



# Complex Networks/ Foundations of Information Systems

5 MAR 2012

Robert J. Bonneau, Ph.D.  
AFOSR/RSL

Air Force Research Laboratory



*Integrity ★ Service ★ Excellence*

# Report Documentation Page

Form Approved  
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE <b>05 MAR 2012</b>		2. REPORT TYPE		3. DATES COVERED <b>00-00-2012 to 00-00-2012</b>	
4. TITLE AND SUBTITLE <b>Complex Networks/Foundations Of Information Systems</b>				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) <b>Air Force Research Laboratory ,Wright-Patterson AFB,45, OH</b>				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT <b>Approved for public release; distribution unlimited</b>					
13. SUPPLEMENTARY NOTES <b>Presented at the Air Force Office of Scientific Research (AFOSR) Spring Review Arlington, VA 5 through 9 March, 2012</b>					
14. ABSTRACT					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT <b>Same as Report (SAR)</b>	18. NUMBER OF PAGES <b>32</b>	19a. NAME OF RESPONSIBLE PERSON
a. REPORT <b>unclassified</b>	b. ABSTRACT <b>unclassified</b>	c. THIS PAGE <b>unclassified</b>			



# 2012 Spring Review



**NAME: Complex Networks/Foundations of Information Systems**

**BRIEF DESCRIPTION OF PORTFOLIO:**

Complex Networks and Foundations of Information Systems uses measured information to assure, manage, predict, and design distributed networks, systems, and architectures

**LIST SUB-AREAS IN PORTFOLIO:**

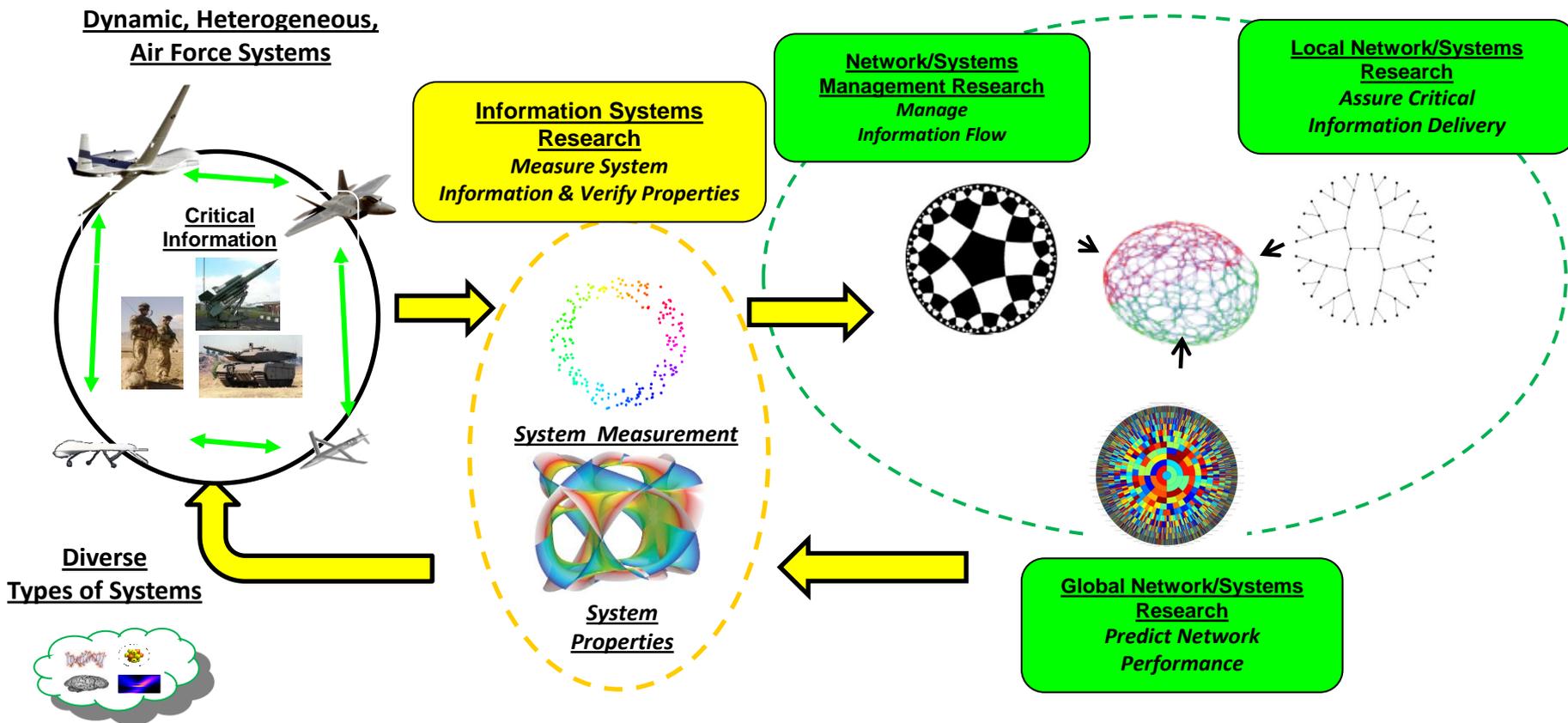
- *Local Network Research*: Coding that assures information delivery and security
- *Network Management Research*: Network and system protocols to maximize information flow
- *Global Network Research*: Predict network performance and design robustness
- *Foundations Information Systems Research (new initiative FY12)*: Measure and verify information system properties



# Complex Networks and Information Systems Roadmap



Complex networks and systems uses measured information to assure, manage, predict, and design distributed networks, systems, and architectures





# Complex Networks and Systems



## Goals:

- Preserve critical information structure and minimize latency over a heterogeneous distributed network and system
- Ensure network and system robustness and stability under a diverse set of resource *constraints* and manage not assuming static models
- Find invariant properties for a given network and system from a distributed set of observations and predict network behavior
- Develop unifying mathematical approach to discovering fundamental principles of networks and system and use them in *network and system design*

## Payoffs:

- Preserve information structures in a network rather than just delivering packets or bits
- Quantify likelihood of a given network management policy to support critical mission functions
- Predict and manage network and system failure comprehensively



# Foundations of Information Systems



## Program Objectives

- **Model** heterogeneous distributed systems using unified mathematical framework **through previous measurement** and validate
- Verify the properties of a given system application through **measurement** of a limited set of system parameters and assess mission risk
- Define general architectural principles of **design** through unified assessment of system operating properties
- Generalize design properties to universal system architectural principles

## Payoff

- Assess and verify properties of a distributed heterogeneous system where there is limited access to its elements
- Assess dynamic Air Force system mission performance and assess risk of failure



# Complex Networks Trends



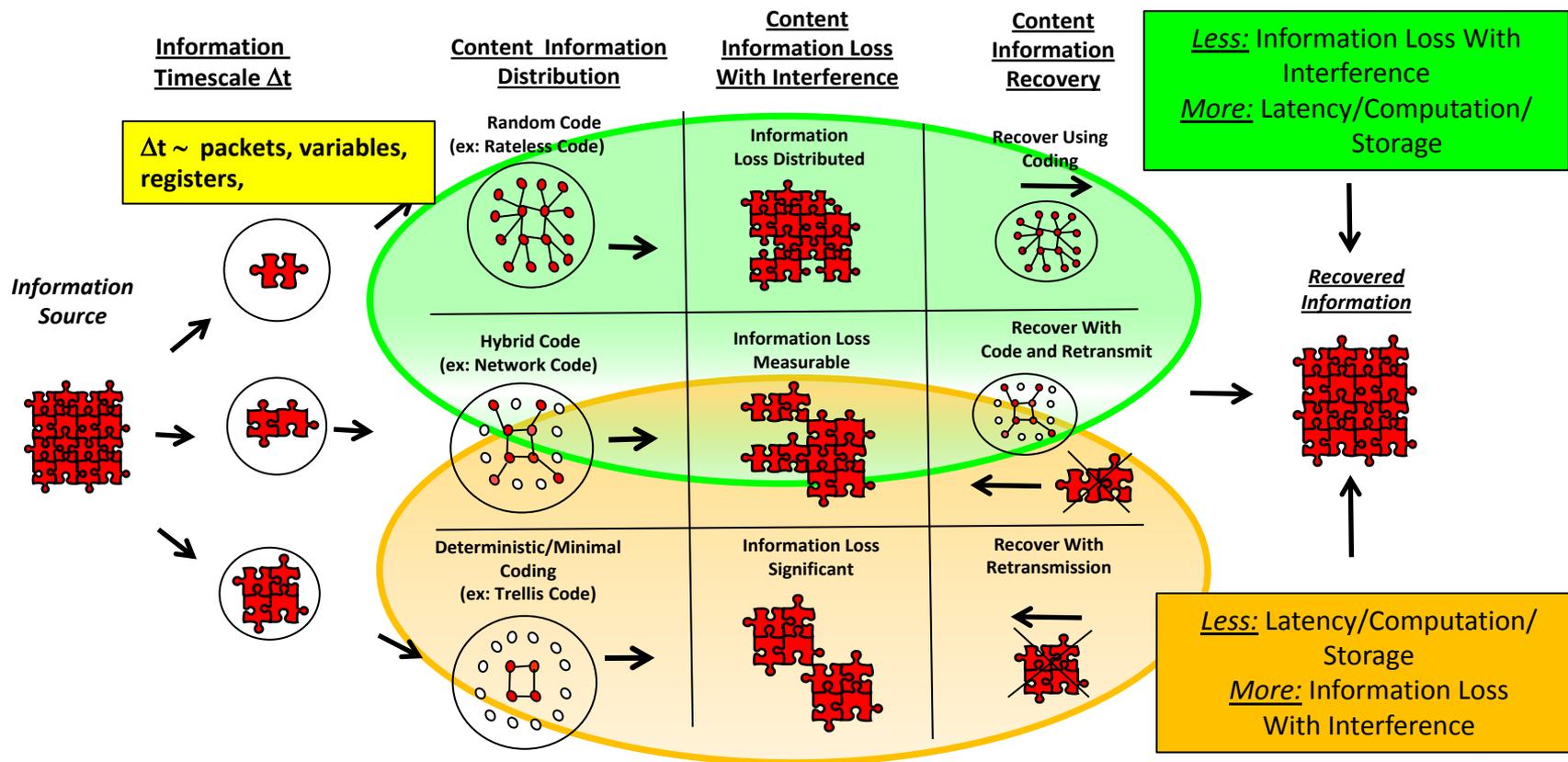
- **Local Network Theory**
  - Geometric and non-binary information coding →
  - Coding information with network performance objectives ↗
  - Integration with verification and quantum methods ↑
- **Network Management**
  - Nonparametric strategies for assessing network performance ↗
  - Distributed strategies for measuring and assessing network information transfer ↗
  - Sparse network management ↑
- **Global Network Theory**
  - Invariant metrics for analysis of network performance ↗
  - Geometric flow analysis for prediction and management of network performance ↗
  - Global state space taxonomy and categorization ↑
- **Information Systems Research**
  - Combined network, software, and hardware analysis ↑
  - Defining correct input data for given mathematical assessment ↑



# Local Network/System Research: Preserving Information Content



- Statistical geometric coding structures are used to transport diverse sets of information in a network and system and preserve its critical structure





# Sparse Approximation for Network Codes

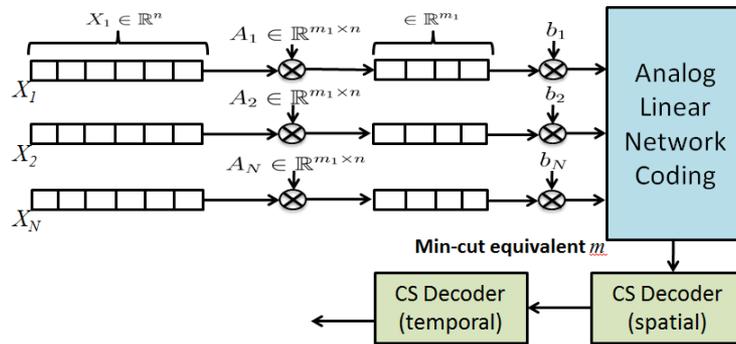


Muriel Medard, MIT

**Approach:** Sparse approximation can be used to construct sparse codes for different classes of networks

**Payoff:** Sparse approximation criteria such as information coherence can be used to guarantee different quality of service over networks without retransmission

## Encoder Architecture



## Decoder Architecture

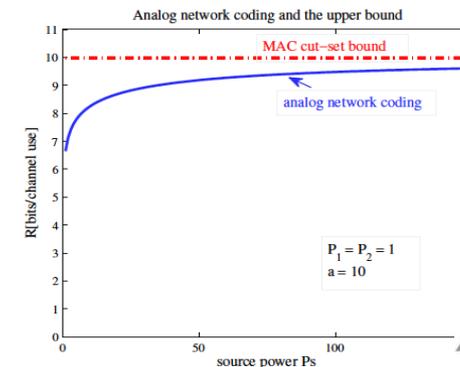
- **Sparse Decoder (LASSO)**

$$\tilde{Y} \in \arg \min_Y \frac{1}{2m} \|Z - GY\|_{l_2}^2 + \xi \|Y\|_{l_1}$$

- If \$G\$ satisfies the RE condition, then

$$\|Y - \tilde{Y}\|_{l_2}^2 \leq \frac{c}{\gamma^2} \frac{k \log(N)}{m} \sigma^2$$

**Sparse decoder can set different guarantees for information recovery over different classes of networks .**





# Minimum Interference Coding

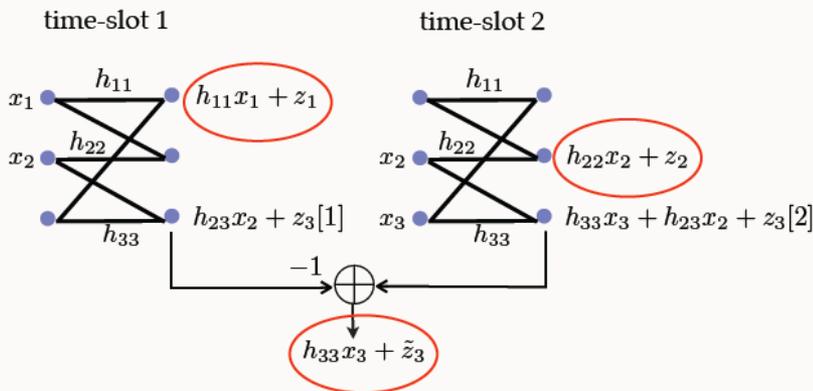
Salman Avestimehr, Cornell



**Approach:** Specific routing path configuration of networks can allow superior throughput of information based on a geometrically structured code

**Payoff:** Information transfer becomes more independent of network protocol performance and matched to time evolving network properties

## Time Sequenced Code For Multiplexed Network



$$R_{\text{sum}} = \frac{1}{2} (C_{11} + C_{22} + C_{33}) \geq \frac{1}{2} C_{\text{sum}}$$

## Network Capacities Can Be Algebraically Defined

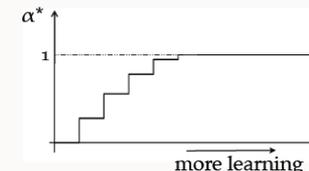
- Definition: A scheme achieves **normalized sum-rate** of  $\alpha'$  if it achieves a sum-rate satisfying

$$\sum_{i=1}^K R_i \geq \alpha C_{\text{sum}}^{\text{(full info)}} - \tau$$

for all networks compatible with  $Q$  (where  $\tau$  is a constant)

## Specific Network Capacity Achieved

- The **normalized sum-capacity**,  $\alpha^*(Q)$ , is the maximum achievable  $\alpha$



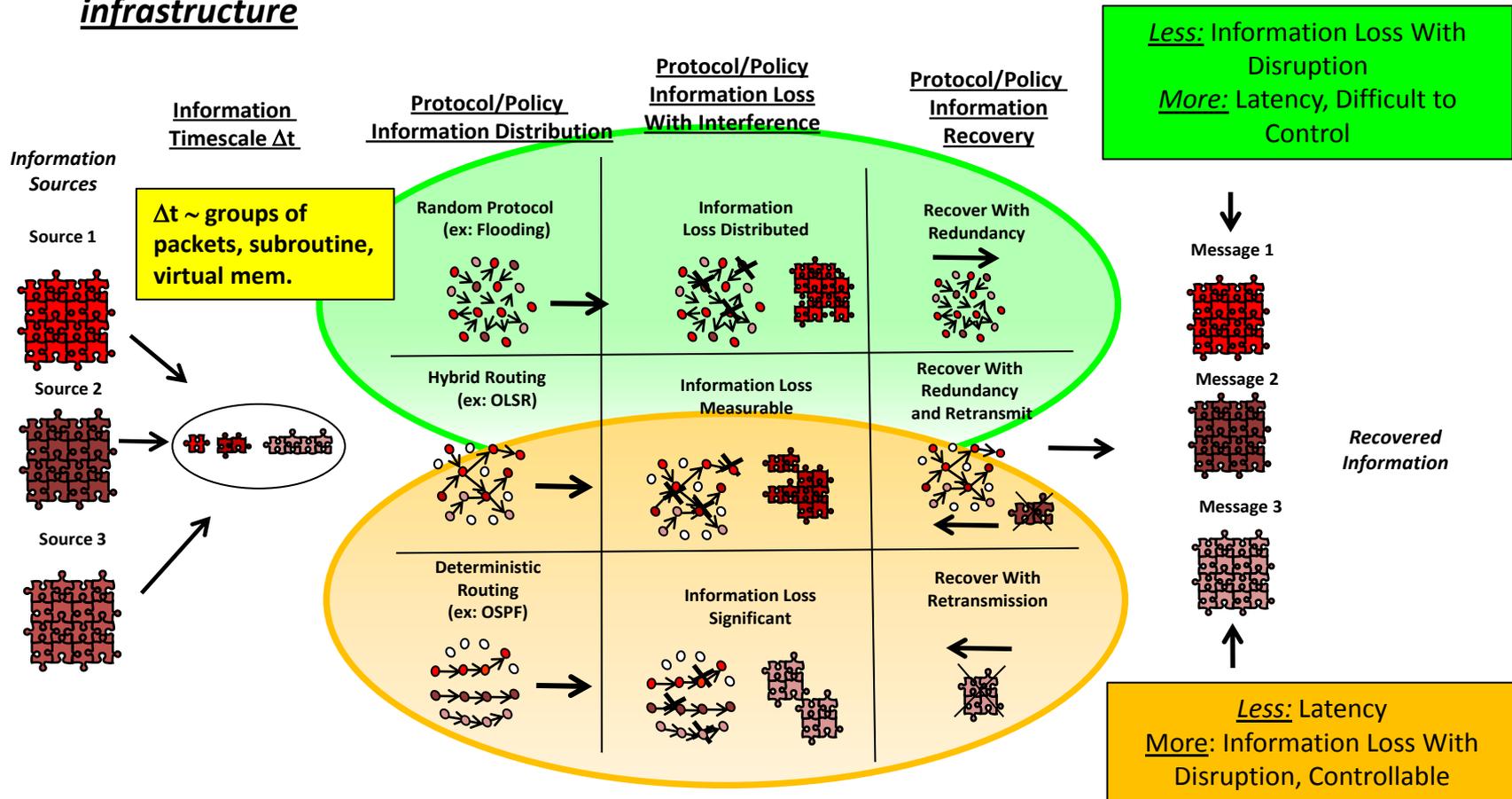


# Network/System Management Research: Guaranteeing Information Transfer



The state of information transfer on a network changes with network and system management policy and protocol

– Particularly important to the Air Force given its unique *heterogeneous mobile infrastructure*





# Complex Network Information Exchange In Random Wireless Environments

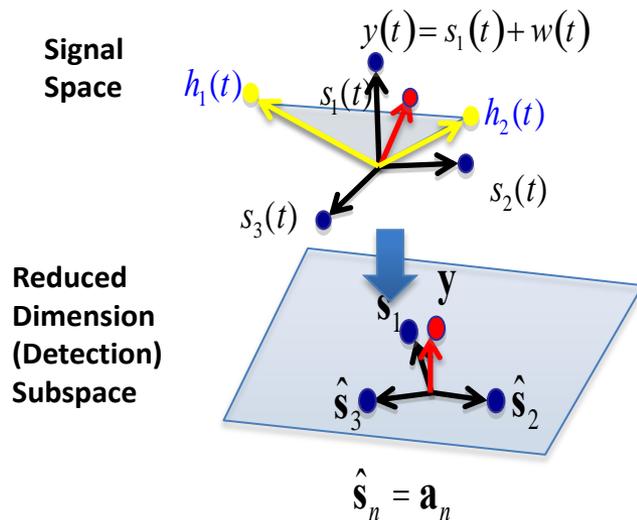
PIs: A. Goldsmith, Yonina Eldar



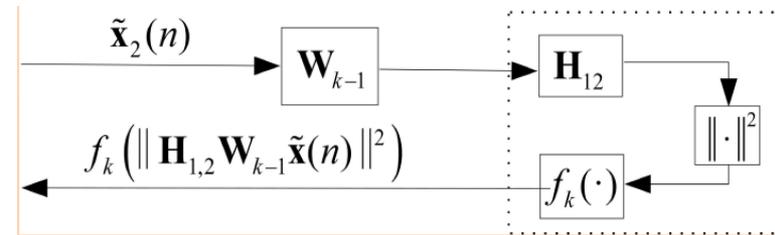
**Approach:** Most physical layer network management routines requires complete and independent measurement of the entire signal space to identify and decode spectral transmission sequences. Sparse approximation with feedback can greatly reduced the overhead of spectral decoding.

**Payoff:** Dramatic reduction the amount of spectrum needed and computational complexity of creating and decoding spectral sequences at the physical layer resulting greater capacity throughput and less information loss.

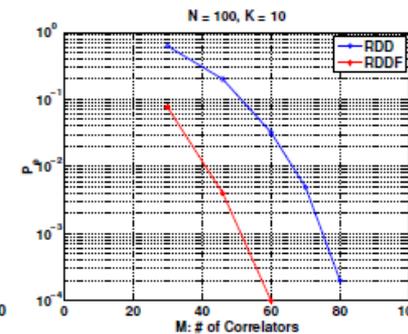
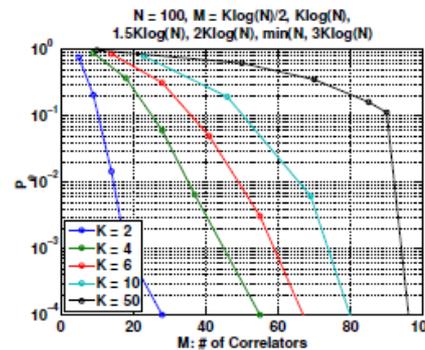
## Sparse Approximation



## Feedback at Decoder



## Greatly Increased Throuput



(a)  $P_e$  vs.  $M$  for different  $K$ , RDD.

(b)  $P_e$  vs.  $M$  for  $K = 10$ , RDD and RDDF.



# Algebraic Spectral Analysis for Resource Network Resource and Stability Analysis

Igor Mezic, UCSB



**Approach:** Modalities of a given network and information system can be discovered and characterized using algebraic spectral analysis

**Payoff:** No analytic model and very little a-priori information is needed to characterize a systems operating characteristics

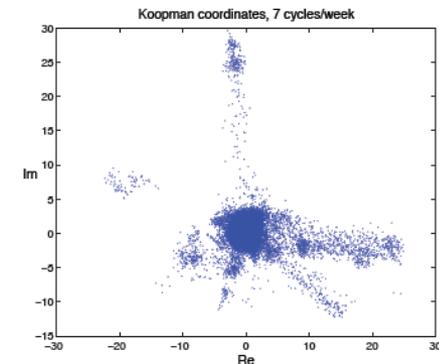
## Network Measurement Projection Operators

Recall: Projection of the function  $v_x$  on the  $j$ -th eigenspace can be obtained as

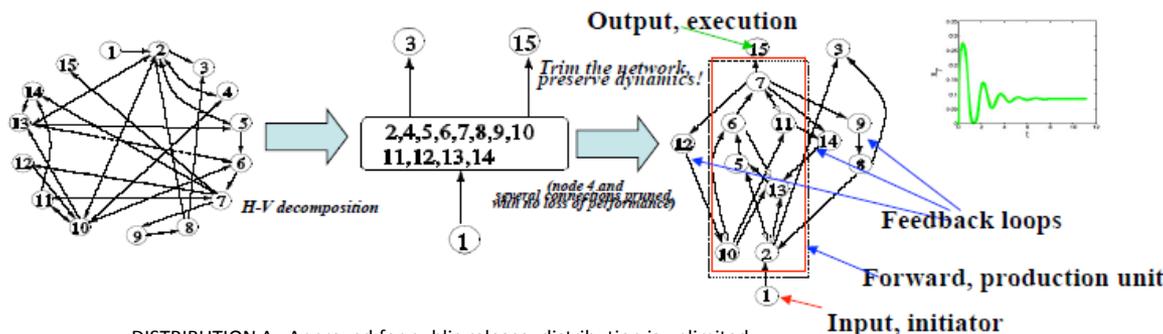
$$P_T^{\omega_j}(v_x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{i2\pi k \omega_j} v_x(T^k(m)) = z(x) f_j(m).$$

$v_x^n |_{H_1}$  is almost-periodic.

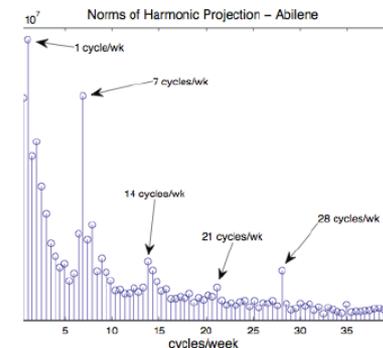
## Resulting Spectral Cluster



## Network Failure Modes Identified



## Actual Failures From Power Grid

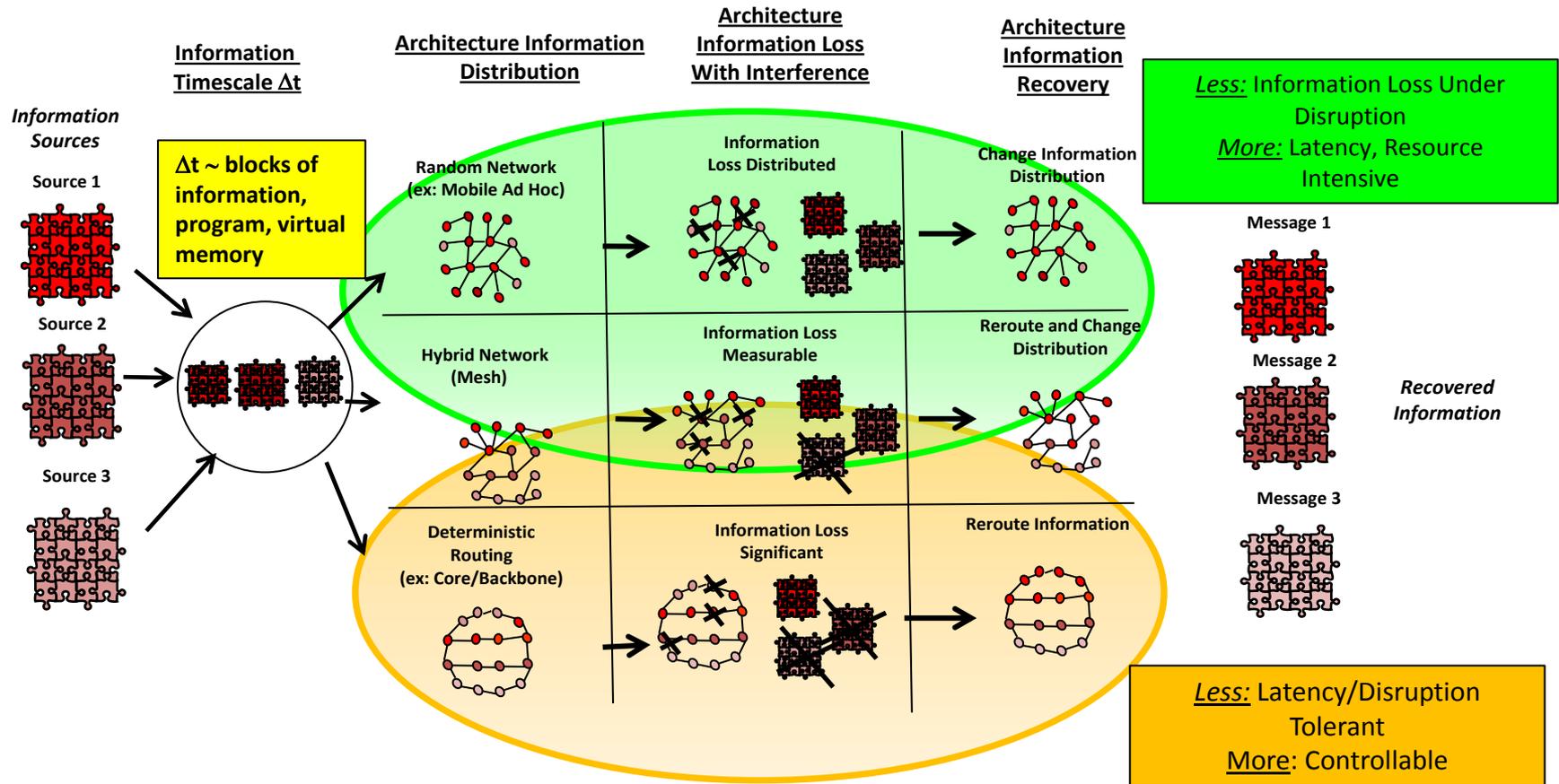




# Global Network/System Research: Architecture Performance Invariants and Prediction



- We wish to develop information invariants that can be used to assess network/system performance





# Application Tomography With Operator Algebras

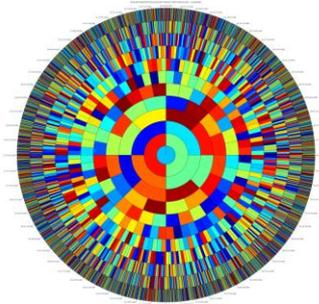
Jones, Rokhlin, Yale, Ness, Bassu Telcordia



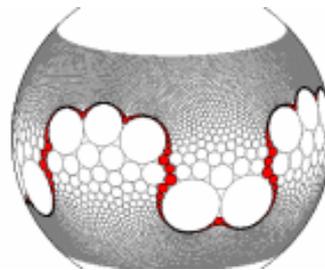
**Approach:** Software applications on clouds are extremely dynamic and can cause system failure due to unpredictable resource usage. Operator algebraic approaches can be used to track these applications.

**Payoff:** Dynamic assessment of resource failures and security threats can be tracked and managed in real time by measuring binary transactions over hardware, software, or network

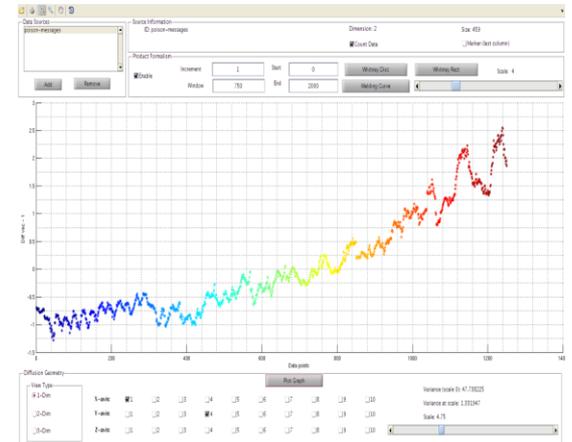
Operator Theoretic  
Network Representation



Geometric and Statistical  
Network Properties



Applications Mapped  
According to Information Flow



- Step 1. Embed the data set in  $\mathbb{R}^d$ .
- Step 2. Choose a value of  $\sigma$  to define a length scale. Build the graph Laplacian matrix  $M = (\exp\{-|x_i - x_j|^2/\sigma\})$
- Step 3. Compute the eigenvectors  $\{\Phi_k\}$ .
- Step 4. Carefully choose a small number of eigenfunctions, e.g.,  $\Phi_3, \Phi_4, \Phi_7$ . The new data set representation is given by the image

$$x_i \rightarrow \{\Phi_3(x_i), \Phi_4(x_i), \Phi_7(x_i)\}$$





# Graphlets

Fan Chung Graham/UCSD



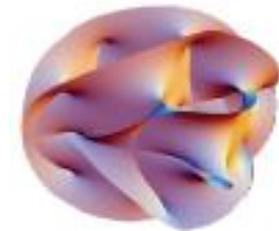
**Approach:** Graphs are typically geometric structures that do not have properties common to continuous analytic models of systems. Graphlets bridge this gap between continuous to discrete geometry

**Payoff:** Network and system data can be decomposed and convolved in multiple dimensions to describe network and system properties (*multi-resolution analysis in multiple dimensions*)

## Composable Set of Graphical Representations



## Continuous System Representation



## Analytic Graphlet Integration

◆  $G_1, G_2, \dots, G_n, \dots \rightarrow \Omega$ ,  $I - \Delta$  has two eigenvalues  $1$  and  $\rho \in (0, 1)$ .

◆  $\mu = \alpha\mu_1 + (1 - \alpha)\mu_2$  for  $\alpha \in (0, 1)$ , and

$$\int_{\Omega} f(x)((I - \Delta)g)(x)\mu(x)$$

$$= \alpha \int_{\Omega} f(x)\mu_1(x) \int_{\Omega} g(x)\mu_1(x) + (1 - \alpha) \int_{\Omega} f(x)\mu_2(x) \int_{\Omega} g(x)\mu_2(x)$$

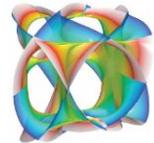
## Enables Data to System Properties



$W_n : [0, 1] \times [0, 1]$



$\Omega_n : [0, 1]^2 \times [0, 1]^2$



$\Omega$





# Foundations of Information Systems



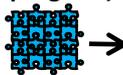
## Measure and verify information system properties among various system constraints

*Heterogeneous Information*

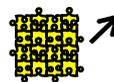
Network States  
(packets, packet blocks, packet groups)



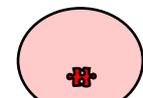
Software States  
(variable, subroutine, program)



Hardware States  
(register, ram, virt. mem)

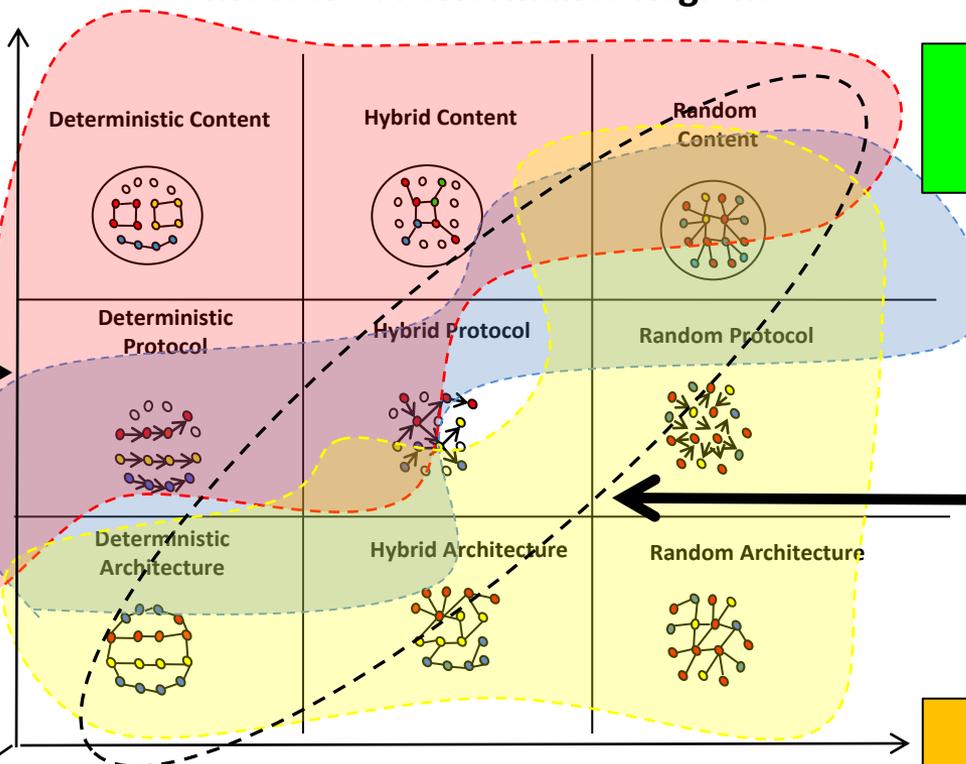


(timescale/level of abstraction)



System Measurements

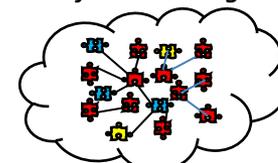
### Measured Performance Regions



*Less:* Information Loss Under Disruption/Live  
*More:* Latency, Resource Intensive/Safe



*Best Integrated Performance Region*

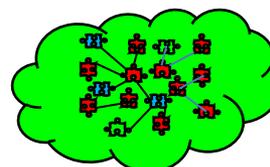
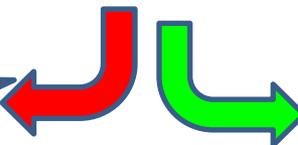
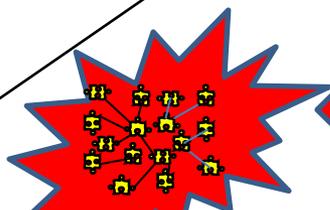


*Less:* Latency/Disruption Tolerant/Safe  
*More:* Controllable/Live

Statistical Properties

Global Properties

Unstable/Un-resourced Insecure



Stable/Resour Secure



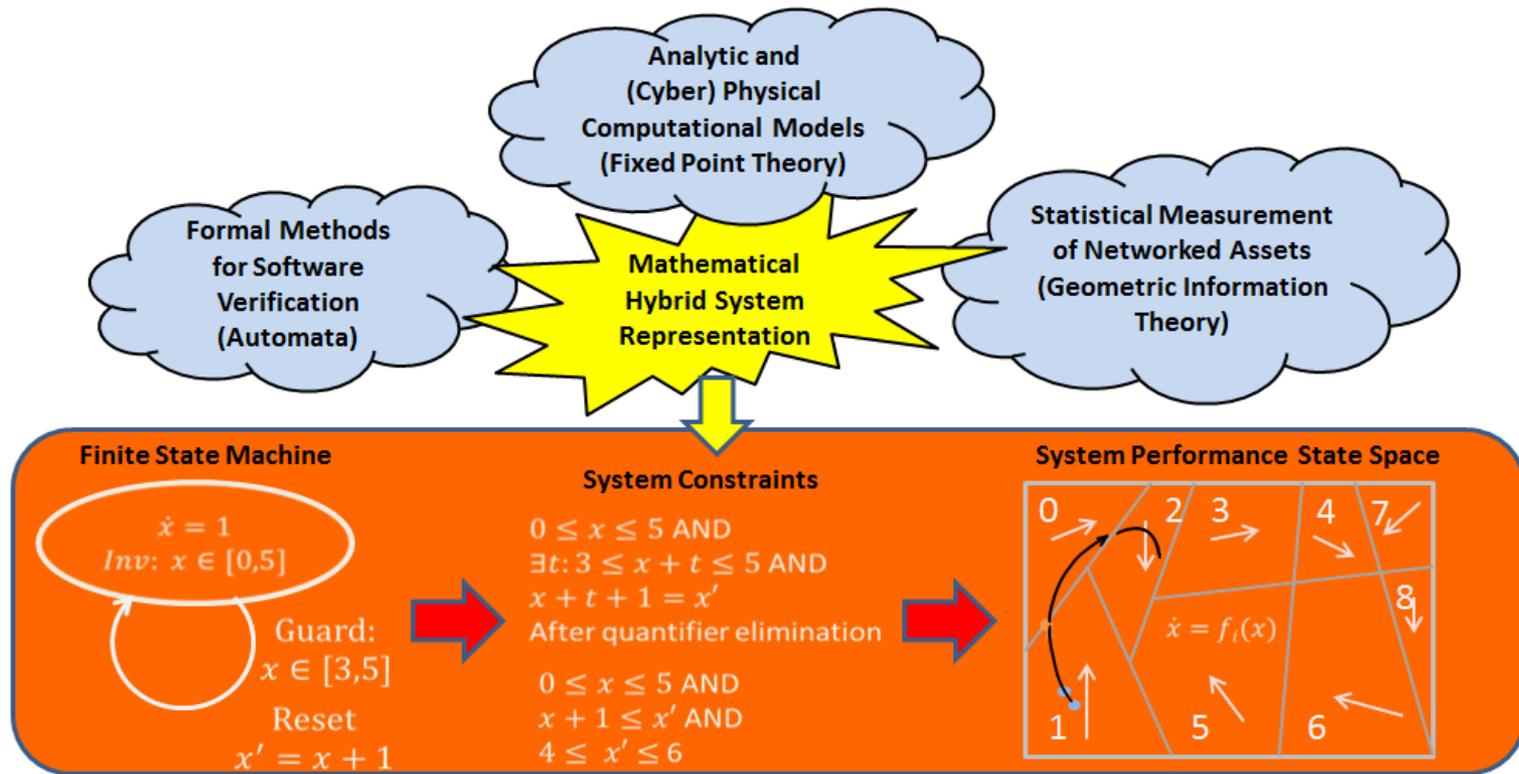
# Mathematical Unification of Systems Information Transaction



Syan Mitra, Gul Agha, University of Illinois Center for Secure Cloud Computing

**Approach:** There has been no unified mathematical approach to network, hardware, and software state space measurement and assessment. These areas can be integrated mathematically

**Payoff:** By combining finite automata theory, fixed point theory, and geometric information theory, the gaps between these areas can be bridged



Collaboration Between AFOSR/AFRL-RI





# Statistical Category Theory of Dynamic System Data

John Harer/Sayan Mukherjee Duke Konstantin Mischaikow/Rutgers



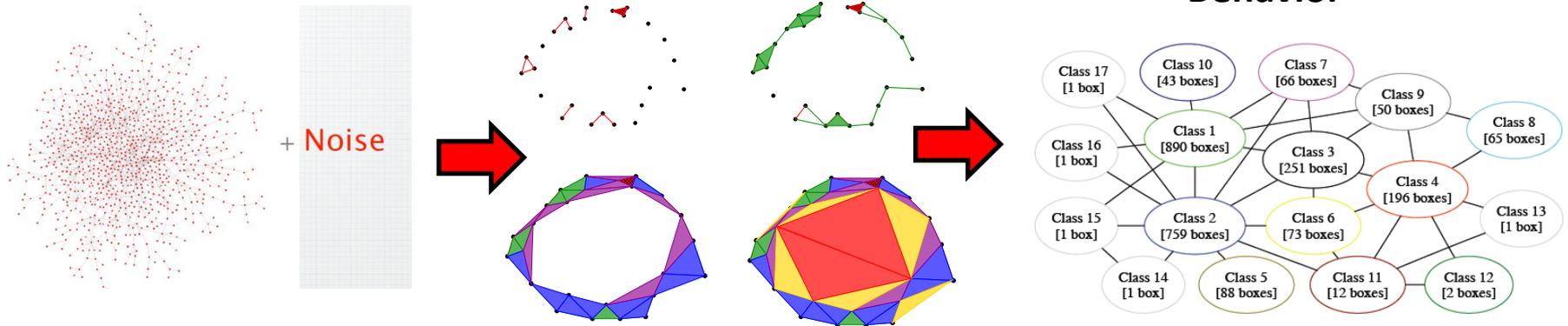
**Approach:** Mathematically going from sampled system data to parameterization requires a series of tools that handle many issues from statistical and topological analysis to category theory

**Payoff:** Data from any arbitrary system can be collected processed and parameterized as if there were an analytic model and its system operating modes characterized

Raw System Data

Persistent Homology

Categorical System Behavior



$f: X \rightarrow X$ , a continuous function on a metric space.

Given compact sets  $P = (P_1, P_0)$  with  $P_0 \subset P_1 \subset X$  define  $f_P: P_1/P_0 \rightarrow P_1/P_0$  by

$$f_P(x) = \begin{cases} f(x) & \text{if } x, f(x) \in P_1 \setminus P_0 \\ [P_0] & \text{otherwise} \end{cases}$$





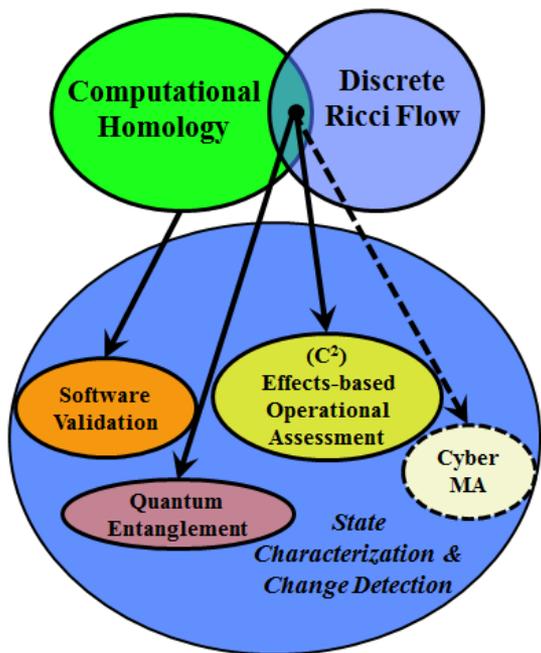
# Computational Homology and Ricci Flow for Verification of Computational Systems

Paul Alsing/AFRL RI, Howard Bramson/Syracuse, Warner Millter, FAU

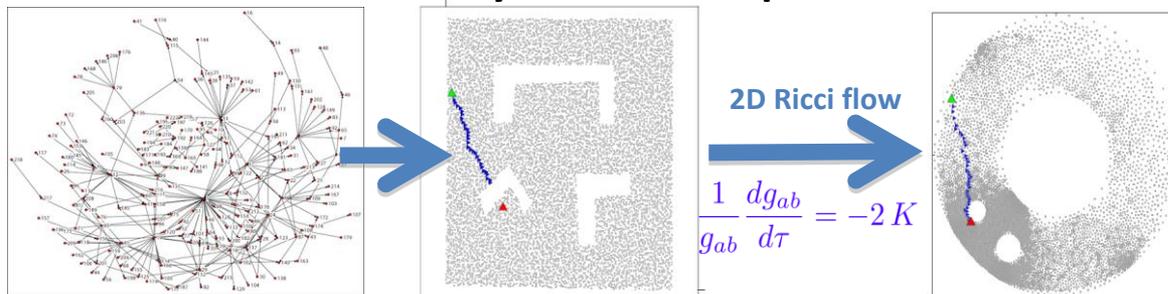


**Approach:** There is no unified mathematical approach to network, hardware, and software state space measurement and assessment

**Payoff:** By combining finite automata theory, fixed point theory, and geometric information theory, the gaps between these areas can be bridged



## Network/System Decomposition



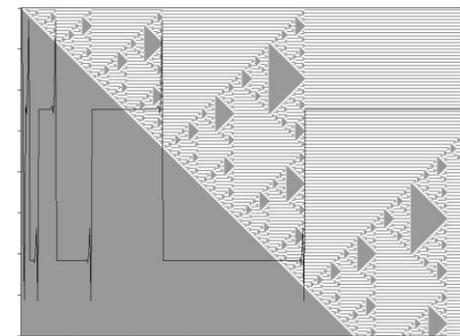
## Geometric System Properties

$$\frac{\dot{\lambda}}{\lambda} = -Ric_\lambda$$

$$\sum_{\lambda|\ell^*} \frac{\dot{\lambda}}{\lambda} V_{e\lambda} = - \sum_{\lambda|\ell^*} Ric_\lambda V_{e\lambda}$$

*Ricci Ve*

$$\sum_{\lambda|\ell^*} m_{e\lambda} \dot{\lambda} = -4 \epsilon_\ell$$





# Complex Networks Transition Activities



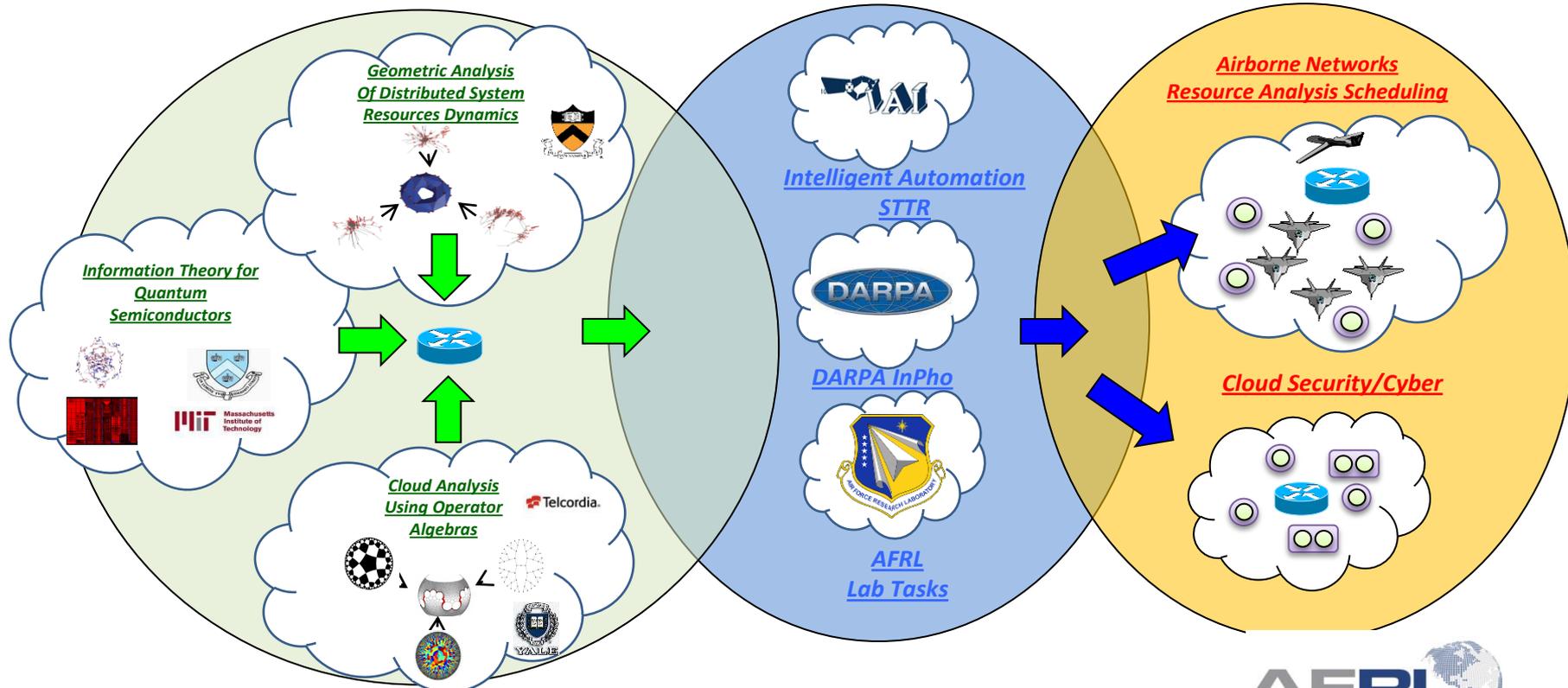
Complex Networks uses advanced mathematical analysis of information systems measurements to resource, verify, and secure distributed Air Force infrastructures

- Princeton MURI – dynamic analysis of packets for airborne network resource management
- Yale/Telcordia commercial grant – real time system security policy verification
- Columbia/MIT – information theory for new quantum semiconductors

## University Research

## Transition Mechanisms

## New Systems Components in Architecture





# Program Impact & Collaboration with Agencies



- **OSTP/NITRD – Co-Chair Large Scale Networks Working Group**
  - New national thrust – Complex Networks and Systems inspired by AFOSR program
- **ASDR&E**
  - Engineered Resilient Systems – Complex Networks and Foundations of Information Systems on Roadmap
- **Secretary of the Air Force Cyber Horizons**
  - Lead for Enabling Technologies
- **DARPA Collaboration/joint program reviews**
  - Graphs – Mathematics of graphs and networks agent
  - Defense Science Office Mathematics Board of Advisors
  - InPho – Information in a photon/quantum network collaborative funding
  - ITMANET – Joint program review and transition agent
- **IARPA – Quantum Computer Science Working Group**
- **ARL/ARO Network Science Board of Advisors**
- **NSF Future Internet, Net-Sci, BECS (Building and Engineering Complex Systems), NETS**



# Other Program Interactions



**Cyber Operations**: Joint University Center of Excellence:  
“Cyber Vision 2025” – Enabling Technologies workshops  
“Secure Cloud Computing” with university and AFRL/RI

**Dynamics and Control** : Verification and Validation of Complex Systems

**Physics and Materials**: New Joint MURI Topic: “Large Scale Integrated Hybrid Nanophotonics”

**Socio-Cultural Analysis**: Social Networks – Joint MURI Topic: “Stable Metrics for Inference in Social Networks ” – UCLA/USC/ASU

**Quantum**: Interaction with quantum network and quantum estimation processes through lab tasks  
- Joint EOARD initiative at Cambridge

**Information Fusion**: Critical feature selection in sensor networks

**Optimization**: Competing optimization requirements.

**Decision**: Networks of neurons.

**Biology**: Systems biological processes as networks.



# Academia/Commercial Outreach



- **Keynote Lecture, American Society of Mechanical Engineers, Complex Systems**
- **Keynote Lecture, International Conference on Complex Networks, 2012**
- **Keynote Speaker IEEE, International Conference on Distributed Computing Systems (ICDCS), Minneapolis, Minn., 2011**
- **Distinguished Lecture: University of Illinois Coordinated Science Laboratory Dec 2011**
- **Track Chair IEEE Milcom: October 2011**
- **Invited Speaker: American Institute for Mathematics Workshop on Geometry of Large Networks, Stanford, Sep 2011**
- **Invited Speaker: Institute for Pure and Applied Mathematics (IPAM), UCLA, Conference on Dynamic Networks Dec 2012**
- **Invited Speaker: Dagstuhl Germany, Mathematics of Network Learning 2011**
- **Invited Speaker: NSF Conference on Cyber Physical Systems, September 2011**
- **Invited Speaker: USCD Information Theory and Applications, LaJolla 2012**
- **Organizer: London Institute of Mathematics: Mathematical Statistical Verification, March 2012**



# Recent Program Awards



- **Vincent Poor:**
  - Member, National Academy of Sciences (elected 2011)
  - Edwin Howard Armstrong Achievement Award, IEEE Communications Society (2011)
  - IET Ambrose Fleming Medal for Achievement in Communications (2010)
  - IET Ambrose Fleming Medal for Achievement in Communications (2010)
- **Robert Calderbank**
  - Elected Dean of Science, Duke University (2011)
- **Joel Tropp:**
  - Alfred P. Sloan Research Fellowship (Mathematics) , 2010
  - Winner, Sixth Vasil A. Popov Prize (Mathematics), 2010
- **Emmanuel Candes:**
  - Collatz Prize (Mathematics) , (ICIAM) 2011
  - Winner, Sixth Vasil A. Popov Prize (Mathematics) , 2010
- **Rob Nowak**
  - IEEE Fellow, 2010
- **Mung Chiang:**
  - IEEE Fellow 2012
  - IEEE Kiyo Tomiyasu Award in 2012
- **Junshan Zhang**
  - IEEE Fellow 2012
- **Jennifer Rexford:**
  - SIGMETRICS “Test of Time” award (Computer Science) (2011)



# Transition Activities



- **AFRL**
  - AFRL/RI – Cyber Vision 2025 workshops/Illinois Center for Secure Cloud Computing/
  - AFRL/RI/RV – DARPA InPho program/DARPA Graphs program
  - AFRL/RI/RV/RH – distributed secure space communications
  - AFRL/RW/RV/RH – verification and validation of complex systems
- **STTR**
  - Intelligent Automation: Transition to ESC of Airborne Networks management – transition to Boeing for test-bed
  - Avirtek: Secure router application interface- AFRL/RI
  - Andro – Joint Spectrum Center Lockheed transition of automated spectrum management tool



# Transition Activities



- Customer/Industry
  - Collaboration with ACC/GCIC, Air Force Spectrum Management Agency on JALIN ICD
  - Collaboration with Boeing, ESC, IAI for transition of coding and routing management protocols baseline CORE tools to Rome Lab for possible integration in CABLE JCTD
  - Briefing to Space Command/Peterson for potential collaboration
  - Interaction with Northrop Grumman/BACN airborne networking program for potential collaboration
- OSD
  - Complex Systems Engineering and Systems 20/20 initiative
  - Software Assurance and Security Initiative
  - Robust Command and Control Initiative
- Commercial
  - New initiatives with Akamai for content distribution analysis
  - Interaction with USFA/DHS/CISCO on router algorithm design



# Transition Activities



- DCT
  - AFRL/RI – Lab tasks/Joint Emulab research center AFRL/RI online January 2010
    - Integration of MURI – “Information Dynamics in Networks” with AFRL Emulab through Princeton/UC Irvine
    - Transition of Yale diffusion map to AFRL/RI for network analysis
    - Network management and coding interaction with ACC – Jim Lehnert Purdue/Len Cimini Delaware/Andrea Goldsmith-Stanford
  - AFRL/RW – weapons tactical data links interaction – Chad Jenkins/Brown
  - AFRL/RH – social network analysis interaction – Michael Mahoney/Stanford
  - AFRL/RV - collaboration with for transitions in network/software policy and management - Larry Carin - Duke
- STTR
  - Transitions between STTR/AFRL/ESC/Boeing under STTR IAI activity – interactions with AFRL/RI
  - STTR ANDRO Computational Research interaction with OSD/NII/NTIA/ARL CERDEC for spectrum planning research – interaction with AFRL/RI
  - Interaction with Princeton/ASU with IAI for integration of STTR work



# Backup

---

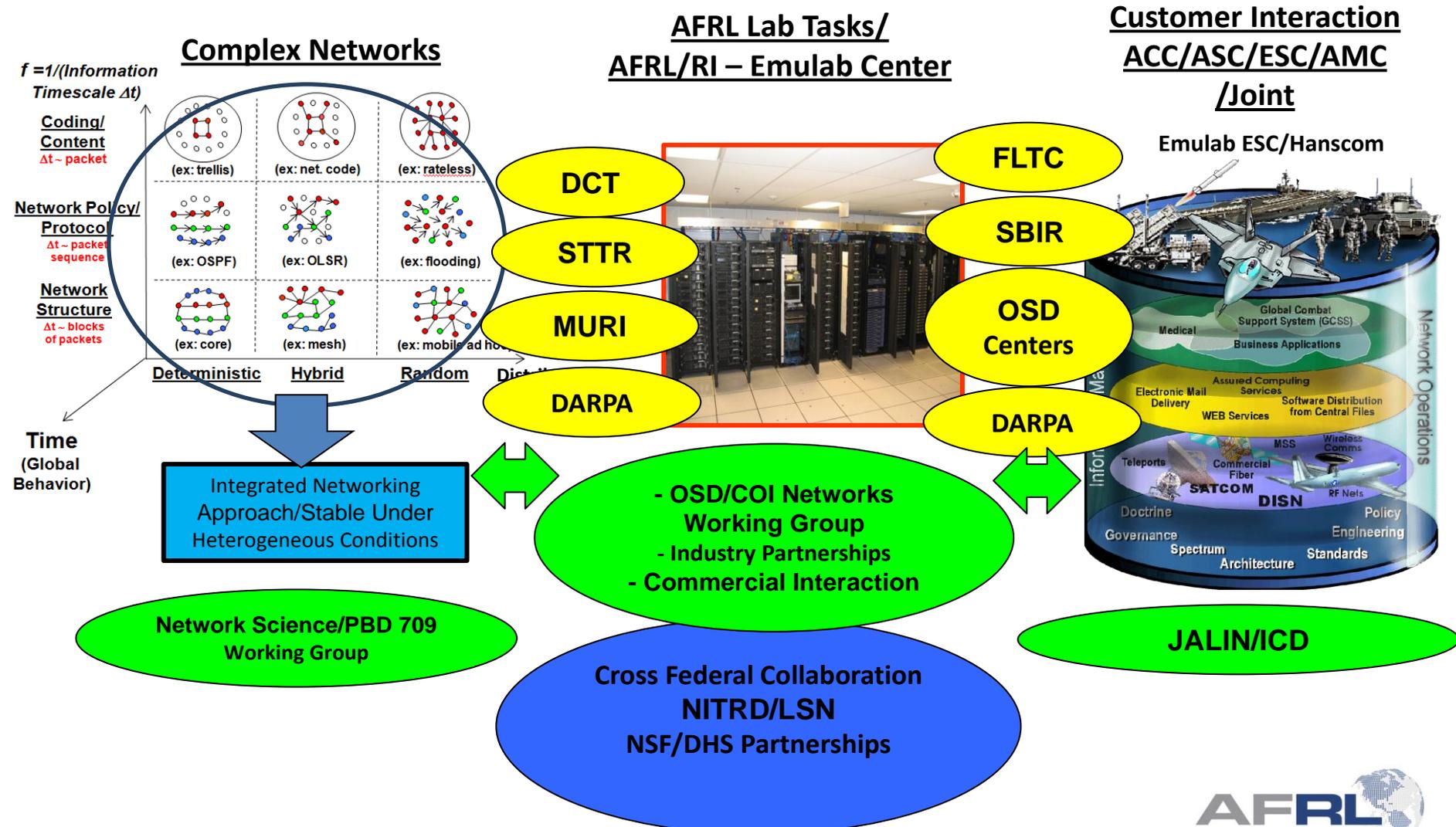




# Complex Networks Transition Organization



- Complex Networks has an integrated transition strategy



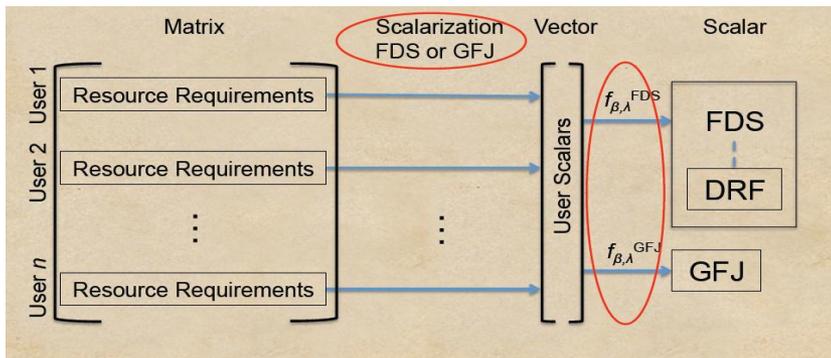


# Axioms of Fairness in Cloud Architectures

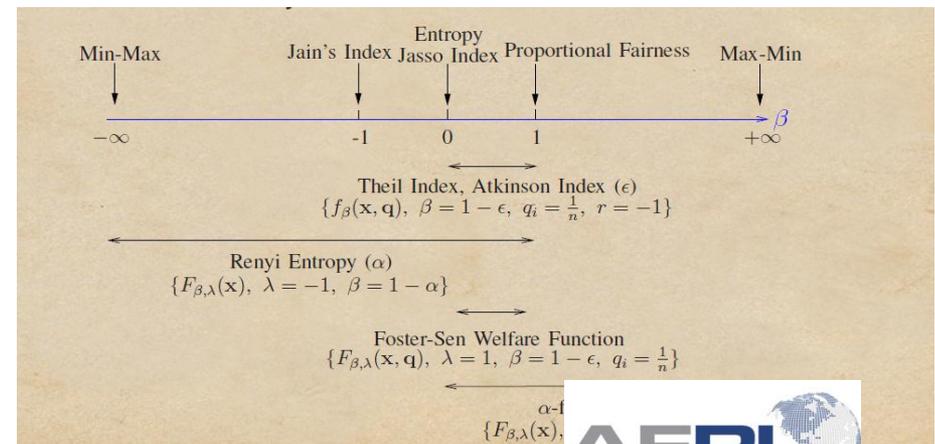
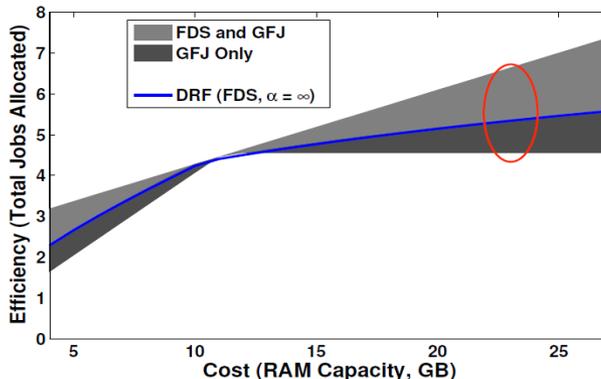


**Approach:** There is no unified mathematical approach to network, hardware, and software state space measurement and assessment

**Payoff:** By combining finite automata theory, fixed point theory, and geometric information theory, the gaps between these areas can be bridged



$$f_{\beta}(\mathbf{x}) = \text{sign}(1 - \beta) \cdot \left[ \sum_{i=1}^n \left( \frac{x_i}{\sum_j x_j} \right)^{1-\beta} \right]^{\frac{1}{\beta}}$$





# Codes Using Algebraic and Linear Programs For Distributed Cloud Resource Allocation



**Approach:** There is no unified mathematical approach to network, hardware, and software state space measurement and assessment

**Payoff:** By combining finite automata theory, fixed point theory, and geometric information theory, the gaps between these areas can be bridged

$$\max r_1 + \dots + r_6$$

Bandwidth constraints

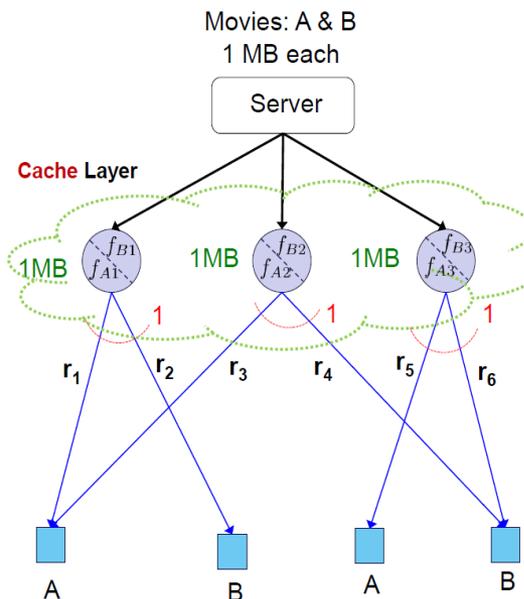
$$\begin{aligned} r_1 + r_2 &\leq 1 \\ r_3 + r_4 &\leq 1 \\ r_5 + r_6 &\leq 1 \end{aligned}$$

$$\begin{aligned} f_{A1} + f_{B1} &\leq 1 \\ f_{A2} + f_{B2} &\leq 1 \\ f_{A3} + f_{B3} &\leq 1 \end{aligned}$$

Storage constraints

$$\begin{aligned} r_1 &\leq f_{A1} \\ r_2 &\leq f_{B1} \\ &\vdots \end{aligned}$$

Availability constraint



KKT conditions

$$\begin{cases} (g_{x_{j i_m}} - (\lambda_j^* + k_{j i_m}^*)) x_{j i_m}^{*[0,+\infty)} = 0 \\ (\sum_{i_m \in \mathbb{N}_j^m} k_{j i_m}^* - l_m \mu_j^*) f_{j i_m}^{*[0,1]} = 0 \\ \lambda_j^* (\sum_{m=1}^M \sum_{i_m \in \mathbb{N}_j^m} x_{j i_m}^* - C_j) = 0 \\ \mu_j^* (\sum_{m=1}^M f_{j i_m}^* V_m - S_j) = 0 \\ k_{j i_m}^* (x_{j i_m}^* - f_{j m}^* r_m) = 0 \end{cases}$$

Primal-Dual Solution

$$\begin{cases} x_{j i_m} = \alpha (g_{x_{j i_m}} - (\lambda_j + k_{j i_m})) x_{j i_m}^{*[0,+\infty)}, \forall j, m, i_m \in \mathbb{N}_{j,m}^c \\ f_{j m} = \beta (\sum_{i_m \in \mathbb{N}_{j,m}^c} k_{j i_m} - l_m \mu_j) f_{j i_m}^{*[0,1]}, \forall j, m \\ \lambda_j = \gamma (\sum_{m=1}^M \sum_{i_m \in \mathbb{N}_{j,m}^c} x_{j i_m} - C_j) \lambda_j^{*[0,+\infty)}, \forall j \\ \mu_j = \delta (\sum_{m=1}^M f_{j i_m} V_m - S_j) \mu_j^{*[0,+\infty)}, \forall j \\ k_{j i_m} = \varepsilon (x_{j i_m} - f_{j m} r_m) k_{j i_m}^{*[0,+\infty)}, \forall j, m, i_m \in \mathbb{N}_{j,m}^c \end{cases}$$

