Optimal Placement of Heterogeneous Sensors in Target Tracking

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Abstract - Target tracking performance is determined by the fidelity of target mobility model (F, Q), tracking sensor measurement quality (R), and sensor-to-target geometry (H). A tracking sensor manager has choices in sensor selection/placement (H), waveform design (R), and filter tuning (F and Q), thus affecting the tracking performance in many ways. This paper concerns with the geometry aspect of sensor placement so as to optimize the tracking performance. Recently, a considerable amount of work has been published on optimal conditions for instantaneous placement of homogeneous sensors (same type and same measurement quality) in which the targets are either assumed perfectly known or the target location uncertainty is averaged out via the expected value of the determinant of the Fisher information matrix. In this paper, we derive conditions for optimal placement of heterogeneous sensors based on maximization of the updated Fisher information matrix from an arbitrary prior characterizing the uncertainty about the initial target location. The heterogeneous sensors can be of the same or different types (ranging sensors, bearing-only sensors, or both). The sensors can also make, over several time steps, multiple independent measurements of different qualities.

Keywords: Performance Evaluation, Fisher Information Matrix, Sensor Placement, Heterogeneous Sensors, Measurement Quality

1. Introduction

Target tracking performance is determined by the fidelity of target mobility model (**F**, **Q**), tracking sensor measurement quality (**R**), and sensor-to-target geometry (**H**). The effectiveness of a sensor update is implicitly dependent of the range to target via signal to noise ratio (SNR) for a ranging sensor while explicit for a bearing-only sensor. A tracking sensor manager has choices in sensor placement (**H**), waveform design (**R**), and filter tuning (**F** and **Q**) to control tracking performance. This paper is concerned with the geometrical aspect of sensor placement so as to optimize the tracking performance.

Placement of *m* sensors around a target has drawn a considerable amount of attention recently. Interesting results are summarized in a recent paper [1] where the determinant of the Fisher information matrix (FIM) is maximized so as to obtain necessary and sufficient conditions for optimal placement of ranging sensors, bearing-only sensors, and time of arrival (TOA) and time difference of arrival (TDOA) sensors, respectively. Similarly, the use of the trace of the Cramer-Rao lower bound (CRLB), which is the inverse of the FIM, is considered in [5, 6, 7] for TDOA measurements.

As pointed out in [2], a majority of work in optimal sensor placement including the references cited above make a fundamental assumption that the target location is *known* perfectly, which is hardly practical but establishes some theoretical bounds. When the uncertainty in target location is characterized by a probability distribution such as truncated radially symmetric distributions, the expected value of the FIM determinant is used in [2] to obtain useful results with a combination of analysis and simulation.

Furthermore, the above references assume homogeneous sensors. That is, they are of the same type and of the same quality for their respective necessary and sufficient conditions of optimal placement to hold.

In this paper, we consider the problem of optimal placement of heterogeneous sensors in tracking of uncertain targets. By heterogeneous, we mean the use of sensors that are different in sensor type and measurement quality. In addition, to account for target uncertainty, we use the updated FIM to derive optimal placement conditions.

Different from previous work, which is mostly formulated in a static snapshot fashion, we consider a tracking application in which the time factor is involved and the sensor placement actually occurs in time and in space. As an example, a number of UAVs receive a handover message from a surveillance aircraft in a layered sensing scenario and they coordinate to track a designated target in a cooperative manner. Since a UAV is flying at a constant altitude and solves for a ground target location, a 2-D model is reasonable and will be considered as in most sensor placement literature.

Assume that the tracking sensors under consideration will make a unit movement per unit time and each movement has an associated cost. Due to higher risk for covertness

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14. ABSTRACT
Target tracking performance is determined by the fidelity of target mobility model (F, Q), tracking sensor measurement quality (R), and sensor-to-target geometry (H). A tracking sensor manager has choices in sensor selection/placement (H), waveform design (R), and filter tuning (F and Q), thus affecting the tracking performance in many ways. This paper concerns with the geometry aspect of sensor placement so as to optimize the tracking performance. Recently, a considerable amount of work has been published on optimal conditions for instantaneous placement of homogeneous sensors (same type and same measurement quality) in which the targets are either assumed perfectly known or the target location uncertainty is averaged out via the expected value of the determinant of the Fisher information matrix. In this paper, we derive conditions for optimal placement of heterogeneous sensors based on maximization of the updated Fisher information matrix from an arbitrary prior characterizing the uncertainty about the initial target location. The heterogeneous sensors can be of the same or different types (ranging sensors, bearing-only sensors, or both). The sensors can also make, over several time steps, multiple independent measurements of different qualities.

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and survivability, it is reasonable to assume that a radial movement to get closer to a target has a higher cost while tangential movement carries a constant cost. A UAV thus faces four choices:

- *Do not move and take independent measurements*. If the current geometry is favorable, the accumulation of independent measurements has the effect of reducing measurement errors. A steady state is soon reached and no further reduction can be expected for this geometry.
- *Move radially toward the target*. This runs a higher cost for movement but has the benefits of increasing SNR and improving the measurement quality, particularly for bearing-only sensors.
- *Move tangentially around the target*. This allows maximum change in viewing geometry if moving in the direction of the principal axis of the error ellipse.
- A combination of all the above.

The above approach decides a move from one time step to next and is therefore a greedy one. The objective is to seek the best outcome in terms of best accuracy in minimum time and cost with least risk.

The rest of the paper is organized as follows. In Section 2, the sensor models, optimality criteria, and information updating equations are introduced. Section 3 presents three strategies for placing two homogenous and heterogeneous sensors, either co-located or distributed. Simulation results are analyzed in Section 4. Finally, concluding remarks are provided in Section 5.

2. Information Updating

Consider a target with an unknown state (*e.g.*, position and velocity) $\mathbf{x}$ and the i-th sensor with a known state $\mathbf{x}_i$. The i-th sensor's measurement is given by

$$z_i = f^{l_i}(\mathbf{x}, \mathbf{x}_i) + v_i \tag{1}$$

where $f^{l_i}(\cdot, \cdot)$ is typically a nonlinear equation, the superscript $l_i \in \{r, \theta, \phi, \ldots\}$ is a label indicating a particular type of nonlinearity for the i-th sensor: range, azimuth, and elevation, and v_i is the sensor measurement error, assumed to be a zero-mean Gaussian with variance (σ^{l_i})^2, denoted by $\mathcal{N}\{0, (\sigma^{l_i})^2\}$. In this paper, the dimension of the state vector $\mathbf{x}$ is n.

Given m such sensors, the relationship between the measurements and target state $\mathbf{x}$ can be represented in matrix-vector form as

$$\mathbf{z} = \mathbf{f}(\mathbf{x}) + \mathbf{v} \tag{2}$$

where

$$\mathbf{z}^T = \begin{bmatrix} z_1 & z_2 & \cdots & z_m \end{bmatrix} \tag{3a}$$

$$\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \tag{3b}$$

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f^{l_1}(\mathbf{x}, \mathbf{x}_1) & \cdots & f^{l_m}(\mathbf{x}, \mathbf{x}_m) \end{bmatrix} \tag{3c}$$

and $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$ where the superscript T stands for vector or matrix transpose.

For an unbiased estimate $\hat{\mathbf{x}}$ of $\mathbf{x}$, the Cramer-Rao lower bound (CRLB) states that:

$$E\{(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T\} \geq CRLB = \mathcal{M}^{-1}(\mathbf{x}) = \mathcal{P}(\mathbf{x}) \tag{4a}$$

where $\mathcal{M}(\mathbf{x})$ is called the Fisher information matrix (FIM) and its ij^{th} element is defined as:

$$[\mathcal{M}(\mathbf{x})]_{ij} = E\{\frac{\partial}{\partial x_i}\ln(\mathbf{f}(\mathbf{x}))^T \frac{\partial}{\partial x_j}\ln(\mathbf{f}(\mathbf{x}))\} \tag{4b}$$

It is clear from (4a) that if $\mathcal{M}(\mathbf{x})$ is singular, no unbiased estimator exists for $\mathbf{x}$ with a finite variance. When the measurement noise is Gaussian, the FIM is given by:

$$\mathcal{M}(\mathbf{x}) = \nabla_{\mathbf{x}}^T \mathbf{f}(\mathbf{x}) \mathbf{R}^{-1} \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}) = \mathbf{H}^T \mathbf{R}^{-1} \mathbf{H} \tag{5}$$

where $\nabla_{\mathbf{x}} \mathbf{f}$ is the Jacobian of the measurement vector $\mathbf{f}$ with respect to $\mathbf{x}$, denoted by $\mathbf{H}$.

In [1], the determinant of the FIM is maximized so as to obtain necessary and sufficient conditions for optimal placement of ranging sensors, bearing-only sensors, and time of arrival (TOA) and time difference of arrival (TDOA) sensors, respectively. Since $\mathbf{f}$ is also a function of $\{\mathbf{x}_i, i = 1, \ldots, m\}$, the optimization can be formulated as:

$$\{\mathbf{x}_1^*, \ldots, \mathbf{x}_m^*\} = \arg \max_{\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}} \det(\mathcal{M}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_m))$$
$$= \arg \max_{\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}} \det(\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H}) \tag{6}$$

Note that the FIM is evaluated at $\mathbf{x}$, which implicitly assumes that the target location is known perfectly. In practice, however, only a coarse estimate of the target state is available. Assume that the prior on the target state is characterized by a distributed denoted by $p(\mathbf{x})$. The expected value of the determinant of the FIM is used in [2] for optimal sensor placement:

$$\{\mathbf{x}_1^*, \ldots, \mathbf{x}_m^*\} = \arg \max_{\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}} \int \det(\mathcal{M}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_m)) p(\mathbf{x}) d\mathbf{x} \tag{7}$$

In this paper, we consider a Gaussian-distributed initial target state as $\mathbf{x} \sim \mathcal{N}(\mathbf{x}_0, \mathbf{P}_0)$ and perform optimal sensor placement based on the updated FIM evaluated around the predicted state $\mathbf{x}_0$.

Given the initial target state distribution as $\mathbf{x} \sim p(\mathbf{x}) = \mathcal{N}(\mathbf{x}_0, \mathbf{P}_0)$, the maximum a posterior (MAP) estimator is actually a nonlinear least squares estimator given by

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} [(\mathbf{z} - \mathbf{f}(\mathbf{x}))^T \mathbf{R}^{-1}(\mathbf{z} - \mathbf{f}(\mathbf{x})) + (\mathbf{x} - \mathbf{x}_0)^T \mathbf{P}_0^{-1}(\mathbf{x} - \mathbf{x}_0)] \tag{8}$$

The covariance of the state estimate can be approximated by linearizing $\mathbf{f}(\mathbf{x})$ about the mean of the prior. Specifically, one assumes that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + \mathbf{H}(\mathbf{x} - \mathbf{x}_0) \tag{9a}$$

$$\mathbf{H}^T = \begin{bmatrix} \mathbf{h}_1 & \cdots & \mathbf{h}_m \end{bmatrix} \tag{9b}$$

$$\mathbf{h}_i = \nabla_{\mathbf{x}} f^{l_i}\Big|_{\mathbf{x}_0} = \left[\frac{\partial f^{l_i}}{\partial x_1} \quad \frac{\partial f^{l_i}}{\partial x_2} \quad \cdots \quad \frac{\partial f^{l_i}}{\partial x_n} \right]^T_{\mathbf{x}_0} \tag{9c}$$

where $\nabla_{\mathbf{x}} f$ stands for the gradient of the scalar function f with respect to $\mathbf{x}$.

By inserting (9) into (8), it is easy to show that the linear least squares estimate of the target state is

$$\hat{\mathbf{x}} = \mathbf{x}_0 + (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H} + \mathbf{P}_0^{-1})^{-1} \mathbf{H}^T \mathbf{R}^{-1}(\mathbf{z} - \mathbf{f}(\mathbf{x}_0)) \tag{10}$$

and the covariance of the state estimate for the posterior distribution is

$$\mathbf{P} = E\{(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T\} = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H} + \mathbf{P}_0^{-1})^{-1} \tag{11}$$

The covariance given by (11) is also the CRLB derived from the joint distribution of the target state and measurements. From (4a), $\mathbf{M} = \mathbf{P}^{-1}$ and the information updating equation is given by

$$\mathbf{M} = \mathbf{M}_0 + \mathbf{H}^T \mathbf{R}^{-1} \mathbf{H}. \tag{12}$$

Usually, the sensors are spatially disparate so that the measurement errors v_i are statistically independent with variance σ_i^2. Then, the information update equation can be rewritten as

$$\mathbf{M} = \mathbf{M}_0 + \sum_{i=1}^m \frac{1}{\sigma_i^2} \mathbf{h}_i \mathbf{h}_i^T = \mathbf{M}_0 + \sum_{i=1}^m \frac{1}{\tilde{\sigma}_i^2} \tilde{\mathbf{h}}_i \tilde{\mathbf{h}}_i^T, \tag{13a}$$

where

$$\tilde{\mathbf{h}}_i = \frac{\mathbf{h}_i}{\|\mathbf{h}_i\|} \quad \text{and} \quad \tilde{\sigma}_i^2 = \frac{\sigma_i^2}{\|\mathbf{h}_i\|^2}. \tag{13b}$$

Inspection of (13) reveals that each sensor provides a rank-one positive definite addition to the information matrix, which increases the "size" of $\mathbf{M}$. The unit-norm $\tilde{\mathbf{h}}_i$ represents the angular orientation of the single sensor contribution in information space, and the reciprocal of $\tilde{\sigma}_i^2$ represents the power (or quality) of the sensor measurement.

Table 1 provides the value of $\tilde{\mathbf{h}}_i$ and $\tilde{\sigma}_i^2$ for 2-D range and bearing measurements in Cartesian, polar (r, θ), and spherical (r, θ, ϕ) coordinates, respectively. In the 2-D case, the angle θ of the LOS from the sensor to target is measured relative to the x-axis. Table 1 reveals that the orientation $\tilde{\mathbf{h}}_i$ of range sensors align with the LOS vector between the sensor and the target. On the other hand, the orientations of bearing sensors are orthogonal to this LOS vector. The quality of the range sensors only depends on the actual measurement variance. However, for a given angular measurement error, the quality of sensor degrades as the distance to the target increases because the corresponding state value is a cross-range quantity. Furthermore, it is reasonable to assume that the measurement error also increases as the target/sensor distance expands due to a decrease in the SNR. Without

accounting for environmental effects, the measurement error for a point target is ideally invariant to the angular coordinates. As a first order approximation, it is reasonable to assume that $\tilde{\mathbf{h}}_i$ and $\tilde{\sigma}_i^2$ only depend on the angular and range coordinates of the sensor, respectively, in a polar coordinate system.

Table 1. Sensor Measurement and Linearized Equations in 2D

(a) 2D Active Sensors with Ranging Measurements

$$f^{r_i}(\mathbf{x}, \mathbf{x}_i) = \sqrt{(x - x_i)^2 + (y - y_i)^2} \qquad \tilde{\sigma}_i^2 = \sigma_i^2$$

$$\tilde{\mathbf{h}}_i = \left[\frac{x_0 - x_i}{\tilde{r}_i} \quad \frac{y_0 - y_i}{\tilde{r}_i} \right]^T = [\cos(\theta_i) \quad \sin(\theta_i)]^T$$

$$\tilde{r}_i = \sqrt{(x_0 - x_i)^2 + (y_0 - y_i)^2}$$

(b) 2D Passive Sensors with Bearing-Only Measurements

$$f^{\theta_i}(\mathbf{x}, \mathbf{x}_i) = \tan^{-1}\left(\frac{y - y_i}{x - x_i} \right) \qquad \tilde{\sigma}_i^2 = \tilde{r}_i^2 \sigma_i^2$$

$$\tilde{\mathbf{h}}_i = \left[-\frac{y_0 - y_i}{\tilde{r}_i} \quad \frac{x_0 - x_i}{\tilde{r}_i} \right]^T = [-\sin(\theta_i) \quad \cos(\theta_i)]^T$$

3. Optimal Placement in Time and Space

For 2D scenarios, a minimum of two sensors are sufficient to obtain an optimal solution. The two sensors can be of the same type, observing a target from two different directions at the same time. Or the two sensors can be of different types, observing the target from the same direction (co-located) at the same time. This is an instantaneous placement with single-look optimality. For stationary or slow-moving targets, sensors can stay at the same place or move to different locations while taking multiple measurements. This is a temporal placement with multiple-look optimality. Conditions for optimal placement are derived below.

3.1 Optimality Criteria

Assume that the number of sensor measurements is constrained to be m. From (13a), it is easy to see that the trace of the updated information matrix for homogeneous measurements is

$$T = \text{trace}(\mathbf{M}_0) + \frac{m}{\tilde{\sigma}^2}$$

This trace of the information matrix accumulates the sensor quality of all sensor updates involved without regard to the effects of angular orientation of the sensors. The following theorem places a lower bound on the achievable updated covariance matrix $\mathbf{P}$.

Theorem 1. Consider an $n \times n$ covariance matrix $\mathbf{P}$. Let $\mathbf{M} = \mathbf{P}^{-1}$ be the information matrix. If $trace(\mathbf{M}) = T$ is constant, then $trace(\mathbf{P}) \geq n^2/T$, $det(\mathbf{P}) \geq n^n/T^n$. The equality occurs if $\mathbf{M} = (T/n)\mathbf{I}$ and $\mathbf{P} = (n/T)\mathbf{I}$.

In short, Theorem 1 states that if the updated covariance matrix has equal eigenvalues, no other configuration of sensors could provide a tighter covariance error when the trace of the information matrix is fixed. In this case, the

error ellipse is actually circular for the 2D cases. Theorem 1 is a generalization of a theorem in [3] and its proof is given in [8].

In general, each measurement can lower one eigenvalue of P_0. For an n-dimensional state vector, it might be possible for n-1 measurements to lower n-1 eigenvalues to the value of the lowest λ_1. However, depending on the spread of the eigenvalues, more measurements might be necessary to achieve a desired spherical error. Even if sensor placement cannot achieve a spherical state error, Theorem 1 provides intuition that it is best to try to achieve as close to uniform eigenvalues in the updated covariance as possible.

3.2 Two Homogeneous Sensors

In this section, we consider two homogeneous sensors that are of the same type and same quality. We will use the unified sensor model (13b) that accounts for both ranging and bearing-only measurements as listed in Table 1(a) and (b). Assume that the prior on the target state is $\mathbf{x} \sim \mathcal{N}(\mathbf{x}_0, \mathbf{P}_0)$. Around $\mathbf{x}_0$, we can linearize the measurement equations for given $\mathbf{x}_1$ and $\mathbf{x}_2$, leading to observation matrices $\mathbf{h}_1$ and $\mathbf{h}_1$, respectively. By a proper coordinate rotation, we can diagonalize $\mathbf{P}_0$ such that we have a diagonal information matrix prior $\mathbf{M}_0 = \mathbf{P}_0^{-1} = diag([a, b])$.

Given the prior $\mathbf{M}_0 = diag([a, b])$, we want to place two sensors at $\widetilde{\mathbf{h}}_1 = [\cos\theta_1, \sin\theta_1]^\mathrm{T}$ and $\widetilde{\mathbf{h}}_2 = [\cos\theta_2, \sin\theta_2]^\mathrm{T}$ with an equal effective measurement error variance $\widetilde{\sigma}^2$ and to take a number of k independent measurements in the placement so as to accumulate the needed gain (accuracy). What are k, θ_1, and θ_2 that result in the maximum updated information matrix?

According to (13a), the updated information matrix can be written as:

$$\mathbf{M} = \mathbf{M}_0 + \frac{k}{\widetilde{\sigma}^2}(\widetilde{\mathbf{h}}_1\widetilde{\mathbf{h}}_1^T + \widetilde{\mathbf{h}}_2\widetilde{\mathbf{h}}_2^T), \, k \geq 1$$

$$= \begin{bmatrix} a + \frac{k}{\widetilde{\sigma}^2}(\cos^2\theta_1 + \cos^2\theta_2) & \frac{k}{\widetilde{\sigma}^2}(\cos\theta_1\sin\theta_1 + \cos\theta_2\sin\theta_2) \\ \frac{k}{\widetilde{\sigma}^2}(\cos\theta_1\sin\theta_1 + \cos\theta_2\sin\theta_2) & b + \frac{k}{\widetilde{\sigma}^2}(\sin^2\theta_1 + \sin^2\theta_2) \end{bmatrix} \quad (14b)$$

To maximize the updated information matrix is equivalent to making (14b) a scaled identity matrix. To do so, we need to find k, θ_1, and θ_2 such that:

$$\cos\theta_1\sin\theta_1 + \cos\theta_2\sin\theta_2 = 0 \quad (15a)$$

$$a + \frac{k}{\widetilde{\sigma}^2}(\cos^2\theta_1 + \cos^2\theta_2) = b + \frac{k}{\widetilde{\sigma}^2}(\sin^2\theta_1 + \sin^2\theta_2) \quad (15b)$$

From the condition (15a), we have:

$$\sin 2\theta_1 + \sin 2\theta_2 = \sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2) = 0 \quad (16a)$$

$$\Rightarrow \begin{cases} \theta_1 + \theta_2 = 0 \\ \theta_1 - \theta_2 = 90^o \end{cases} \quad (16b)$$

The condition in the second row of (16b) is not valid in view of (15b) unless $a = b$. When $a = b$, the prior is

circular. The condition indicates that two placements such that $\theta_1 - \theta_2 = 90°$ provide an optimal solution, which is consistent with our intuition.

Bringing the first condition (16b) into (15b) gives:

$$\sin^2\theta_1 = \frac{1}{2}\left(\frac{\widetilde{\sigma}^2(a-b)}{2k} + 1\right) \quad (17a)$$

$$\theta_1 = -\theta_2 = \sin^{-1}\sqrt{\frac{1}{2}\left(\frac{\widetilde{\sigma}^2(a-b)}{2k} + 1\right)} \quad (17b)$$

That is, the optimal placement is for the two sensors at $\theta_1 = -\theta_2$ (symmetric about a principal axis of the error ellipse). The condition for such an optimal placement is:

$$\left|\frac{1}{2}\left(\frac{\widetilde{\sigma}^2(a-b)}{2k} + 1\right)\right| \leq 1 \quad (18a)$$

$$\left|\frac{\widetilde{\sigma}^2(a-b)}{2k}\right| \leq 1 \quad (18b)$$

It is easy to verify that when $a = b$, the angular sector between the sensors is:

$$\theta_1 - \theta_2 = 2\sin^{-1}\sqrt{\frac{1}{2}} = 90° \quad (19)$$

which is consistent with the second condition (16b).

When $|\widetilde{\sigma}^2(a-b)| > 2$, there is no single look update (i.e., $k = 1$) to achieve an instantaneous optimality. However, a number of independent updates can be used to accumulate the required gain to ensure a solution. The number is given by:

$$k^* = \sup\left\{k \geq \left[\frac{\widetilde{\sigma}^2|a-b|}{2}\right] + 1\right\} \quad (20)$$

where $sup\{\cdot\}$ is the smallest of all integers that satisfy the condition and $[\cdot]$ stands for the least integer.

When the condition for a solution (18b) is met, the angular placement θ_1 varies as a function of k. In the limit k goes to infinity, the placement solution is the same as (19) with $\theta_1 = -\theta_2 = 45^o$. That is, the two sensors maintain an angular separation of 90° and symmetric about a principal axis of the error ellipse.

The above analysis leads to an optimal strategy to the placement of two cooperative sensors. From (15b), we have:

$$\cos^2\theta_1 - \sin^2\theta_1 + \cos^2\theta_2 - \sin^2\theta_2 = \frac{(b-a)\widetilde{\sigma}^2}{k} \quad (21a)$$

$$\cos 2\theta_1 + \cos 2\theta_2 = \frac{(b-a)\widetilde{\sigma}^2}{k} \quad (21b)$$

Adding the sums of (16a) squared and (21b) squared gives:

$$\cos 2\theta_1 \cos 2\theta_2 + \sin 2\theta_1 \sin 2\theta_2 = \frac{(b-a)^2 \tilde{\sigma}^2}{2k^2} - 1 \qquad (21c)$$

$$\cos 2(\theta_1 - \theta_2) = \frac{(b-a)^2 \tilde{\sigma}^4}{2k^2} - 1 \qquad (21d)$$

$$\theta_1 - \theta_2 = \frac{1}{2}\cos^{-1}\left[\frac{(b-a)^2 \tilde{\sigma}^4}{2k^2} - 1\right] \qquad (21e)$$

From the first condition of (16b), $\theta_1 + \theta_2 = 0$, we finally have:

$$\theta_1 = -\theta_2 = \frac{1}{4}\cos^{-1}\left[\frac{(b-a)^2 \tilde{\sigma}^4}{2k^2} - 1\right] \qquad (21f)$$

The condition that $\left|\frac{(b-a)^2 \tilde{\sigma}^4}{2k^2} - 1\right| \le 1$ requires $k \ge \frac{|b-a|\tilde{\sigma}^2}{2}$, which leads to the same condition as (20). It can be shown that (21f) and (17b) are equivalent.

Note that the optimal placement condition derived in this section is for a generalized sensor with $\tilde{\mathbf{h}}$ and $\tilde{\sigma}^2$, which can represent either a ranging error or a bearing-only sensor. In the latter case, if a sensor's LOS vector to target has an angle φ with respect to the x-axis, then the observation vector's angle is given by $\theta = \varphi \pm \pi/2$. The equivalent measurement error is given by $\tilde{\sigma}^2 = r\sigma^2$ as listed in Table 1.

3.3 Two Co-Located Heterogeneous Sensors

In this section, we consider two sensors co-located on a same sensor platform. The two sensors are of different types with different measurement qualities, one providing ranging and the other bearing-only measurements. Given a prior about the unknown target $\mathbf{x} \sim \mathcal{N}(\mathbf{x}_0, \mathbf{P}_0)$, we want to determine where to place the sensor platform $\mathbf{x}_1$ relative to $\mathbf{x}_0$ and to take how many independent measurements so as to obtain maximum updated information matrix.

Again, the coordinate axes can be rotated to align with the principal axes of $\mathbf{P}_0$, leading to the prior in the diagonal form denoted by $\mathbf{M}_0 = \mathbf{P}_0^{-1} = diag([a, b])$. Around $\mathbf{x}_0$, we can linearize the measurement equations for the two sensors, leading to observation matrices $\mathbf{h}_1 = [\cos\theta, \sin\theta]^T$ and $\mathbf{h}_2 = [\sin\theta, \cos\theta]^T$ with measurement errors σ_1^2 and σ_2^2, respectively. What are k and θ that result in the maximum updated information matrix?

To start, the updated information matrix is:

$$\mathbf{M} = \mathbf{M}_0 + \frac{k}{\sigma_1^2}\mathbf{h}_1\mathbf{h}_1^T + \frac{k}{\sigma_2^2 r^2}\mathbf{h}_2\mathbf{h}_2^T \qquad (22)$$

where $r = \|\mathbf{x}_1 - \mathbf{x}_0\|_2$ is the predicted range from the sensor to target. Eq. (22) can be further written as:

$$\mathbf{M} = \begin{bmatrix} a + k\left(\frac{\cos^2\theta}{\sigma_1^2} + \frac{\sin^2\theta}{r^2\sigma_2^2}\right) & k\left(\frac{\cos\theta\sin\theta}{\sigma_1^2} - \frac{\sin\theta\cos\theta}{r^2\sigma_2^2}\right) \\ k\left(\frac{\cos\theta\sin\theta}{\sigma_1^2} - \frac{\sin\theta\cos\theta}{r^2\sigma_2^2}\right) & b + k\left(\frac{\sin^2\theta}{\sigma_1^2} + \frac{\cos^2\theta}{r^2\sigma_2^2}\right) \end{bmatrix} \quad (23)$$

The determinant of $\mathbf{M}$ is given by:

$$\det(\mathbf{M}) = \left(a + k\left(\frac{\cos^2\theta}{\sigma_1^2} + \frac{\sin^2\theta}{r^2\sigma_2^2}\right)\right)\left(b + k\left(\frac{\sin^2\theta}{\sigma_1^2} + \frac{\cos^2\theta}{r^2\sigma_2^2}\right)\right)$$
$$- k^2\left(\frac{\cos\theta\sin\theta}{\sigma_1^2} - \frac{\sin\theta\cos\theta}{r^2\sigma_2^2}\right)^2 \qquad (24a)$$

$$= \left(a + k\left(\frac{1}{\sigma_1^2} + \left(\frac{1}{r^2\sigma_2^2} - \frac{1}{\sigma_1^2}\right)\sin^2\theta\right)\right)\left(b + k\left(\frac{1}{r^2\sigma_2^2} + \left(\frac{1}{\sigma_1^2} - \frac{1}{r^2\sigma_2^2}\right)\sin^2\theta\right)\right)$$
$$- k^2\left(\left(\frac{1}{\sigma_1^2} - \frac{1}{r^2\sigma_2^2}\right)\frac{\sin 2\theta}{2}\right)^2 \qquad (24b)$$

The goal is to maximize $\det(\mathbf{M})$ (equivalent to minimize $\det(\mathbf{P})$). Since the second term on the right hand side of (24b) is always negative, it can be nullified when $\theta = 0^\circ$ or 90°.

When $\theta = 0^\circ$ or 90°, (24b) becomes:

$$\det(\mathbf{M}) = \begin{cases} \left(a + \frac{k}{\sigma_1^2}\right)\left(b + \frac{k}{r^2\sigma_2^2}\right) & \theta = 0^\circ \\[2mm] \left(a + \frac{k}{r^2\sigma_2^2}\right)\left(b + \frac{k}{\sigma_1^2}\right) & \theta = 90^\circ \end{cases} \qquad (25)$$

The difference between the two values of (25) can be evaluated, for $a > b$, as:

$$\left(a + \frac{k}{\sigma_1^2}\right)\left(b + \frac{k}{r^2\sigma_2^2}\right) - \left(a + \frac{k}{r^2\sigma_2^2}\right)\left(b + \frac{k}{\sigma_1^2}\right) = k(a-b)\left(\frac{1}{r^2\sigma_2^2} - \frac{1}{\sigma_1^2}\right) \quad (26a)$$

$$= \begin{cases} > 0, & r^2\sigma_2^2 < \sigma_1^2 \quad \Rightarrow \theta = 0^\circ \\[2mm] < 0, & r^2\sigma_2^2 > \sigma_1^2 \quad \Rightarrow \theta = 90^\circ \end{cases} \qquad (26b)$$

The above analysis indicates that the number of updates k does not affect the optimal placement and the optimal sensor pointing is either along the first principal axis or perpendicular to it. When $a > b$, that is, the largest eigenvalue along the y-axis, the sensor should be placed at $\theta = 0^\circ$ if $r^2\sigma_2^2 < \sigma_2^2$, that is, the cross-ranging error (along the y-axis) is smaller than the ranging error (along the x-axis). This strategy advocates updating the largest error direction with the best measurement. In essence, sensors are added such that the posterior $\mathbf{M}$ remains diagonal and the eigenvalue spread is as small as possible.

3.4 Two Heterogeneous Sensors

In this section, we consider a more general case with two heterogeneous sensors. We start with diagonal information matrix prior $\mathbf{M}_0 = diag([a, b])$, which is obtained by rotating the coordinate axes along with the principal axes of an arbitrary prior.

Two sensors are placed at $\tilde{\mathbf{h}}_1 = [\cos\theta_1, \sin\theta_1]^T$ and $\tilde{\mathbf{h}}_2 = [\cos\theta_2, \sin\theta_2]^T$ with their effective measurement errors being $\tilde{\sigma}_1^2$ and $\tilde{\sigma}_2^2$, respectively. Assume a number of k independent updates are taken in the placement to accumulate the needed gain (accuracy).

The resulting updated information matrix can be written as:

$$\mathbf{M} = \mathbf{M}_0 + k(\frac{1}{\widetilde{\sigma}_1^2}\widetilde{\mathbf{h}}_1\widetilde{\mathbf{h}}_1^T + \frac{1}{\widetilde{\sigma}_2^2}\widetilde{\mathbf{h}}_2\widetilde{\mathbf{h}}_2^T), \ k \geq 1 \quad (27a)$$

$$= \begin{bmatrix} a + k(\frac{\cos^2\theta_1}{\widetilde{\sigma}_1^2} + \frac{\cos^2\theta_2}{\widetilde{\sigma}_2^2}) & \frac{k\cos\theta_1\sin\theta_1}{\widetilde{\sigma}_1^2} + \frac{k\cos\theta_2\sin\theta_2}{\widetilde{\sigma}_2^2} \\ \frac{k\cos\theta_1\sin\theta_1}{\widetilde{\sigma}_1^2} + \frac{k\cos\theta_2\sin\theta_2}{\widetilde{\sigma}_2^2} & b + k(\frac{\sin^2\theta_1}{\widetilde{\sigma}_1^2} + \frac{\sin^2\theta_2}{\widetilde{\sigma}_2^2}) \end{bmatrix} \quad (27b)$$

To make (27b) a scaled identity matrix, we need find k, θ_1, and θ_2 such that:

$$\frac{k\cos\theta_1\sin\theta_1}{\widetilde{\sigma}_1^2} + \frac{k\cos\theta_2\sin\theta_2}{\widetilde{\sigma}_2^2} = 0 \quad (28a)$$

$$a + k(\frac{\cos^2\theta_1}{\widetilde{\sigma}_1^2} + \frac{\cos^2\theta_2}{\widetilde{\sigma}_2^2}) = b + k(\frac{\sin^2\theta_1}{\widetilde{\sigma}_1^2} + \frac{\sin^2\theta_2}{\widetilde{\sigma}_2^2}) \quad (28b)$$

From the condition (28a), we have:

$$\frac{\sin 2\theta_1}{\widetilde{\sigma}_1^2} + \frac{\sin 2\theta_2}{\widetilde{\sigma}_2^2} = 0 \quad (29a)$$

$$\frac{\sin^2 2\theta_1}{\widetilde{\sigma}_1^4} + \frac{2\sin 2\theta_1 \sin 2\theta_2}{\widetilde{\sigma}_1^2\widetilde{\sigma}_2^2} + \frac{\sin^2 2\theta_2}{\widetilde{\sigma}_2^4} = 0 \quad (29b)$$

From (28b), we have:

$$\frac{\cos 2\theta_1}{\widetilde{\sigma}_1^2} + \frac{\cos 2\theta_2}{\widetilde{\sigma}_2^2} = \frac{b-a}{k} \quad (30a)$$

$$\frac{\cos^2 2\theta_1}{\widetilde{\sigma}_1^4} + \frac{2\cos 2\theta_1 \cos 2\theta_2}{\widetilde{\sigma}_1^2\widetilde{\sigma}_2^2} + \frac{\cos^2 2\theta_2}{\widetilde{\sigma}_2^4} = \left(\frac{b-a}{k}\right)^2 \quad (30b)$$

The sum of (29b) and (30b), after easy manipulation, leads to:

$$\cos 2(\theta_1 - \theta_2) = \left(\frac{b-a}{k}\right)^2 \frac{\widetilde{\sigma}_1^2\widetilde{\sigma}_2^2}{2} - \frac{\widetilde{\sigma}_1^4 + \widetilde{\sigma}_2^4}{2\widetilde{\sigma}_1^2\widetilde{\sigma}_2^2} \quad (31a)$$

$$2\theta_1 = 2\theta_2 + \alpha \quad (31b)$$

$$\alpha = \cos^{-1}\left[\left(\frac{b-a}{k}\right)^2 \frac{\widetilde{\sigma}_1^2\widetilde{\sigma}_2^2}{2} - \frac{\widetilde{\sigma}_1^4 + \widetilde{\sigma}_2^4}{2\widetilde{\sigma}_1^2\widetilde{\sigma}_2^2}\right] \quad (31c)$$

Brining (31b) to (29a) gives:

$$\left(\frac{\cos\alpha}{\widetilde{\sigma}_1^2} + \frac{1}{\widetilde{\sigma}_2^2}\right)\sin 2\theta_2 + \frac{\sin\alpha}{\widetilde{\sigma}_1^2}\cos 2\theta_2 = 0 \quad (32a)$$

$$\cos\beta\sin 2\theta_2 + \sin\beta\cos 2\theta_2 = 0 \quad (32b)$$

$$\sin(2\theta_2 + \beta) = 0 \quad (32c)$$

$$\theta_2 = -\frac{\beta}{2}, \ \theta_1 = \frac{\alpha-\beta}{2} \quad (32d)$$

where

$$\cos\beta = \frac{A}{A^2+B^2}, \ \sin\beta = \frac{B}{A^2+B^2} \quad (33a)$$

$$A = \frac{\cos\alpha}{\widetilde{\sigma}_1^2} + \frac{1}{\widetilde{\sigma}_2^2}, \ B = \frac{\sin\alpha}{\widetilde{\sigma}_1^2} \quad (33b)$$

The above analysis shows a closed form solution for 2-D optimal placement of 2 sensors. However, there may be no single look update (i.e., $k = 1$) to achieve an instantaneous optimality.

But a number of independent updates can be used to accumulate the required gain to ensure a solution. From (31a), the number is given by:

$$\left[\frac{|a-b|\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right] + 1 \leq k < \left[\left|\frac{(a-b)\sigma_1^2\sigma_2^2}{\sigma_1^2-\sigma_2^2}\right|\right] + 1 \quad (34)$$

The upper bound in (34) only limits k to achieve a scaled identity information matrix, not on the ability to further increase the information matrix, which has no limit on k.

Within the bounds in (34), for each k, there is a pair of optimal angular placements $\{\theta_1, \theta_2\}$ as given by (32d) and (32e).

Since the optimal placement condition is derived for generalized sensors, $\widetilde{\mathbf{h}}$ and $\widetilde{\sigma}^2$ can represent either a ranging error or a bearing-only sensor. Again in the latter case, if a sensor's LOS vector to target has an angle φ with respect to the x-axis, then the observation vector's angle is given by $\theta = \varphi \pm \pi/2$. The equivalent measurement error is given by $\widetilde{\sigma}^2 = r\sigma^2$ as listed in Table 1.

3.5 Generalization of Existence Conditions

In [1], a condition on ranges of bearing-only sensors is formulated (see (43) in Theorem 9, [1]), which determines if it is possible or not to get all the eigenvalues equal so as to maximize the determinant of the FIM. If not, it requires that one sensor, which is much closer to target than the rest, be $\pi/2$ to all other sensors that must stay collinear.

The condition for the remaining sensors to be collinear is equivalent to a single sensor having the same LOS vector with an equivalent range to target. It is also equivalent to a single sensor making multiple independent observations so that the resulting variance-weighted inverse range is comparable to the closer sensor. Either equivalence enables FIM to be diagonalized. The condition becomes a two-sensor scenario, which is similar to what is presented in Section 3.1 and 3.3.

A generalization of the condition (43) of Theorem 9 in [1] is as follows. For m sensors with variance $\widetilde{\sigma}_i^2$, $i = 1, \ldots, m$, The conditions of Theorem 1 can be achieved for k measurement if

$$\frac{|a-b|}{k} \leq \sum_{i=1}^{m}\frac{1}{\widetilde{\sigma}_i^2} \quad (35a)$$

and for sensor j that produces the smallest measurement error variance $\widetilde{\sigma}_j^2$,

$$\frac{1}{\widetilde{\sigma}_j^2} \leq \frac{|a-b|}{k} + \sum_{i=1,i\neq j}^{m}\frac{1}{\widetilde{\sigma}_i^2} \quad (35b)$$

Note the first condition (35a) puts a lower bound on a possible k and the second term (35b) put the upper bound. For two sensors, this is equivalent to (34).

4. Simulation Examples

In this section, three examples are used to illustrate the optimal placement conditions derived in previous sections.

Example 1. When the prior covariance is not circular, placing sensors evenly all around the error ellipse is not an optimal approach. Similarly, optimizing the angular orientation of a sensor, one measurement at a time, is not an optimal strategy, either. These two points are illustrated by the following numerical example in contrast to the optimal condition given in Section 3.2.

Consider the prior covariance $\mathbf{P}_0 = [20\ 0;\ 0\ 1]$ and two sensor updates with equal effective measurement error $\tilde{\sigma}^2 = 1$. Following the optimal single sensor update strategy of selecting the angular orientation that aligns with the maximum eigenvector, the updated error covariance is

$$\tilde{\mathbf{h}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{P} = \begin{bmatrix} 0.9524 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \tilde{\mathbf{h}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \mathbf{P} = \begin{bmatrix} 0.9524 & 0 \\ 0 & 0.5 \end{bmatrix}$$

This choice of angular positions for the two sensors surrounds the target and the resulting performance measures for $\mathbf{P}$ are $tp_1 = trace(\mathbf{P}) = 1.4524$ and $dp_1 = det(\mathbf{P}) = 0.4762$. The corresponding performance measures for $\mathbf{M}$ are $tm_1 = trace(\mathbf{M}) = 3.05$ and $dm_1 = det(\mathbf{M}) = 2.30$.

However, it is possible to position the sensors to provide a tighter covariance error. Namely,

$$\tilde{\mathbf{h}}_1 = \begin{bmatrix} \sqrt{0.7375} \\ \sqrt{0.2625} \end{bmatrix}, \tilde{\mathbf{h}}_2 = \begin{bmatrix} \sqrt{0.7375} \\ -\sqrt{0.2625} \end{bmatrix} \Rightarrow \mathbf{P} = \begin{bmatrix} 0.6557 & 0 \\ 0 & 0.6557 \end{bmatrix}$$

From (36), it is easy to see that the placement angles are $\theta_1 = -\theta_2 = 30.8203^\circ$. For this placement, the performance measures for $\mathbf{P}$ are $tp_2 = trace(\mathbf{P}) = 1.3115$ and $dp_2 = det(\mathbf{P}) = 0.4300$. The corresponding performance measures for $\mathbf{M}$ are $tm_2 = trace(\mathbf{M}) = 3.05$ and $dm_2 = det(\mathbf{M}) = 2.3256$. Clearly, under the same measurement quality ($tm_1 = tm_2 = 3.05$), the second choice is better ($dm_2 = 2.3256 > dm_1 = 2.3$).

Fig. 1 shows the color-coded surface of the determinant of the updated FIM (14a) as a function of for θ_1 and θ_2 for this example with $\mathbf{M}_0 = [0.05\ 0;\ 0\ 1]$ and $\tilde{\sigma}^2 = 1$. Dark red areas indicate large values whereas dark blue areas indicate small values.

As shown, there are four pairs of maximum values. The first pair occurs when $\theta_1 = 30.8203^\circ$ and $\theta_2 = 149.1797^\circ$ and 329.1797°, respectively. Note that the two values of θ_2 are off by 180°, thus being along the same direction. For the second pair, the peaks appear at $\theta_1 = 149.1797^\circ$ and $\theta_2 = 30.8203^\circ$ and 210.8203°, respectively. Again the two values of θ_2 are off by 180°. It is clear that the value of θ_1 for the second pair is in fact that of θ_2 in the first pair. θ_1 and θ_2 switch their positions in the two pairs. Finally, the third and fourth pairs are repeated patterns of the first and second pairs with 180° in θ_1.
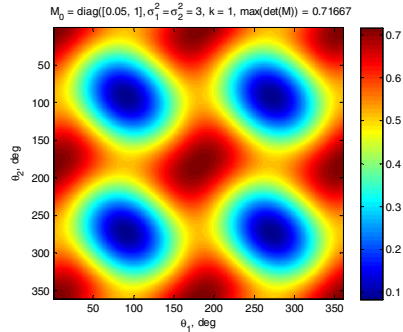

Fig. 1. Determinant of Updated FIM vs. θ_1 and θ_2 for Given $\mathbf{M}_0$ and $\tilde{\sigma}^2$ (Example 1)
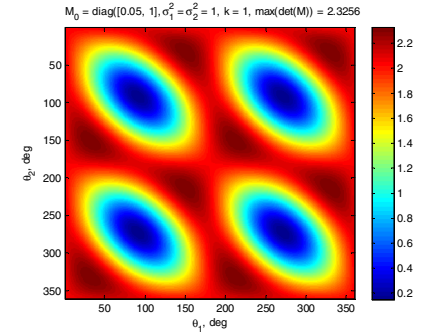
Fig. 2. Determinant of Updated FIM vs. θ_1 and θ_2 for Given $\mathbf{M}_0$ and $\tilde{\sigma}^2$ (Example 2, $k = 1$)
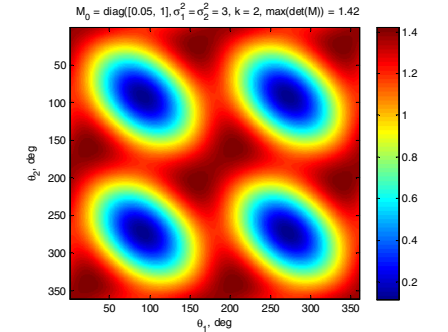
Fig. 3. Determinant of Updated FIM vs. θ_1 and θ_2 for Given $\mathbf{M}_0$ and $\tilde{\sigma}^2$ (Example 2, $k = 2$)
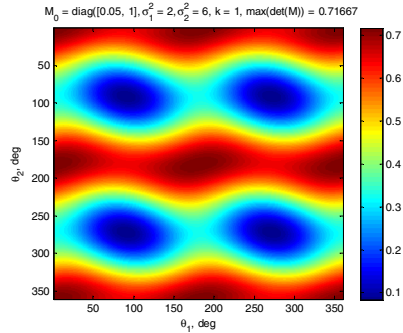
Fig. 4. Determinant of Updated FIM vs. θ_1 and θ_2 for Given $\mathbf{M}_0$ and $\tilde{\sigma}^2$ (Example 3, $k = 1$)
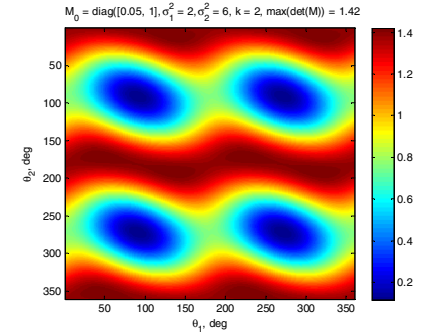
Fig. 5. Determinant of Updated FIM vs. θ_1 and θ_2 for Given $\mathbf{M}_0$ and $\tilde{\sigma}^2$ (Example 3, $k = 2$)
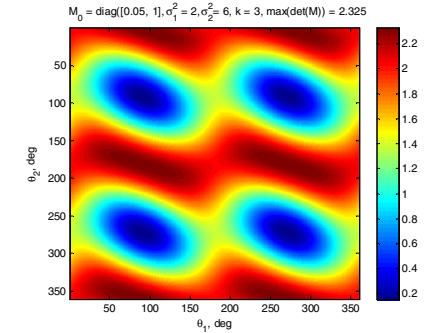
Fig. 6. Determinant of Updated FIM vs. θ_1 and θ_2 for Given $\mathbf{M}_0$ and $\tilde{\sigma}^2$ (Example 3, $k = 3$

Example 2. In the second example, we still have $\mathbf{M}_0 = $ [0.05 0; 0 1] but $\tilde{\sigma}^2 = 3$. It is easy to verify from (18) that there is no single-look optimal placement. The color-coded surface of the determinant of the updated FIM as a function of for θ_1 and θ_2 for this example with $k = 1$ is shown in Fig. 2. The largest values occur at $\theta_1 = 0^\circ$ ($\theta_1 = 180^\circ$) and $\theta_2 = 0^\circ$ and 180°, respectively, meaning along the largest eigenvalue direction.

However, if the sensors take two independent measurements per placement, the resulting updated FIM is shown in Fig. 3. The first pair of peaks occur at ($\theta_1 = 22.2807^\circ$, $\theta_2 = 337.7193^\circ$) and ($\theta_1 = 22.2807^\circ$, $\theta_2 = 157.7193^\circ$) where the two values θ_2 are off by 180°, pointing to the same direction.

The second pair of peaks occur when θ_1 and θ_2 just switch their values, that is, ($\theta_1 = 157.7193^\circ$, $\theta_2 = 202.2807^\circ$) and ($\theta_1 = 157.7193^\circ$, $\theta_2 = 22.2807^\circ$). Again, the two values θ_2 are off by 180°, pointing to the same direction.

The third and fourth pairs repeat the first and second pairs every 180° as discussed in Example 1 and they actually point to the same direction.

Example 3. In the third example, we still have $\mathbf{M}_0 = $ [0.05 0; 0 1] but $\tilde{\sigma}_1^2 = 2$ and $\tilde{\sigma}_2^2 = 6$. It is easy to verify from (34) that there is no single-look optimal placement. The color-coded surface of the determinant of the updated FIM as a function of for θ_1 and θ_2 for this example with $k = 1$ is shown in Fig. 4. As in Example 2 (Fig. 2), when the optimal condition is not met, the peaks occur at $\theta_1 = 0^\circ$ ($\theta_1 = 180^\circ$) and $\theta_2 = 0^\circ$ and 180°, respectively, meaning along the largest eigenvalue direction.

However, if the sensors take two independent measurements per placement, the resulting updated FIM is shown in Fig. 5 for $k = 2$. The first pair of peaks occur at ($\theta_1 = 43.7034^\circ$, $\theta_2 = -10.4142^\circ$) and ($\theta_1 = 43.7034^\circ$, $\theta_2 = 169.5858^\circ$) where again the two values θ_2 are off by 180°. The second to fourth pairs of peaks appear following the same patterns as discussed in previous examples.

When $k \geq 3$, the optimal condition (34) does not hold. Indeed, the resulting updated FIM with three independent measurements is shown in Fig. 6. As shown, when the optimal condition is not met, the peaks occur at $\theta_1 = 90^\circ$ ($\theta_1 = 270^\circ$) and $\theta_2 = 0^\circ$ and 180°, respectively. This indicates the desired updates are along the two eigenvalue directions.

5. Conclusions

In this paper we derived conditions for optimal placement of two heterogeneous sensors in tracking of uncertain targets over several time steps. The conditions were derived based on the maximization of the updated Fisher information matrix from an arbitrary prior characterizing the uncertainty about the initial location of a target. The two heterogeneous sensors can be of the same or different types such as ranging sensors, bearing-only sensors, or both. The sensors can also offer measurements of different qualities. The results presented in this paper complement other recently published work on instantaneous optimal placement that considered same sensor types and same measurement qualities for targets at perfectly known locations or with the target location uncertainty averaged out via the expected value of the determinant of the Fisher information matrix.

Future work will consider the cases where two cooperative mobile sensors take their measurements neither at the same rate nor at the same time where two numbers of independent update k_1 and k_2 can used to account for this asynchronous nature of operations with distributed sensors. Another effort is to extend the results of this paper to $m > 2$ sensors. In the spirit of [2], one may take the expectation of a matrix measure of choice and one choice is the trace of the inverse of $\mathbf{M}$. It is of great interest to bring in such aspects as placement costs and risks into consideration, which will be another direction of our future study.

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