128}$$

The magnetic Gauß law

$$\frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{\imath}}} \left(\sqrt{\tilde{g}}\, \tilde{b}^{\tilde{\imath}} \right) = 0, \tag{129}$$

is derived similarly.

6.4 Maxwell's Microscopic Equations in Convective Coordinates

Many of the same techniques can be applied to transform the microscopic Maxwell equations to convective coordinates. That this needs to be discussed at all springs from the fact that these equations directly incorporate the contravariant components of the electric field vector and the covariant components of the magnetic flux density, in spite of the fact that these cannot be derived from the usual process of raising or lowering indices. (That this is the case is already clear from, for instance, equation 124; the coordinate change formula for the electric field should not involve the magnetic field if the electric field is a true independent tensor quantity.)

On the other hand, two of the microscopic Maxwell equations are identical to their macroscopic counterparts because they are homogeneous. These two equations, the Gauß's law for the magnetic field (equation 129) and the Faraday law (equation 123) therefore are not altered. The two equations containing sources, however, are more complicated and we turn to them now.

6.4.1 Gauß's Law for the Electric Field

Integrating Gauß's law for the electric field in microscopic form (equation 101) over an arbitrary volume, and using Stokes's theorem to convert the field integral to a surface integral yields

$$\epsilon_0 \oint\limits_{\partial\Omega} \delta^{ij} e_j \,\mathrm{d}a_i = \int\limits_{\Omega} q \,\mathrm{d}v \tag{130}$$

Using equations 86 and 90, these areas and volumes can be converted to convective coordinates resulting in

$$\epsilon_0 \oint\limits_{\partial\Omega} \delta^{ij} \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} e_j \,\mathrm{d}\tilde{a}_{\tilde{\imath}} = \int\limits_{\Omega} q \,\mathrm{d}\tilde{v}. \tag{131}$$

Using the divergence theorem again to convert the left-hand side of this equation back into a volume integral, and defining the convected total charge in the obvious way (i.e., $\tilde{q} \doteq q$), we find

$$\epsilon_0 \int\limits_{\Omega} \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{\imath}}} \left[\sqrt{\tilde{g}}\, \delta^{ij} \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} e_j \right] \mathrm{d}\tilde{v} = \int\limits_{\Omega} \tilde{q} \,\mathrm{d}\tilde{v}. \tag{132}$$

Finally, recognizing the arbitrariness of the integration volume, and substituting the expression for the electric field from the inverse of equation 124 (while invoking the velocity transformation of equation 91), we find the final microscopic form of the Gauß law for the electric field:

$$\epsilon_0 \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{\imath}}} \left\{ \sqrt{\tilde{g}} \left[\tilde{g}^{\tilde{\imath}\tilde{\jmath}} \left(\tilde{e}_{\tilde{\jmath}} + \sqrt{\tilde{g}}\, \epsilon_{\tilde{\jmath}\tilde{k}\tilde{m}} \tilde{\psi}^{\tilde{k}} \tilde{b}^{\tilde{m}} \right) \right] \right\} = \tilde{q}. \tag{133}$$

We can derive the definition of the electric displacement density in terms of the electric field and polarization density from this equation. By expressing the total charge in terms of its free and bound components in equation 131 (while substituting the expression for electric field from equation 133), then proceeding on the left-hand side as before, we can write

$$\epsilon_0 \int\limits_\Omega \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{\imath}}} \left\{ \sqrt{\tilde{g}} \left[\tilde{g}^{\tilde{\imath}\tilde{\jmath}} \left(\tilde{e}_{\tilde{\jmath}} + \sqrt{\tilde{g}}\, \epsilon_{\tilde{\jmath}\tilde{k}\tilde{m}} \tilde{\psi}^{\tilde{k}} \tilde{b}^{\tilde{m}} \right) \right] \right\} \mathrm{d}\tilde{v} = \int\limits_\Omega q_\mathrm{F}\, \mathrm{d}v - \int\limits_\Omega \frac{\partial p^i}{\partial x^i} \mathrm{d}v. \tag{134}$$

Now, following the same steps as before to convert the rightmost integral first to a surface integral and back, changing to the convected coordinate system along the way, the right-hand side becomes

$$\int\limits_\Omega q_\mathrm{F}\, \mathrm{d}v - \int \frac{\partial p^i}{\partial x^i} \mathrm{d}v = \int\limits_\Omega \tilde{q}_\mathrm{F}\, \mathrm{d}\tilde{v} - \int\limits_\Omega \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{\imath}}} \left(\sqrt{\tilde{g}} \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} p^i \right) \mathrm{d}\tilde{v}. \tag{135}$$

Thus, defining the convected polarization density through the standard tensorial transformation

$$\tilde{p}^{\tilde{\imath}} \doteq \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} p^i, \tag{136}$$

we may write that in the convected coordinates

$$\tilde{q} = \tilde{q}_\mathrm{F} - \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{\imath}}} \left(\sqrt{\tilde{g}}\, \tilde{p}^{\tilde{\imath}} \right), \tag{137}$$

so that

$$\epsilon_0 \int\limits_\Omega \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{\imath}}} \left\{ \sqrt{\tilde{g}} \left[\tilde{g}^{\tilde{\imath}\tilde{\jmath}} \left(\tilde{e}_{\tilde{\jmath}} + \sqrt{\tilde{g}}\, \epsilon_{\tilde{\jmath}\tilde{k}\tilde{m}} \tilde{\psi}^{\tilde{k}} \tilde{b}^{\tilde{m}} \right) + \tilde{p}^{\tilde{\imath}} \right] \right\} \mathrm{d}\tilde{v} = \int\limits_\Omega \tilde{q}_\mathrm{F}\, \mathrm{d}\tilde{v}. \tag{138}$$

Comparison of this equation with equation 128, coupled with the realization that the volume of integration is arbitrary, leads to the final form of the convected constitutive law:

$$\tilde{d}^{\tilde{m}} = \epsilon_0 \tilde{g}^{\tilde{m}\tilde{\imath}} \left(\tilde{e}_{\tilde{\imath}} + \sqrt{\tilde{g}}\, \epsilon_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \tilde{\psi}^{\tilde{\jmath}} \tilde{b}^{\tilde{k}} \right) + \tilde{p}^{\tilde{m}}. \tag{139}$$

This is not of the form $\mathbf{d} = \epsilon_0 \mathbf{e} + \mathbf{p}$ seen in $\mathfrak{s}$. Therefore, if we are to believe this equation, we must believe that the laws of physics depend on the observer, in direct contradiction to the principle of relativity. The reader may object that this particular equation results only from our decision to force the macroscopic Maxwell equations to have the same form in different coordinate systems. This, of course, is true, but if we attempted to force the constitutive relation to hold in all systems, we would find the macroscopic Maxwell equations would be of different form in each system. We conclude from this that electromagnetics is incompatible with the Galilean relativity assumed in standard Newtonian physics.

6.4.2 The Ampère-Maxwell Law

We finally turn to the convective form of equation 104, the microscopic Ampère-Maxwell law. Integrating this equation, and using Stokes's theorem and the definition of the convected time derivative (equation 100), we find

$$\oint_{\partial\Gamma} \left[\frac{1}{\mu_0} b_i - \epsilon_0 \epsilon_{ijk} v^j \left(\delta^{km} e_m\right)\right] \mathrm{d}x^i = \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma} \delta^{ij} e_j \, \mathrm{d}a_i + \int_{\Gamma} \left[j^i - \epsilon_0 \frac{\partial}{\partial x^k} \left(\delta^{jk} e_k\right) v^i\right] \mathrm{d}a_i. \quad (140)$$

The integrals are now converted over to convective coordinates using the usual formulas, giving

$$\oint_{\partial\Gamma} \left[\frac{1}{\mu_0} b_i - \epsilon_0 \epsilon_{ijk} v^j \left(\delta^{km} e_m\right)\right] \frac{\partial x^i}{\partial \tilde{x}^{\tilde{\imath}}} \mathrm{d}\tilde{x}^{\tilde{\imath}}$$

$$= \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma} \delta^{ij} e_j \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} \mathrm{d}\tilde{a}_{\tilde{\imath}} + \int_{\Gamma} \left[j^i - \epsilon_0 \frac{\partial}{\partial x^k} \left(\delta^{jk} e_k\right) v^i\right] \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} \mathrm{d}\tilde{a}_{\tilde{\imath}}. \quad (141)$$

From the last term of this equation, we immediately recognize the convective total current

$$\tilde{j}^{\tilde{\imath}} = \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} \left(j^i - q v^i\right). \quad (142)$$

Now, the left-hand side of equation 141 can be manipulated algebraically to show that

$$\left[\frac{1}{\mu_0} b_i - \epsilon_0 \epsilon_{ijk} v^j \left(\delta^{km} e_m\right)\right] = \frac{1}{\mu_0} \tilde{g}_{\tilde{\imath}\tilde{m}} \tilde{b}^{\tilde{m}} + \epsilon_0 \sqrt{\tilde{g}}\, \epsilon_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \tilde{\psi}^{\tilde{\jmath}} \tilde{g}^{\tilde{k}\tilde{m}} \left(\tilde{e}_{\tilde{m}} + \sqrt{\tilde{g}}\, \epsilon_{\tilde{m}\tilde{n}\tilde{p}} \tilde{\psi}^{\tilde{n}} \tilde{b}^{\tilde{p}}\right). \quad (143)$$

Inserting these relations into equation 141, and once again invoking the inverse of equation 124 and recognizing the arbitrariness of the integration volume, gives the final form of the microscopic Ampère-Maxwell law:

$$\frac{1}{\sqrt{\tilde g}}\epsilon_{\tilde\imath\tilde\jmath\tilde k}\frac{\partial}{\partial\tilde x^{\tilde\jmath}}\left\{\sqrt{\tilde g}\left[\frac{1}{\mu_0}\tilde g_{\tilde k\tilde\ell}\tilde b^{\tilde\ell}+\epsilon_0\sqrt{\tilde g}\,\epsilon_{\tilde k\tilde m\tilde n}\tilde\psi^{\tilde m}\tilde g^{\tilde n\tilde p}\left(\tilde e_{\tilde p}+\sqrt{\tilde g}\,\epsilon_{\tilde p\tilde r\tilde s}\tilde\psi^{\tilde r}\tilde b^{\tilde s}\right)\right]\right\}$$
$$=\epsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde g^{\tilde\imath\tilde\jmath}\left(\tilde e_{\tilde\jmath}+\sqrt{\tilde g}\,\epsilon_{\tilde\jmath\tilde k\tilde\ell}\tilde\psi^{\tilde k}\tilde b^{\tilde\ell}\right)\right]+\tilde\jmath^{\tilde\imath}. \quad (144)$$

This result is inconsistent with the principle of relativity; it looks nothing like its spatial version.

Finally, we need to derive the magnetic constitutive relations in convective coordinates from this equation. We proceed in a manner similar to that for the electric relationship. To begin, we can separate the current and charge into their component parts using equations 112 and 108, and insert these definitions into the integral of equation 144. Thus, equation 144 is written as

$$\oint_{\partial\Gamma}\left[\frac{1}{\mu_0}\tilde g_{\tilde\imath\tilde\ell}\tilde b^{\tilde\ell}+\epsilon_0\sqrt{\tilde g}\,\epsilon_{\tilde\imath\tilde\jmath\tilde k}\tilde\psi^{\tilde\jmath}\tilde g^{\tilde k\tilde\ell}\left(\tilde e_{\tilde\ell}+\sqrt{\tilde g}\,\epsilon_{\tilde\ell\tilde r\tilde s}\tilde\psi^{\tilde r}\tilde b^{\tilde s}\right)\right]\mathrm{d}\tilde x=$$
$$\epsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma\tilde g^{\tilde\imath\tilde\jmath}\left(\tilde e_{\tilde\jmath}+\sqrt{\tilde g}\,\epsilon_{\tilde\jmath\tilde r\tilde s}\tilde\psi^{\tilde r}\tilde b^{\tilde s}\right)\mathrm{d}\tilde a_{\tilde\imath}$$
$$+\int_\Gamma\frac{\partial\tilde x^{\tilde\imath}}{\partial x^i}\left\{j^i_{\mathrm F}+\frac{\partial p^i}{\partial t}+\epsilon^{ijk}\left[\frac{\partial m_k}{\partial x^j}+\frac{\partial}{\partial x^j}\left(\epsilon_{k\ell m}p^\ell v^m\right)\right]-\left(q_{\mathrm F}-\frac{\partial p^j}{\partial x^j}\right)v^i\right\}\mathrm{d}\tilde a_{\tilde\imath}. \quad (145)$$

The conduction current is immediately recognizable here, and so can be simplified out of the messy last integral. The resulting expression reads

$$\oint_{\partial\Gamma}\left[\frac{1}{\mu_0}\tilde g_{\tilde\imath\tilde\ell}\tilde b^{\tilde\ell}+\epsilon_0\sqrt{\tilde g}\,\epsilon_{\tilde\imath\tilde\jmath\tilde k}\tilde\psi^{\tilde\jmath}\tilde g^{\tilde k\tilde\ell}\left(\tilde e_{\tilde\ell}+\sqrt{\tilde g}\,\epsilon_{\tilde\ell\tilde r\tilde s}\tilde\psi^{\tilde r}\tilde b^{\tilde s}\right)\right]\mathrm{d}\tilde x^{\tilde\imath}$$
$$=\epsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma\tilde g^{\tilde\imath\tilde\jmath}\left(\tilde e_{\tilde\jmath}+\sqrt{\tilde g}\,\epsilon_{\tilde\jmath\tilde r\tilde s}\tilde\psi^{\tilde r}\tilde b^{\tilde s}\right)\mathrm{d}\tilde a_{\tilde\imath}$$
$$+\int_\Gamma\frac{\partial\tilde x^{\tilde\imath}}{\partial x^i}\left\{\frac{\partial p^i}{\partial t}+\epsilon^{ijk}\left[m_{k,j}+\frac{\partial}{\partial x^j}\left(\epsilon_{k\ell m}p^\ell v^m\right)\right]+p^j_{\ ,j}v^i\right\}\mathrm{d}\tilde a_{\tilde\imath}+\int_\Gamma\tilde\jmath^{\tilde\imath}_{\mathrm C}\mathrm{d}\tilde a_{\tilde\imath}, \quad (146)$$

where

$$\tilde\jmath^{\tilde\imath}_{\mathrm C}\doteq\frac{\partial\tilde x^{\tilde\imath}}{\partial x^i}j^i_{\mathrm C}=\frac{\partial\tilde x^{\tilde\imath}}{\partial x^i}\left(j^i_{\mathrm F}-q_{\mathrm F}v^i\right)=\tilde\jmath^{\tilde\imath}_{\mathrm F}+\tilde q_{\mathrm F}\tilde\psi^{\tilde\imath}. \quad (147)$$

Next, by transforming the Levi-Civita symbol and invoking Stokes's theorem, we can show that

$$\int_\Gamma\frac{\partial\tilde x^{\tilde\imath}}{\partial x^i}\epsilon^{ijk}\frac{\partial m_k}{\partial x^j}\,\mathrm{d}\tilde a_{\tilde\imath}=\oint_{\partial\Gamma}m_i\frac{\partial x^i}{\partial\tilde x^{\tilde\imath}}\mathrm{d}\tilde x^{\tilde\imath}, \quad (148)$$

from which we gather that

$$\tilde{m}_{\tilde{k}} \doteq \frac{\partial x^k}{\partial \tilde{x}^{\tilde{k}}} m_k. \tag{149}$$

Finally, we invoke the definition of the convected time derivative (equation 100) to simplify the terms involving the polarization:

$$\int_{\Gamma} \frac{\partial \tilde{x}^{\tilde{\imath}}}{\partial x^i} \left[\frac{\partial p^i}{\partial t} + \epsilon^{ijk} \frac{\partial}{\partial x^j} \left(\epsilon_{k\ell m} p^\ell v^m \right) + \frac{\partial p^j}{\partial x^j} v^i \right] \mathrm{d}\tilde{a}_{\tilde{\imath}} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma} \tilde{p}^{\tilde{\imath}} \,\mathrm{d}\tilde{a}_{\tilde{\imath}}. \tag{150}$$

Substituting all of these equations back into equation 146 finally yields

$$\begin{aligned} &\oint_{\partial\Gamma} \left[\frac{1}{\mu_0} \tilde{g}_{\tilde{\imath}\tilde{\ell}} \tilde{b}^{\tilde{\ell}} + \epsilon_0 \sqrt{\tilde{g}}\, \epsilon_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \tilde{\psi}^{\tilde{\jmath}} \tilde{g}^{\tilde{k}\tilde{\ell}} \left(\tilde{e}_{\tilde{\ell}} + \sqrt{\tilde{g}}\, \epsilon_{\tilde{\ell}\tilde{r}\tilde{s}} \tilde{\psi}^{\tilde{r}} \tilde{b}^{\tilde{s}} \right) - \tilde{m}_{\tilde{\imath}} \right] \mathrm{d}\tilde{x}^{\tilde{\imath}} \\ &\qquad = \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma} \left[\tilde{g}^{\tilde{\imath}\tilde{\jmath}} \left(\tilde{e}_{\tilde{\jmath}} + \sqrt{\tilde{g}}\, \epsilon_{\tilde{\jmath}\tilde{r}\tilde{s}} \tilde{\psi}^{\tilde{r}} \tilde{b}^{\tilde{s}} \right) + \tilde{p}^{\tilde{\imath}} \right] \mathrm{d}\tilde{a}_{\tilde{\imath}} + \int_{\Gamma} \tilde{\jmath}^{\tilde{\imath}}_{\mathrm{C}} \,\mathrm{d}\tilde{a}_{\tilde{\imath}}. \end{aligned} \tag{151}$$

Recognizing the expression for $\tilde{d}^{\tilde{\imath}}$ buried in here, we find that the constitutive relationship

$$\tilde{h}_{\tilde{\imath}} = \frac{1}{\mu_0} \tilde{g}_{\tilde{\imath}\tilde{\ell}} \tilde{b}^{+} \epsilon_0 \sqrt{\tilde{g}}\, \epsilon_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \tilde{\psi}^{\tilde{\jmath}} \tilde{g}^{\tilde{k}\tilde{\ell}} \left(\tilde{e}_{\tilde{\ell}} + \sqrt{\tilde{g}}\, \epsilon_{\tilde{\ell}\tilde{m}\tilde{n}} \tilde{\psi}^{\tilde{m}} \tilde{b}^{\tilde{n}} \right) - \tilde{m}_{\tilde{\imath}} \tag{152}$$

recovers equation 119. Notice that once again, this equation does not resemble its spatial form. Finally, a careful accounting of the dispensation of all of the terms in the above demonstrates that

$$\tilde{\jmath}^{\tilde{\imath}} = \tilde{\jmath}^{\tilde{\imath}}_{\mathrm{C}} \epsilon^{\tilde{\imath}\tilde{\jmath}\tilde{k}} \frac{\partial \tilde{m}_{\tilde{k}}}{\partial \tilde{x}^{\tilde{\jmath}}} + \frac{\mathrm{d}\tilde{p}^{\tilde{\imath}}}{\mathrm{d}t}. \tag{153}$$

The convective contribution of the polarization density present in the spatial domain is absent here as it is bound to the material that is not moving in the this frame.

7. Conclusions

This report has documented the formulation of physical laws in moving reference frames, with extra attention paid to the Maxwell equations of electromagnetic theory. The first part of the report, making up its bulk, describes how the expression of physical theories change with changing coordinate systems. In particular, this portion of the work discusses what sort of relationships physical quantities must obey so that the resulting theory is consistent when the coordinate system that forms the basis for its description is curvilinear, nonorthogonal, and time-varying. In particular, the discussion shows that two different sets of basis vectors are convenient for the expression of physical law in nonorthogonal systems: A covariant set (called $\tilde{\mathbf{u}}_{\tilde{\imath}}$ in the text), in which each basis vector points in the direction of variation of a given coordinate holding all other coordinates fixed, and a contravariant set (called $\tilde{\mathbf{u}}^{\tilde{\imath}}$ in the text) pointing orthogonal to the constant coordinate surfaces. In orthonormal systems, these bases are identical, but in nonorthogonal systems they are merely biorthogonal.

These different bases are chosen merely because they are useful for the expression of physical laws, and, in general, have no deep physical significance in and of themselves. Changing from one system to the other is merely a mathematical change of basis; for the underlying physical theory to be meaningful, such a transformation must be physically irrelevant. (A theory that said otherwise would imply that the world is affected by our description of it!) When the Maxwell equations are formulated in the material coordinate systems of continuum mechanics this simple consistency requirement fails, and for this reason, the literature is filled with a profusion of contradictory theories about how the Maxwell equations should be expressed in the coordinate systems used by continuum mechanics other than the spatial.

The second "half" of this report seeks to clarify the expression of the Maxwell equations in coordinate systems relevant to continuum mechanics. In so doing, it subtly illustrates two primary points:

- The consistency or inconsistency of a theory cannot be determined by its expression in *material* or *"Lagrangian"* coordinates, but must be determined in a coordinate system describing the same physics as the spatial system, such as the *convective* coordinate system.
- Under standard Newtonian physics, invoking the Galilean transformation for moving systems, the convective form of the Maxwell laws *cannot* be consistently formulated, resulting in the confusion in the literature. Consistent formulation requires the theory of relativity and is the subject of the sequel to this report.

We discuss each of these points in turn.

The first point is perhaps most clearly understood as it results from the definition of material coordinates as merely labels for material points; they function as a way of tracking individual particles on their journey through space and time. For this reason, not only need they not correspond to the state of the body in question at the initial time, they need not match the physical arrangement of the body at any time whatsoever. They need merely represent a potential state of the body so that the mapping describing its motion retains some simple topological consistency. As a mere labeling of points, there is no need for any physical theory to be satisfied in material coordinates; the configuration they describe is a fiction. This leaves the theorist at a loss in distinguishing between different expressions of the theory in the material frame, since there is no reason for preferring one transformation to any other.

In this regard, we also note that for precisely the same reason, the experimentalist cannot resolve the issue. The "value of the electric field in material coordinates" is not something that can be measured since it refers to a coordinate system that may never have existed in the lab. This is not a mere matter of accessibility either; there is simply no field that can be measured or even suitably manipulated *a posteriori* into a quantity that can be coherently called the material electric field.

The second point is made clear from section 6. By forcing the macroscopic Maxwell equations to have the same form in convective coordinates, some variables are forced to be covariant and others are forced to be contravariant. While the Maxwell equations still stand in a recognizable form with the proper choices, the resulting constitutive laws are bizarre. The equations transforming fields between frames are coupled, which should not be the case if the electric and magnetic fields are independent physical entities. Worse still, indices cannot be "raised" or "lowered" in the usual way, a situation literally tantamount to saying that the physics of the situation depends on a choice of basis, i.e., on an arbitrary issue of mathematical depiction. These problems (with the description in convective coordinates) spring from the constancy of the speed of light relative to all observers predicted by the Maxwell relations but not by Newtonian physics, and they cannot be remedied without appeal to the special theory of relativity.

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