Trust, Opinion Diffusion and Radicalization in Social Networks

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***Abstract*—Gossiping models have increasingly been applied to study social network phenomena. In this context, this paper is specifically concerned with modeling how opinions of social agents can be radicalized if the agents interact more strongly with neighbors that share their beliefs. In our model, each agent's belief is represented by a vector of probabilities that a given state is true. The agents average their opinions with that of their neighbors over time, giving more weight to opinions that are closer to their current beliefs. The increasing trust that may exist among likeminded agents is modeled through a weight that is a monotonically decreasing function of the distance in opinion. We consider a *continuous* (soft) and a *discontinuous* (hard) model for the weight and analyze the convergence properties.**

I. Introduction

The concept of convergent social behavior or belief is a well-documented feature of social phenomenon in a number of diverse fields, ranging from *herding* [1] in economics and finance to *fad* and *trend* [2] in social psychology and the *Bandwagon effect* [3] in political science. The techniques used to elucidate this phenomenon can be generally divided into two classes, *i.e.*, *Bayesian* models and *non-Bayesian* models. Bayesian models [1], [2] describe individuals as *rational* agents: opinions (or beliefs) of social agents are probabilities of a given state, conditioned on all the available information; individuals observe and update their beliefs using Bayes rule. In many scenarios, the relevant information is dispersed throughout a network; social agents only observe a fraction of the total information, usually in the form of their past experiences. When the information available to an agent is not directly observable to others and agents do not know the structure of the social network, it is highly impractical to learn the state of nature in a Bayesian fashion. In contrast, non-Bayesian models [4], [5], [6], [7], [8], [9] use a simple and heuristic local updating rule to capture the opinion dynamics over a complex network topology. For example, in the *Hegselmann-Krause* model [7], [8], opinions (which are represented by a real number) are updated synchronously, as an average of all other opinions that differ from its own by less than a confidence level ϵ. Other studies that have investigated the effects of using simple pair-wise interactions between neighboring agents whose opinions differ by less than a threshold are [9] and [10]. Specifically, Deffuant et al. in [9] model the network on a square grid: each agent can only communicate with its four immediate neighbors. Weisbuch in [10] extends this topology to a scale free network model.

In our previous work [11], [12], we have introduced a model slightly more general than the HK model, to study random gossiping type of interactions among social agents in an arbitrary network. Interactions among pairs of these agents occur at random. The opinion distance after the interaction cannot be larger than the distance prior to the interaction. The amount of change in opinion is proportional to the *degree* of trust between agents and inversely proportional to the opinion distance between agents. To characterize how different trust models will effect the evolution of opinions in a society, in [11], [12] we proposed two interaction models: a *soft-interaction* model and a *hard-interaction* model. In the former, trust always exists between two interacting agents, while in the latter, trust only exists when the opinion distance between agents is below a threshold. In this paper, we extend the work in [11], [12] by analyzing the convergence properties of the two interaction models, finding their local rates of convergence. We prove that, under the soft-interaction model, the average opinion distance and the average squared opinion distance always converge at an exponential rate. Under the hard-interaction model, we show that there exists a phase transition from a society of radicalized opinions to one with a consistent opinion at a critical opinion distance threshold.

II. Model

In our model, agents are treated as nodes $\mathcal{V} = \{1, \cdots, n\}$ in an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the edge set $\mathcal{E}$ connects agents who are able to interact. Without loss of generality, it is assumed that $\mathcal{G}$ is connected in the sense that there exists a path joining directly or indirectly any two nodes in $\mathcal{V}$. The random interactions between agents are modeled through a time-invariant vector $\mathbf{p}$, describing the probability of each agent initiating an interaction, and a stochastic matrix $\mathbf{P} = [P_{ij}]$, specifying the probability that an initiator interacts with one of its neighbors. Hence, the probability of the pair $(i, j) \in \mathcal{E}$ interacting is $\overline{P}_{ij} = p_i P_{ij} + p_j P_{ij}$. Moreover, $[\overline{P}_{ij}]$ has the same sparsity structure as the graph $\mathcal{G}$, that is, $\overline{P}_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $\overline{P}_{ij} = 0$ otherwise. This implies that each edge in $\mathcal{E}$ is activated infinitely often.

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14. ABSTRACT
Gossiping models have increasingly been applied to study social network phenomena. In this context, this paper is specifically concerned with modeling how opinions of social agents can be radicalized if the agents interact more strongly with neighbors that share their beliefs. In our model, each agent?s belief is represented by a vector of probabilities that a given state is true. The agents average their opinions with that of their neighbors over time, giving more weight to opinions that are closer to their current beliefs. The increasing trust that may exist among likeminded agents is modeled through a weight that is a monotonically decreasing function of the distance in opinion. We consider a continuous (soft) and a discontinuous (hard) model for the weight and analyze the convergence properties.

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Here we recall the modeling assumptions introduced in [11]. Suppose that there are q possible outcomes of an experiment. The opinion or belief of the ith agent is expressed as a q-dimensional vector $\mathbf{x}_i = [x_{i1}, \cdots, x_{in}]$ in which $x_{i\ell}$ is the probability that the ℓth outcome is believed to be true by agent i. Hence, the belief space of dimension q is $\mathcal{X} = \{\mathbf{x} | \sum_{\ell=1}^{q} x_\ell = 1 \text{ and } x_\ell \in [0,1]\}$. Since our interaction models depend on the opinion distance between agents, we define the distance metric $d(\mathbf{x}_i, \mathbf{x}_j) : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}^+$ to be a proper geometric distance. Thus, with respect to the norm $\|\mathbf{x}\| := d(\mathbf{x}, \mathbf{0})$, the belief space $\mathcal{X}$ is bounded. Using the triangular inequality, the distance $d(\mathbf{x}_i, \mathbf{x}_j) \le 2 \sup_i \|\mathbf{x}_i\| := d_{\text{max}}$. For notational convenience, let $d_{ij}[k]$ denote the distance $d(\mathbf{x}_i[k], \mathbf{x}_j[k])$ after k network-wide interactions have occurred.

Recall that agents in the network interact at random. If trust exists between interacting agents, then their beliefs will move closer to each other. Formally, the degree of trust between connected agents is captured by the function $\rho(d) \in [0, \frac{1}{\sup \epsilon_k}]$ according to the following nonlinear model:

$$(\mathbf{a1}) \quad d_{ij}[k+1] = (1 - \epsilon_k \rho(d_{ij}[k])) d_{ij}[k] \ , \tag{1}$$

where the trust function

$(\mathbf{a2})$ $\rho(d)$ is a non-increasing function of d;

And for technical reasons, the step-size ϵ_k satisfies

$(\mathbf{a3})$ $\epsilon_k : \ \sum_{k=1}^{\infty} \epsilon_k \to \infty \ , \ \sum_{k=1}^{\infty} \epsilon_k^2 < \infty.$

As the system in (1) evolves, we say that *consensus* is achieved when all the agents attain the same belief vector as measured by the distance metric. More precisely, $d_{ij}[k] = 0$ for some k and for $\forall i, j \in \mathcal{V}$. In contrast, the society is said to exhibit *radicalized* beliefs if the agents are divided into subgroups: agents attain consensus within the subgroup; no subgroup has influence on the other sub-groups' beliefs.

Remark 1: The curious reader may be wondering how a pair of interacting agents update their beliefs to reflect the system in (1). Indeed, we have many choices. One simple choice is to have the beliefs move closer to each other through their shortest path. More precisely, let $\mathbf{x}'$ denote a agent's belief after an update, then the displacement of the beliefs is $d(\mathbf{x}_i', \mathbf{x}_j') = d(\mathbf{x}_i, \mathbf{x}_j) - d(\mathbf{x}_i, \mathbf{x}_i') - d(\mathbf{x}_j, \mathbf{x}_j')$.

III. Analysis

This section studies in detail the asymptotic convergence property of the system by first deriving an ordinary differential equation (ODE) of the stochastic approximation of (1) as follows: under $(a3)$,

$$\dot{d}_{ij}(t) = -\rho(d_{ij}(t)) d_{ij}(t) \ , \tag{2}$$

where $\dot{d}_{ij}$ denotes the derivative of d_{ij} with respect to a continuous time variable t, replacing the discrete time variable k. For convenient, the time variable t is not explicitly shown in the rest of this section. Recall that $\overline{P}_{ij}$ represents the probability that the pair $(i,j) \in \mathcal{E}$ interacts. We can define an averaged distance variable $\overline{d}$ over the edge set $\mathcal{E}$ as follows

$$\overline{d} := \sum\nolimits_{(i,j)\in\mathcal{E}} \overline{P}_{ij} d_{ij} \ .$$

From (2), the ODE of the average belief distance $\overline{d}$ is

$$\dot{\overline{d}} = -\sum\nolimits_{(i,j)\in\mathcal{E}} \overline{P}_{ij} \rho(d_{ij}) d_{ij} \ . \tag{3}$$

Note that the convergence property of the above expression depends on how the trust function $\rho(d)$ is modeled and is investigated in the following two sub-sections.

A. Soft-Interaction Model

As mentioned earlier, trust always exists between agents under the soft-interaction model. Formally, we impose:

$(\mathbf{a4})$ $\rho(d)$ is C^2-differentiable for $\forall d \in (0, d_{\text{max}})$;

$(\mathbf{a5})$ $\lim_{d \to d_{\text{max}}} \rho(d) = \rho_{\text{min}} > 0$;

$(\mathbf{a6})$ $\rho(d)d$ is concave for $\forall d \in [0, d_{\text{max}}]$.

The following lemma as proved in [11] implies asymptotic belief convergence.

Lemma 1: Under $(a1) - (a6)$, $\exists \alpha \in (0, \frac{1}{2}]$ such that the dynamics of $\overline{d}$ is upper and lower bounded by

$$-\rho(\overline{d}) d \le \dot{\overline{d}} \le -\alpha \rho(\overline{d}) \overline{d}.$$

Because of the strictly positive condition imposed on $\rho(d)$, both the upper bound and lower bound systems have one and only one stable equilibrium at $\overline{d} = 0$ and thus (3) will always converge to $\overline{d} = 0$, implying that $d_{ij} = 0$ for $\forall (i,j) \in \mathcal{E}$. Since the network $\mathcal{G}$ is assumed to be connected, then for any two agents $\ell, m \in \mathcal{V}$, their opinion distance equals $d_{\ell,m} = 0$, as the opinion distances along any path joining the two nodes ℓ and m are zero. Therefore, the soft-interaction model always lead to belief consensus.

Notice that since $\rho(d)$ is differentiable, the rate of convergence can be computed locally. In fact, the system locally resembles the form of the logistic equation and the local rate of convergence is exponential (See Corollary 1). Recall that a system is said to converge exponentially to d^* if $\exists C, r > 0$ such that $|d(t) - d^*| \le Ce^{-rt}|d(0) - d^*|$. Note that $d^* = 0$ in our case.

Corollary 1: Let $r(\overline{d})$ be the local rate of convergence in the neighborhood of $\overline{d}$. Under $(a1) - (a6)$, $\exists \alpha \in (0, \frac{1}{2}]$ such that the local rate of converge is exponential and is bounded by

$$\alpha[\rho(\overline{d}) - \overline{d}\dot{\rho}(\overline{d})] \le r(\overline{d}) \le \rho(\overline{d}) - \overline{d}\dot{\rho}(\overline{d}).$$

Proof: See Appendix. ■

Lemma 1 and Corollary 1 imply that the average belief distance between agents asymptotically approaches zero with a rate that is locally exponential. We will prove next that the average squared distance as expressed below also approaches zero.

Define the average squared distance as

$$\overline{d^2} := \sum\nolimits_{(i,j)\in\mathcal{E}} \overline{P}_{ij} d_{ij}^2 \ .$$

Property 1: Under $(a2)$ and $(a6)$, the function $g(a) = \rho(\sqrt{a})a$ is concave for $\forall a \in [0, d_{\text{max}}^2]$.

Proof: See Appendix. ■

Using Property 1, the following lemma indicates that the average squared distance in belief converges to zero.

Lemma 2: Under $(a1)-(a6)$, $\exists\eta \in (0, \frac{1}{2}]$ such that the dynamics of $\overline{d^2}$ is upper and lower bounded by

$$-2\rho\left(\sqrt{\overline{d^2}}\right)\overline{d^2} \le \dot{\overline{d^2}} \le -\eta\rho\left(\sqrt{\overline{d^2}}\right)\overline{d^2}.$$

Proof: See Appendix. ■

Since $\rho(\sqrt{\overline{d^2}})$ is strictly positive, both the upper and lower bound systems have only one equilibrium at $\overline{d^2} = 0$. Therefore, the average squared distance in belief will asymptotically approach zero. Following a procedure similar to that in the proof of Corollary 1, the rate of convergence can be computed as stated below.

Corollary 2: Under $(a1)-(a6)$, the local rate of convergence in the neighborhood of $\overline{d^2}$ is exponential and is bounded by

$$\eta\big[\rho(\sqrt{\overline{d^2}}) - \overline{d^2}\dot{\rho}(\sqrt{\overline{d^2}})\big] \le r(\overline{d^2}) \le 2\big[\rho(\sqrt{\overline{d^2}}) - \overline{d^2}\dot{\rho}(\sqrt{\overline{d^2}})\big].$$

In summary, as long as trust exists between any pair of connected agents, as in the case of the soft-interaction model, agents will keep updating their beliefs until consensus is attained. On the other hand, the next sub-section investigates the convergence property of the system under the hard-interaction model.

B. Hard-Interaction Model

Recall that hard-interaction model describes the situation in which agents only decide to trust their neighbors whose opinions differ by less than a threshold. To be precise,

$(\mathbf{a7})$ $\tau :\ d \ge \tau \to \rho(d) = 0;$

In addition, the following assumptions replace $(a4)-(a6)$ of the soft-interaction model.

$(\mathbf{a8})$ $\rho(d)$ is C^2-differentiable for $\forall d \in (0, \tau)$;
$(\mathbf{a9})$ $\rho(0)/\rho(\tau^-) \le \beta < \infty$;
$(\mathbf{a10})$ $\rho(d)d$ is concave for $\forall d \in [0, \tau]$.

Intuitively, if a society has a large threshold τ, *i.e.* is more open-minded, it is possible that it will asymptotically attain consensus. In contrast, if a society is close-minded with a small value of τ, it will fail to converge and several opinion clusters will emerge. Hence, the asymptotic belief profile varies with τ. As the value of τ increases, a society may transition from one with radicalized beliefs to a society with a consistent belief. This suggests the existence of a phase transition. Indeed, the following lemma as proved in [11] provides some insights on how large τ needs to be for a society to reach consensus.

Lemma 3: Under $(a1)-(a3)$ and $(a7)-(a10)$, a necessary condition for the system in (3) to converge almost surely is $\tau > \overline{d}(0)$.

However, the above condition is not sufficient. To show its insufficiency, consider a society $\mathcal{H}$: there are two groups $\mathcal{H}_1$ and $\mathcal{H}_2$ existing in $\mathcal{H}$, *i.e.* $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$; $d_{ij} = 0$ if i and j are in the same group; $d_{ij} = \tau + \epsilon$ if i and j are in different groups. When $\sum_{\mathcal{H}_1 \times \mathcal{H}_2} \overline{P}_{ij} < \frac{\tau}{\tau+\epsilon}$, one can verify that $\overline{d}(0) < \tau$. Hence, $\tau > \overline{d}(0)$ is not a sufficient condition.

IV. Simulations

A. Soft-Interaction Model

Fig. 1 illustrates the evolution of a belief profile over time under the soft-interaction model. Specifically, the underlying network graph $\mathcal{G}$ consists of $n = 100$ agents randomly distributed over an unit diameter disk and is generated using a random geometric graph (RGG), *i.e.* $\mathcal{G} = \mathcal{G}(n, r)$, with radius of communication $r = 0.6$. Note that $r = 1$ implies a fully connected RGG and $r = 0$ corresponds to a completely disconnected network. The choice of RGG is arbitrary since our analysis applies to any type of network topology. The initial belief profile is uniformly distributed in $\mathcal{X}$ of dimension 3. The distance d_{ij} between agents' beliefs is defined using the L_2 norm and thus $d_{\max} = 2\sup_i \|\mathbf{x}_i\|_2 = 2$. Agents update their beliefs through the shortest path connecting them with a constant step-size $\epsilon_k = 1$ and a trust function $\rho(d) = 0.5 - 0.2d$. This means that if agent i initiates an interaction with agent j, then they will adjust their beliefs to $d(\mathbf{x}_i, \mathbf{x}_i') = \mu(d_{ij})d_{ij}$ and $d(\mathbf{x}_j, \mathbf{x}_j') = \gamma(d_{ij})d_{ij}$, respectively, where $\mu(d_{ij}) + \gamma(d_{ij}) = \rho(d_{ij})$. We use the uniform communication scheme to model the rate of interaction between agents. Let $\mathcal{N}_i$ be the set of neighbors of agent i. Uniform communication corresponds to homogeneous rates $p_i = 1/n$ for $\forall i \in \mathcal{V}$ with $P_{ij} = 1/|\mathcal{N}_i|$ uniform across neighbors. As the system evolves, the belief profile asymptotically converges a single belief as seen from Fig. 1. Fig. 2 shows the convergence rate of $\overline{d}$ (*i.e.*, solid line) averaged over 300 trials. It is then compared with its local upper bound and lower bound (*i.e.*, dotted lines) around various values of $\overline{d}$ as derived in Corollary 1. Observe that the actual rate of convergence always lies in between its analytical bounds. Fig. 3 shows two histograms of 300 belief profiles at time zero (left) and after the dynamics have stabilized (right). Each belief profile is associated with a randomly generated RGG.

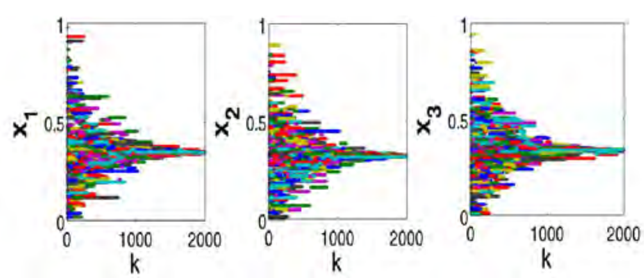

Fig. 1. The evolution of a belief profile with time: each line segment corresponds to a node and a segment terminates when a node interacts and changes its belief.

B. Hard-Interaction Model

We ran a suite of 300 trials; each trial starts with an uniformly distributed random initial belief profile in $\mathcal{X}$ of dimension 3, over a randomly generated RGG similar to that in the soft-interaction model. The trust function $\rho(d)$ equals 1 if $d < \tau$ and 0 otherwise. The underlying communication graph and updating rule are the same as in the soft-interaction model. Networks of three sizes $n = 50, 100, 200$ were simulated. We define the effective graph as $\mathcal{G}_{\text{eff}} = (\mathcal{V}, \mathcal{E}_{\text{eff}})$, with

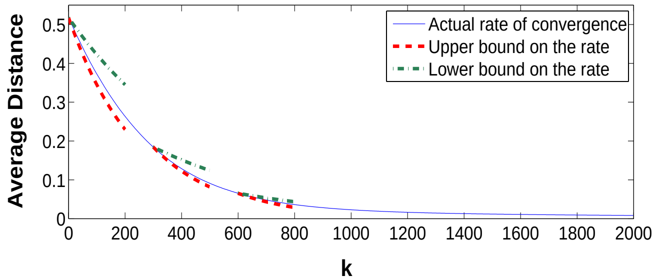

Fig. 2. The rate of convergence (solid line) is locally bounded from above and from below by two exponential rate functions (dotted lines).

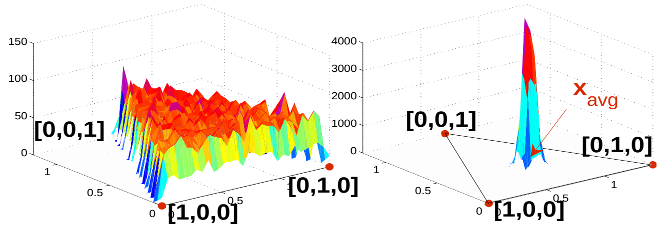

Fig. 3. A histogram of 300 belief profiles at time zero (left) and after the dynamics have stabilized (right). Each initial belief profile is uniformly distributed in $\mathcal{X}$ of dimension 3.

$\mathcal{E}_{\text{eff}} := \{(i,j) \in \mathcal{E} | d_{ij}[k] < \tau\}$ where k is sufficiently large. Figure 3 shows the algebraic connectivity of the effective graph; it is clear that the network converges with a probability one if τ is sufficiently above $\overline{d}(0)$.

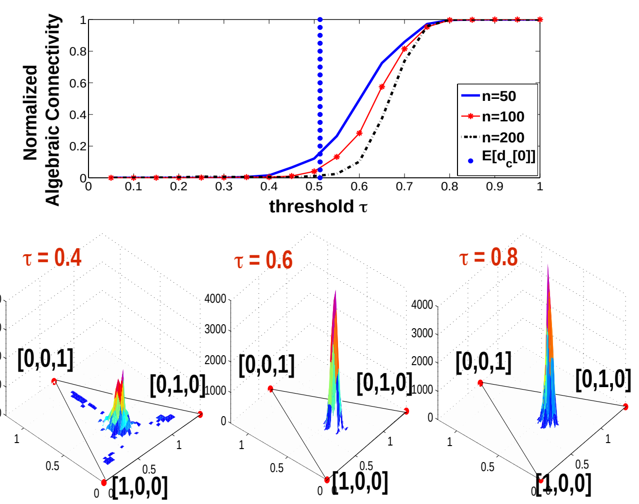

Fig. 4. Phase transition (top) of a society from radicalized beliefs to a consistent belief. Histogram (bottom) of 300 asymptotic opinion profiles with $n = 100$ at $\tau = 0.4, 0.6, 0.8$.

C. Heterogeneous Model

It is also of interest to learn how a belief profile evolves in a heterogeneous model, that is, one in which a fraction of the population uses the hard-interaction model and the rest uses the soft-interaction model. We want to investigate how the critical threshold shifts as the number of agents using the hard-interaction model decreases. Fig. 5 shows two simple heterogeneous interaction models compared with the hard-interaction model in a network of $n = 50$ agents. Specifically, h represents the number of agents using the hard-interaction model and s is the number of agents using the soft-interaction model. One observes that, on average, as the society becomes more open-minded (*i.e.* s increases and h decreases), the critical threshold shifts down.

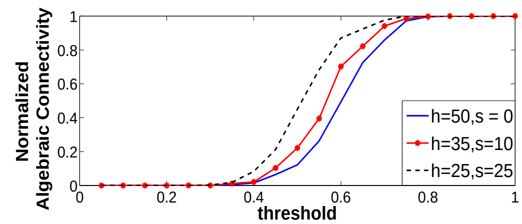

Fig. 5. Heterogeneous Model

V. Conclusion

In this paper, we have extended the analysis of the convergence properties of both the soft-interaction model and hard-interaction model, which were originally proposed in [11]. Under the soft-interaction model, we have proved that the average belief distance converges to zero; moreover, the average squared distance also converges. The rate of convergence is locally exponential. Under the hard-interaction, we have showed the existence of a phase transition from radicalized beliefs to a consistent belief at a critical threshold τ, which is closely related to the averaged initial belief distance.

VI. Appendix: Proofs

Corollary 1: Suppose that $\overline{d}(t+s)$ is in a neighborhood of $\overline{d}(t)$ provided that s is small. Hence, Taylor's expansion gives $\rho\left(\overline{d}(t+s)\right) \approx \rho\left(\overline{d}(t)\right) + \left[\overline{d}(t+s) - \overline{d}(t)\right] \dot{\rho}\left(\overline{d}(t)\right)$, where $\dot{\rho}(\overline{d}) = d\rho/d\overline{d}$. Since $\dot{\rho}\left(d(t)\right) \leq 0$ given $(a2)$, the upper bound system $\dot{\overline{d}}(t+s) \leq -\alpha\rho(\overline{d}(t+s))\overline{d}(t+s)$ becomes

$$\dot{\overline{d}}(t+s) \leq \begin{cases} -\alpha\rho(\overline{d}(t))\overline{d}(t+s) & \text{if } \dot{\rho}\left(\overline{d}(t)\right) = 0; \\ -\alpha f(s;t) & \text{if } \dot{\rho}\left(\overline{d}(t)\right) < 0. \end{cases}$$

where $f(s;t) = \left[\rho\left(\overline{d}(t)\right) - \overline{d}(t)\dot{\rho}\left(\overline{d}(t)\right)\left(1 - \frac{\overline{d}(t+s)}{\overline{d}(t)}\right)\right]\overline{d}(t+s)$. In the first case, $\rho(\overline{d}(t))$ is locally constant and hence, the local rate of convergence around $\overline{d}(t)$ is exponential and is lower bounded by $r(\overline{d}(t)) \geq \alpha\rho\left(b(t)\right)$. In the second case, define $\xi(s;t) := \overline{d}(t+s)/\overline{d}(t)$. The dynamics of ξ become

$$\dot{\xi}(s;t) = \frac{\dot{\overline{d}}(t+s)}{\overline{d}(t)} \leq -\alpha R(t)\xi(s;t)\left(1 - \frac{\xi(s;t)}{K(t)}\right) \quad (4)$$

where $R(t) = \rho\left(\overline{d}(t)\right) - \overline{d}(t)\dot{\rho}\left(\overline{d}(t)\right)$ and $K(t) = 1 + \frac{\rho\left(\overline{d}(t)\right)}{-\overline{d}(t)\dot{\rho}\left(\overline{d}(t)\right)} > 1$. Note that the dynamics on ξ resemble the logistic equation. Provided that $\xi(0;t) = 1$ and $R(t) \neq 0$, the solution to (4) is

$$\xi(s;t) \leq \frac{K(t)e^{-R(t)s}}{K(t) - 1 + e^{-R(t)s}} \overset{s\uparrow}{\approx} \left(\frac{K(t)}{K(t)-1}\right) e^{-\alpha R(t)s},$$

where $\alpha R(t)$ is the rate of convergence of the upper bound system. Combining the results from both cases yields $r\left(\overline{d}(t)\right) \geq \alpha R(t)$. Similarly, using the lower bound system $\dot{\overline{d}}(t) \geq -\rho(\overline{d}(t))\overline{d}(t)$, one can also find an upper bound on the rate of convergence around $\overline{d(t)}$, that is, $r\left(\overline{d}(t)\right) \leq R(t)$.

Property 1: $g(a)$ is concave if its second derivative is negative. Taking the first derivative of $g(a)$ yields $\dot{g}(a) = \rho\left(\sqrt{a}\right) + \frac{\sqrt{a}}{2}\dot{\rho}\left(\sqrt{a}\right)$. Taking the second derivative yields

$$\ddot{g}(a) = \frac{\ddot{\rho}\left(\sqrt{a}\right)\sqrt{a} + 2\dot{\rho}\left(\sqrt{a}\right)}{4\sqrt{a}} + \frac{\dot{\rho}\left(\sqrt{a}\right)}{4\sqrt{a}} \quad . \tag{5}$$

$(a6)$ implies that the second derivative of $\rho(d)d$ is negative, *i.e.*, $\ddot{\rho}(d)d + 2\dot{\rho}(d) < 0$ for $\forall d \in [0, d_{\text{max}}]$. Let $a = d^2$, then for $\forall a \in [0, d^2_{\text{max}}]$, the first term in (5) is negative. Moreover, $\dot{\rho}\left(\sqrt{a}\right) \leq 0$ from $(a2)$. Hence, $\ddot{g}(a) < 0$ and $g(a)$ is concave.

Lemma 2: From $(a1)$, we have

$$d^2_{ij}[k+1] - d^2_{ij}[k] = -2\epsilon_k \rho(d_{ij}[k]) d^2_{ij}[k] + \epsilon^2_k \rho^2(d_{ij}[k]) d^2_{ij}[k].$$

Since $\rho(d) \leq \frac{1}{\sup \epsilon_k} \leq \frac{1}{\epsilon_k}$ for $\forall d \in [0, d_{\text{max}}]$, then $\epsilon_k \rho(d_{ij}[k]) \leq 1$ and thus

$$-2\epsilon_k \rho(d_{ij}[k]) d^2_{ij}[k] \leq d^2_{ij}[k+1] - d^2_{ij}[k] \leq -\epsilon_k \rho(d_{ij}[k]) d^2_{ij}[k].$$

Stochastic approximation implies that

$$-2 \sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij}\rho(d_{ij})d^2_{ij} \leq \dot{\overline{d^2}} \leq -\sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij}\rho(d_{ij})d^2_{ij} \quad . \tag{6}$$

Let $a_{ij} = d^2_{ij}$ and $\overline{a} := \sum_{(i,j)\in\mathcal{E}} \overline{P}_{ij} a_{ij}$. Hence, $\overline{a} = \overline{d^2}$. Using Property 1 and Jensen's inequality, we get

$$\sum\nolimits_{(i,j)\in\mathcal{E}} \overline{P}_{ij}\rho(\sqrt{a_{ij}})a_{ij} \leq \rho\left(\sqrt{\overline{a}}\right)\overline{a} = \rho\left(\sqrt{\overline{d^2}}\right)\overline{d^2} \quad . \tag{7}$$

Hence, $\dot{\overline{d^2}}$ is lower bounded by $\dot{\overline{d^2}} \geq -2\rho\left(\sqrt{\overline{d^2}}\ \right)\overline{d^2}$.

To prove the upper bound, let $S = \{(\ell, m) \in \mathcal{E}_c | d^2_{\ell m} \leq \overline{d^2}\}$, $S^c = \{(i,j) \in \mathcal{E}_c | d^2_{ij} > \overline{d^2}\}$ s.t. $S \cup S^c = \mathcal{E}_c$. From (6), $\dot{\overline{d^2}} \leq f_c(\overline{d^2}) := -\sum_{(i,j)\in S^c} \overline{P}_{ij}\rho(d_{ij})\overline{d^2}$ and $\dot{\overline{d}} \leq f(\overline{d^2}) := -\sum_{(\ell,m)\in S} \overline{P}_{\ell m}\rho(\sqrt{\overline{d^2}})d^2_{\ell m} = -\left(\overline{d^2} - \sum_{(i,j)\in S^c} \overline{P}_{ij} d^2_{ij}\right)\rho(\sqrt{\overline{d^2}})$. A weighted sum yields

$$\dot{\overline{d}} \leq \eta f(\sqrt{\overline{d^2}}) + (1-\eta) f_c(\sqrt{\overline{d^2}}) = -\eta\rho(\sqrt{\overline{d^2}})\overline{d^2} - g(\overline{d^2}), \tag{8}$$

where $g(\overline{d^2}) = \sum_{(i,j)\in S^c} \overline{P}_{ij}\left[(1-\eta)\rho(d_{ij})\overline{d^2} - \eta\rho(\sqrt{\overline{d^2}})d^2_{ij}\right]$. The term $g(\overline{d^2})$ will vanish if $\eta = \frac{1}{1+\theta}$, where $\theta = \frac{\sum_{(i,j)\in S^c} \overline{P}_{ij} d^2_{ij} \rho(\sqrt{\overline{d^2}})}{\sum_{(i,j)\in S^c} \overline{P}_{ij}\rho(d_{ij})\overline{d^2}} \geq 1$ because for $\forall (i,j) \in S^c$, $d^2_{ij} > \overline{d^2}$ and $\rho(\sqrt{\overline{d^2}}) \geq \rho(d_{ij})$. Moreover, by $(a5)$, $\theta \leq \frac{\rho(\sqrt{\overline{d^2}})}{\sum_{(i,j)\in S^c} \overline{P}_{ij}\rho(d_{ij})} < \infty$. Therefore, $\exists \eta \in \left(0, \frac{1}{2}\right]$ such that $\dot{\overline{d^2}}$ is upper bounded by $\dot{\overline{d^2}} \leq -\eta\rho(\sqrt{\overline{d^2}})\overline{d^2}$.

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