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REMOTE SENSING OF LAYERED RANDOM MEDIA USING THE RADIATIVE TRANSFER THEORY

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The Radiative Transfer (RT) approach is widely used in applications involving scattering from layered random media with rough interfaces. Although this approach involves approximations, in most applications they are not explicitly stated or well understood. In order to better understand the RT approach to our problem, we adopt a statistical wave approach and then transition to the RT equations. The geometry of our problem consists of a multi-layer discrete random medium with rough boundaries which are planar on the average. The random medium in each layer consists of a homogeneous background medium in which discrete scatterers are randomly distributed. The regions above and below the random medium stack are homogeneous. Using the Greens functions of the problem without the volumetric fluctuations we represent our problem as a system of integral equations. Employing the T-matrix description we first average with respect to volumetric fluctuations and then use the Twersky approximation to obtain a system of integral equations. We next average with respect to surface fluctuations, apply the weak surface correlation approximation and arrive at a closed system of integral equations for the first and second moments of the electric fields. We use the Wigner transforms to translate the coherence functions to radiant intensities, which are the fundamental quantities in the RT approach. On applying the quasi-static field approximation we hence arrive at a system of equations identical to those used in the RT approach. From this study we learn that there are more conditions involved in the RT approach than widely believed to be sufficient.

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2. Summary

The Radiative Transfer (RT) approach is widely used in applications involving scattering from layered random media with rough interfaces (F.T. Ulaby, R.K. Moore and AK Fung, *Microwave Remote Sensing*, Vol. 3, Artech House, 1986). Although this approach involves approximations, in most applications they are not explicitly stated or well understood. The RT approach for random media with non-scattering boundaries has been well studied (M.I. Mishchenko, *Appl. Optics*, 41, 7114-34, 2002). In contrast our problem has scattering boundaries which are randomly rough. In order to better understand the RT approach to our problem we adopt a statistical wave approach and then transition to the RT equations. The geometry of our problem consists of a multi-layer discrete random medium with rough boundaries which are planar on the average. The random medium in each layer consists of a homogeneous background medium in which discrete scatterers are randomly distributed. The statistical characteristics of the random medium in each layer are independent of each other and independent of the statistics describing the rough interfaces. The regions above and below the random medium stack are homogeneous. Using the Greens functions of the problem without the volumetric fluctuations we represent our problem as a system of integral equations. Employing the T-matrix description we first average with respect to volumetric fluctuations and then use the Twersky approximation to obtain a system of integral equations. We next average with respect to surface fluctuations, apply the weak surface correlation approximation and arrive at a closed system of integral equations for the first and second moments of the electric fields. We use the Wigner transforms to translate the coherence functions to radiant intensities, which are the fundamental quantities in the RT approach. On applying the quasi-static field approximation we hence arrive at a system of equations identical to those used in the RT approach. From this study we learn that there are more conditions involved in the RT approach than widely believed to be sufficient.

3. Introduction

The radiative transfer (RT) theory is widely used in remote sensing problems [14,28,20,15,1,30]. Often the model of layered random medium with rough interfaces is used. Multiple scattering processes in this structure are well represented by the RT equations. Although quite successful in numerous applications in various disciplines, it is known that the RT approach involves approximations. Often people in the remote sensing community are not quite familiar with the approximations involved in the RT approach and hence there has been inappropriate use of the RT approach in the literature. Since the phenomenological RT theory [7,27] was first developed for light scattering in planetary atmospheres the RT conditions prevalent in the atmospheric context has been popularly identified as sufficient conditions for employing the RT theory. However, we notice that the RT theory has been used for a variety of different problems [18,24,19,26] in various applications with complex geometries. It is not clear whether the classical conditions associated with the RT theory are sufficient in all situations. In this paper we will review the approximations involved and clarify misconceptions. In order to better understand the RT approach we employ the more rigorous statistical wave theory to the problem and hence make the transition to the RT equations. In this process we clarify and explain the assumptions or approximations involved in the RT approach. By following this procedure we found that there are more conditions embedded in the RT approach than widely believed to be sufficient. For our study we have considered a multi-layer random medium composed of discrete scatterers (see Figure 1). By considering several special cases of this general problem we show that the number of conditions implied in the RT approach reduces with simpler geometries. Our conclusions are not just for the case discrete random media. We show similar conditions apply for the random continuum as well.

The report is organized as follows. First we describe the geometry of the problem. Next we give the radiative transfer approach to the problem. The next section is on the statistical wave approach to the problem. This occupies the major part of the paper. It deals with the derivation and analysis of the first and second moments of the wave functions. A transition is next made to RT equations. Next we enumerate and discuss the conditions implied in the RT approach.

4. Description of the Problem

The geometry of the problem is shown in Figure 1. We have an *N*-layer random medium stack with rough interfaces which, on the average, are parallel planes. Let ε_j be the permittivity of the background medium, and let ε_{js} be the permittivity of the scatterers in the *j*-th layer. The location, orientation, and shape of the scatterers are random functions characterizing the fluctuations. We assume the the volumetric fluctuations in each layer are statistically independent of each other. Let N_j be the number of scatterers, and let ρ_j be the density of the scatterers (number of scatterers per unit volume) of the *j*-the layer. The permeability of all the layers is that of free space. The randomly rough interfaces are given as $z = z_j + \zeta_j(\mathbf{r}_\perp)$. Then ζ_j are zero-mean isotropic stationary random processes independent of volumetric fluctuations of the problem. Let $z_0 = 0$, and let d_j be the thickness of the *j*-th layer.

Let $z_0 = 0$, and let d_j be the thickness of the *j*-th layer. The media above and below the stack are homogeneous with parameters ε_j, k_0, and ε_{N+1}, k_{N+1}, respectively. This system is excited by a monochromatic electromagnetic plane wave and we are interested in formulating the resulting multiple scattering process.

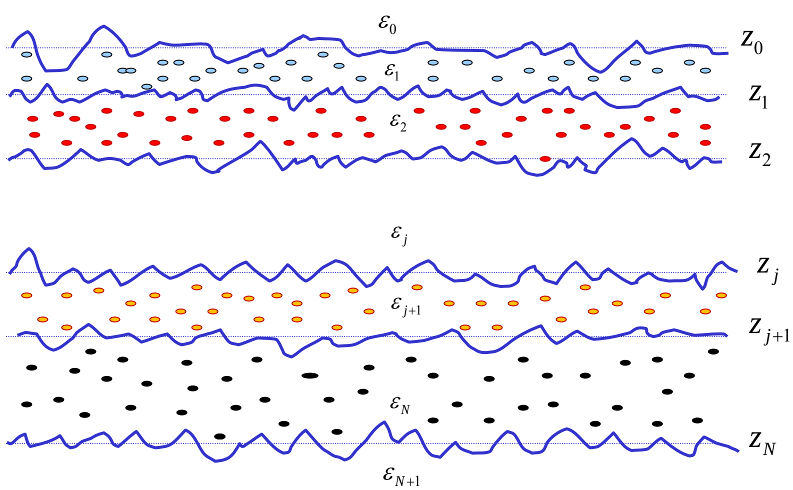

Figure 1. Geometry of the Problem

5. Radiative Transfer Approach

Multiple scattering in a complex environment is well described by the radiative transfer theory. This theory is not only conceptually simple but also very efficient. The fundamental quantity here is the specific intensity $\mathbf{I}$ which is governed by the following equation [7,27,10]

$$\hat{s}\cdot\nabla\mathbf{I}(\mathbf{r},\hat{s})+\bar{\bar{\gamma}}\mathbf{I}(\mathbf{r},\hat{s})=\int\bar{\mathbf{P}}(\hat{s},\hat{s}')\mathbf{I}(\mathbf{r},\hat{s}')d\Omega' \tag{1}$$

One may regard this equation as a statement of conservation of energy density $\mathbf{I}$ which is a phase-space quantity at position $\mathbf{r}$ and direction $\hat{s}$. $\bar{\bar{\gamma}}$ is the extinction matrix which is a measure of loss of energy due to scattering in other directions. $\bar{\bar{\mathbf{P}}}$ is the phase matrix representing increase in energy density due to scattering from neighbouring elements. Ω is the solid angle subtended by $\hat{s}$. Given the statistical characteristics of the medium one can calculate the phase function using the single scattering theory for elements that constitute the random medium of the layer [31,11,6,17]. The extinction matrix is hence calculated using the optical theorem. The specific intensity in each layer is governed by an equation similar to (1). Since our layer problem has translational invariance in azimuth the RT equation for the *m*-th layer takes the following form,

$$\cos\theta\frac{d}{dz}\mathbf{I}_m(z,\hat{s})+\bar{\bar{\gamma}}_m\mathbf{I}_m(z,\hat{s})=\int_{\Omega_m}\int\bar{\mathbf{P}}_m(\hat{s},\hat{s}')\mathbf{I}_m(z,\hat{s}')d\Omega' \tag{2}$$

where the subscript *m* denotes that the quantity corresponds to that of the *m*-th layer and θ is the elevation angle of $\hat{s}$. This set of RT equations is complemented by a set of boundary conditions which is in turn based on energy conservation considerations. To be more precise, we impose the condition that the energy flux density at each interface is conserved. This leads to the following boundary conditions on the *m*-th interface

$$\mathbf{I}_m^u(z_m,\hat{s})=\int\mathbf{R}_{\mathbf{m+1},m}(\hat{s},\hat{s}')\,\mathrm{I}_m^d(z_{m,}\hat{s}')d\Omega'+\int\mathbf{X}_{m,\mathbf{m+1}}(\hat{s},\hat{s}')\,\mathrm{I}_{m+1}^u(z_{m,}\hat{s}')d\Omega' \tag{3}$$

The boundary conditions on the *(m-1)*-th interface are given as

$$\mathbf{I}_m^d(z_{m-1},\hat{s})=\int\bar{\mathbf{R}}_{\mathbf{m-1},m}(\hat{s},\hat{s}')\,\mathrm{I}_m^u(z_{m-1,}\hat{s}')d\Omega'+\int\bar{\mathbf{X}}_{m,\mathbf{m-1}}(\hat{s},\hat{s}')\,\mathrm{I}_{m-1}^d(z_{m-1,}\hat{s}')d\Omega' \tag{4}$$

where $\bar{\mathbf{R}}_{mn}$ and $\bar{\mathbf{X}}_{mn}$ are the local reflection and transmission Mueller matrices. To be more specific, $\bar{\mathbf{R}}_{mn}$ represents the reflection matrix of waves incident from medium *n* on the interface separating medium *m* and medium *n*. The superscripts *u* and *d* indicate whether the intensity corresponds to a wave travelling upwards or downwards. These Mueller matrices are often calculated using some asymptotic theory such as the Kirchhoff approximation [33,5,30,29]. The integration in these expressions are over a solid angle (hemisphere) corresponding to $\hat{s}'$. Suppose we have a time-harmonic electromagnetic plane wave incident on this stack from above. Then the downward travelling intensity in Region 0 is

$$\mathbf{I}_0^d(z,\hat{s}) = \mathbf{B}_0\delta(\cos\theta_0 - \cos\theta_i)\delta(\phi_0 - \phi_i) \tag{5}$$

where $\mathbf{B}_0$ is the intensity of the incident plane wave and $\{\theta_i, \varphi_i\}$ describes its direction. Since there is no source or scatterer in Region N+1,

$$\mathbf{I}_{N+1}^u(z,\hat{s}) = 0$$

Notice again that these boundary conditions represent conservation of intensity at the interfaces. We should point out that the radiative transfer approach as applied to a particular problem is only a model based on certain assumptions. Since the RT theory is used in a variety of applications, the particular assumptions involved are described in terms of different terminologies, specific to the discipline where it is used. One good way to understand in more general terms the RT approach and the underlying assumptions is to compare it with the more rigorous wave approach. For the case of an unbounded random medium this kind of study was carried out in the 1970s [3,2]. From that study we learn that the radiative transfer theory can be applied under the following conditions:

1. Quasi-stationary field approximation
2. Sparse distribution
3. Statistical homogeneity of the medium fluctuations

These are the well-known conditions that we associate with the RT approach. However, our problem has bounded structures and, further, they are randomly rough. The question is this: are the above conditions sufficient to apply the RT approach for our problem? This is the motivation for this paper. We follow the wave approach to this problem, derive the equations for the intensities, and hence make the transition to the RT equations. This procedure enables us to better understand the necessary conditions for using the RT approach for our problem.

6. Statistical Wave Approach

The following are the equations that govern the waves in the layer structure:

$$\nabla\times\nabla\times\mathbf{E}_j - k_j^2\mathbf{E}_j = v_j\mathbf{E}_j \qquad j = 1,\cdots,N \tag{6}$$

where

$$v_j \equiv \sum_{i=1}^{N_j} \mathrm{Q}_{\mathrm{ji}}(\mathbf{r}) \tag{7a}$$

$$Q_{ji}(\mathbf{r}) = \begin{cases} k_{js}^2 - k_j^2 & \mathbf{r}\in V_{ji} \\ 0 & \text{otherwise} \end{cases} \tag{7b}$$

$$k_{js}^2 = \omega^2\mu\varepsilon_{js} \qquad k_j^2 = \omega^2\mu\varepsilon_j \tag{7c}$$

V_{ji} is the domain of the *i*-th scatterer and N_j is the number of scatterers in layer *j*. Thus v_j represents the volumetric fluctuations in Region *j*. For the homogeneous regions above and below we have

$$\nabla\times\nabla\times\mathbf{E}_0 - k_0^2\mathbf{E}_0 = 0 \tag{8a}$$

$$\nabla\times\nabla\times\mathbf{E}_{N+1} - k_{N+1}^2\mathbf{E}_{N+1} = 0 \tag{8b}$$

The boundary conditions at the *j*-th interface are

$$\hat{\mathbf{n}}\times\mathbf{E}_j(\mathbf{r}_\perp,\zeta_j) = \hat{\mathbf{n}}\times\mathbf{E}_{j+1}(\mathbf{r}_\perp,\zeta_j) \tag{9a}$$

$$\hat{\mathbf{n}}\times\nabla\times\mathbf{E}_j(\mathbf{r}_\perp,\zeta_j) = \hat{\mathbf{n}}\times\nabla\times\mathbf{E}_{j+1}(\mathbf{r}_\perp,\zeta_j) \tag{9b}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the *j*-th interface with normal pointing into the medium *j*. This system is complemented by the radiation conditions well away from the stack. We assume that we know the solution to the problem without volumetric fluctuations, and denote it as $\breve{\mathbf{E}}$. Let the Green's functions to this problem be denoted as $\breve{\bar{\mathbf{G}}}_{ij}$

These Green's functions are governed by the following set of equations:

$$\nabla\times\nabla\times\breve{\bar{\mathbf{G}}}_{jk}(\mathbf{r},\mathbf{r}') - k_j^2\breve{\bar{\mathbf{G}}}_{jk}(\mathbf{r},\mathbf{r}') = \bar{\mathbf{I}}\delta_{jk}\delta(\mathbf{r}-\mathbf{r}') \tag{10a}$$

$$\hat{\mathbf{n}}\times\breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r}_\perp,\zeta_j;\mathbf{r}')=\hat{\mathbf{n}}\times\breve{\bar{\bar{\mathbf{G}}}}_{(j+1)k}(\mathbf{r}_\perp,\zeta_j;\mathbf{r}') \tag{10b}$$

$$\hat{\mathbf{n}}\times\nabla\times\breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r}_\perp,\zeta_j;\mathbf{r}')=\hat{\mathbf{n}}\times\nabla\times\breve{\bar{\bar{\mathbf{G}}}}_{(j+1)k}(\mathbf{r}_\perp,\zeta_j;\mathbf{r}') \tag{10c}$$

where $\bar{\mathbf{I}}$ is unit dyad. The boundary conditions above are for the *j*-th interface. Similarly on the (*j*-1)-th interface we have the following boundary conditions.

$$\hat{\mathbf{n}}\times\breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r}_\perp,\zeta_{j-1};\mathbf{r}')=\hat{\mathbf{n}}\times\breve{\bar{\bar{\mathbf{G}}}}_{(j-1)k}(\mathbf{r}_\perp,\zeta_{j-1};\mathbf{r}') \tag{11a}$$

$$\hat{\mathbf{n}}\times\nabla\times\breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r}_\perp,\zeta_{j-1};\mathbf{r}')=\hat{\mathbf{n}}\times\nabla\times\breve{\bar{\bar{\mathbf{G}}}}_{(j-1)k}(\mathbf{r}_\perp,\zeta_{j-1};\mathbf{r}') \tag{11b}$$

Using these Green's functions and the radiation conditions the wave functions can be represented as

$$\mathbf{E}_j(\mathbf{r})=\breve{\mathbf{E}}_j(\mathbf{r})+\sum_{k=1}^{N}\int_{\Omega_k}\breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r},\mathbf{r}')v_k(\mathbf{r}')\mathbf{E}_k(\mathbf{r}')d\mathbf{r}' \qquad j=0,1,\ldots,N+1 \tag{12}$$

where $\Omega_k=\{\mathbf{r}'_\perp;\zeta_k<z'<\zeta_{k-1}\}$. Note that $v_0=v_{N+1}=0$.

In order to carry out multiple scattering analysis with a distribution of discrete scatterers it is convenient to employ the concept of transition operator $\bar{\mathbf{T}}$ [13,34,32]. Suppose we know the electric field $\mathbf{E}^l$ incident on the *l*-th scatterer. We introduce the transition operator $\bar{\mathbf{T}}^l$ such that the electric field scattered by the *l*-th scatterer is given as $\bar{\mathbf{T}}^l\mathbf{E}^l$. Using this concept (12) may be expressed as

$$\mathbf{E}_j(\mathbf{r})=\breve{\mathbf{E}}_j(\mathbf{r})+\sum_{k=1}^{N}\sum_{l=1}^{N_k}\int_{\Omega_k}dr'\int_{\Omega_k}dr''\breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r},\mathbf{r}')\bar{\mathbf{T}}_k^l(\mathbf{r}',\mathbf{r}'')\mathbf{E}_k^l(\mathbf{r}'') \qquad j=0,1,\ldots,N+1 \tag{13}$$

Note that $\bar{\mathbf{T}}_k^l$ depends only on the *l*-th scatterer. To proceed further with the multiple scattering analysis it is expedient to use symbolic representation of (13).

$$\mathbf{E}_j=\breve{\mathbf{E}}_j+\breve{\bar{\bar{\mathbf{G}}}}_{jk}\bar{\mathbf{T}}_k^l\mathbf{E}_k^l \tag{14}$$

First we average (14) w.r.t. volumetric fluctuations to get

$$\left\langle\mathbf{E}_j\right\rangle_v=\breve{\mathbf{E}}_j+\breve{\bar{\bar{\mathbf{G}}}}_{jk}\left\langle\bar{\mathbf{T}}_k^l\mathbf{E}_k^l\right\rangle_v \tag{15}$$

where the subscript v denotes volumetric averaging. Since there are N_k scatterers in the k-th layer

$$\left\langle \overline{\mathbf{T}}_k^l \mathbf{E}_k^l \right\rangle_v = \int \overline{\mathbf{T}}_k^l \mathbf{E}_k^l \, p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}; \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_k}) d\mathbf{r} d\mathbf{s} \tag{16}$$

where p is the joint probability density function of finding the scatterers at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}$ with orientations $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_k}$. We assume that the positions and orientations are independent of each other. In other words

$$p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}; \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_k}) = p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}) \, p(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_k}) \tag{17}$$

Furthermore, assume that the orientation of the particle at position $\mathbf{r}_1$ is independent of the orientation of all other particles, which means

$$p(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_k}) = \sum_{j=1}^{N_k} p(\mathbf{s}_j) \tag{18}$$

We next express the joint position probability density function as

$$p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}) = p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_l', \ldots, \mathbf{r}_{N_k} \left| \mathbf{r}_l \right|) \, p(\mathbf{r}_l) \tag{19}$$

where $p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_l', \ldots, \mathbf{r}_{N_k} \mid \mathbf{r}_l)$ is the conditional pdf of finding scatterers at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}$ by fixing the l-th scatterer is at $\mathbf{r}_l$. The prime in $\mathbf{r}_l'$ denotes that $\mathbf{r}_l$ should be omitted in the argument list of the conditional probability density function. Substituting this relation in (16) we obtain

$$\left\langle \overline{\mathbf{T}}_k^l \mathbf{E}_k^l \right\rangle = \left\langle \overline{\mathbf{T}}_k^l \left\langle \mathbf{E}_k^l \right\rangle^l \right\rangle \tag{20}$$

where $\left\langle \mathbf{E}_k^l \right\rangle^l$ denotes conditional average with scatterer l fixed at $\mathbf{r}_l$. If N_k is large and the distance between scatterers is large then we can approximate

$$\left\langle \mathbf{E}_k^l \right\rangle^l \approx \left\langle \mathbf{E}_k^l \right\rangle \tag{21}$$

This is called the Foldy's approximation and is applicable for sparse media. Under this approximation (15) becomes

$$\left\langle \mathbf{E}_j \right\rangle_v \approx \breve{\mathbf{E}}_j + \rho_k \breve{\overline{\mathbf{G}}}_{jk} \left\langle \overline{\mathbf{T}}_k \right\rangle \left\langle \mathbf{E}_k \right\rangle_v \tag{22}$$

where ρ_k is the density of the scatterers in layer *k*. Now operate the above equation by $\nabla\times\nabla\times\overline{\mathbf{I}}-k_j^2\overline{\mathbf{I}}$ to obtain

$$\nabla\times\nabla\times\left\langle\mathbf{E}_k\right\rangle_v-k_j^2\left\langle\mathbf{E}_k\right\rangle_v=\rho_j\left\langle\overline{\mathbf{T}}_j\right\rangle\left\langle\mathbf{E}_j\right\rangle_v \tag{23}$$

Next average this over surface fluctuations to get

$$\nabla\times\nabla\times\left\langle\mathbf{E}_j\right\rangle_{vs}-k_j^2\left\langle\mathbf{E}_j\right\rangle_{vs}=\rho_j\left\langle\overline{\mathbf{T}}_j\right\rangle\left\langle\mathbf{E}_j\right\rangle_{vs} \tag{24}$$

Note that the transition operators are independent of surface fluctuations. From this we see that

$$\nabla\times\nabla\times\left\langle\mathbf{E}_j\right\rangle_{vs}-k_j^2\left\langle\mathbf{E}_j\right\rangle_{vs}=0 \qquad j=0,N+1 \tag{25}$$

which means that the coherent propagation constants in the regions above and below the layered stack are unaffected by the fluctuations of the problem. However they indeed get modified in the layered stack region. On writing (24) as

$$\left\{\nabla\times\nabla\times\overline{\mathbf{I}}-k_j^2-\rho_j\left\langle\overline{\mathbf{T}}_j\right\rangle\right\}\left\langle\mathbf{E}_j\right\rangle_{vs}=0 \tag{26}$$

we infer that $\chi_j=\sqrt{k_j^{\,2}+\rho_j\left\langle\overline{\mathbf{T}}_j\right\rangle}$ represents the mean propagation constant, in operator form, for coherent waves in layer *j*.

Since the problem is statistically homogeneous in azimuth the mean fields in our system have the following form:

$$\left\langle\mathbf{E}_j^p(\mathbf{r})\right\rangle_{vs}=\exp(i\mathbf{k}_{\perp i}\cdot\mathbf{r}) \left\{A_j^p(\mathbf{k}_{\perp i})\mathbf{p}^+\exp\left[iq_j^p z\right]+B_j^p(\mathbf{k}_{\perp i})\mathbf{p}^-\exp\left[-iq_j^p z\right]\right\} \qquad j=1,2,\ldots,N \tag{27}$$

$$\left\langle\mathbf{E}_0^p(\mathbf{r})\right\rangle_{vs}=\exp(i\mathbf{k}_{\perp i}\cdot\mathbf{r})\left\{\mathbf{p}_0^-\ \exp\left[-ik_{0zi}^p z\right]+R^p(\mathbf{k}_{\perp i})\mathbf{p}_0^+\exp\left[ik_{0zi}^p z\right]\right\} \tag{28}$$

$$\left\langle\mathbf{E}_{N+1}^p(\mathbf{r})\right\rangle_{vs}=\exp(i\mathbf{k}_{\perp i}\cdot\mathbf{r})X^p(\mathbf{k}_{\perp i})\mathbf{p}_{N+1}^-\exp\left[-ik_{(N+1)zi}^p z\right] \tag{29}$$

where the superscript *p* stands for the polarization, either horizontal or vertical. $\mathbf{p}$ is the unit vector representing polarization. q_j is the *z*-component of χ_j. The subscript *i* is used to indicate that the wave vector is in the incident direction. *R* and *X* denote respectively the mean reflection and transmission coefficient of the stack. A_j and B_j denote respectively the mean coefficients of up-going and down-going waves in the *j*-th layer.

Based on this we can formulate the waves averaged w.r.t. volumetric fluctuations as

$$\left\langle \mathbf{E}_j^p(\mathbf{r}) \right\rangle_v = \frac{1}{4\pi^2} \int d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp \cdot \mathbf{r}) \left\{ A_j^{pq}(\mathbf{k}_\perp, \mathbf{k}_{\perp i}) q_j^+ \, \mathrm{e}^{iq_j z} + B_j^{pq}(\mathbf{k}_\perp, \mathbf{k}_{\perp i}) q_j^- \, \mathrm{e}^{-iq_j z} \right\} \tag{30}$$

$$\left\langle \mathbf{E}_0^p(\mathbf{r}) \right\rangle_v = \exp(i\mathbf{k}_\perp \cdot \mathbf{r}) \exp\left[-ik_{0zi} z\right] \mathbf{p}_0^- + \frac{1}{4\pi^2} \int \exp(i\mathbf{k}_\perp \cdot \mathbf{r}) R^{pq}(\mathbf{k}_\perp, \mathbf{k}_{\perp i}) q_0^+ \exp\left[ik_{0z} z\right] d\mathbf{k} \tag{31}$$

and

$$\left\langle \mathbf{E}_{N+1}^p(\mathbf{r}) \right\rangle_v = \frac{1}{4\pi^2} \int \exp(i\mathbf{k}_\perp \cdot \mathbf{r}) X^{pq}(\mathbf{k}_\perp, \mathbf{k}_{\perp i}) q_{N+1}^- \exp\left[-ik_{(N+1)z} z\right] d\mathbf{k}_\perp \tag{32}$$

where A_j, B_j, R and X are now integral operators representing scattering from rough interfaces. The boundary conditions associated with the above equations at the *j*-th interface are

$$\hat{\mathbf{n}} \times \left\langle \mathbf{E}_j(\mathbf{r}_\perp, \zeta_j) \right\rangle_v = \hat{\mathbf{n}} \times \left\langle \mathbf{E}_{j+1}(\mathbf{r}_\perp, \zeta_j) \right\rangle_v \qquad j = 1, 2, \ldots, N \tag{33}$$

and

$$\hat{\mathbf{n}} \times \nabla \times \left\langle \mathbf{E}_j(\mathbf{r}_\perp, \zeta_j) \right\rangle_v = \hat{\mathbf{n}} \times \nabla \times \left\langle \mathbf{E}_{j+1}(\mathbf{r}_\perp, \zeta_j) \right\rangle_v \qquad j = 1, 2, \ldots, N \tag{34}$$

The above system may be solved either numerically or by any one of the asymptotic methods available in rough surface scattering theory [5,4,33] to evaluate the mean coefficients that appear in (27)-(29).

We proceed now to the analysis of the second moments, by starting with (12). For convenience we write it in symbolic form as

$$\mathbf{E}_j = \breve{\mathbf{E}}_j + \sum_{k=1}^{N} \breve{\overline{\mathbf{G}}}_{jk} \overline{\mathbf{T}}_k^l \mathbf{E}_k^l \tag{35}$$

We take the tensor product of this equation with its complex conjugate and average w.r.t. volumetric fluctuations and obtain

$$\left\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \right\rangle = \left\langle \mathbf{E}_j \right\rangle_v \otimes \left\langle \mathbf{E}_j^* \right\rangle_v + \sum_{k=1}^{N} \sum_{k'=1}^{N} \sum_{l=1}^{N_k} \sum_{l'=1}^{N_{k'}} \left\langle \overline{\mathbf{G}}_{jk} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{jk'}^* \right\rangle_v \mathbf{K}_{kk'll'} \left\langle \mathbf{E}_k^l \otimes \mathbf{E}_k^{l'*} \right\rangle_v \tag{36}$$

where $\mathbf{K}$ is the intensity operator of the volumetric fluctuations. Employing the weak fluctuation approximation we approximate $\mathbf{K}$ by its leading term

$$\mathbf{K}_{kk'll'} \approx \left\langle \overline{\mathbf{T}}_k^l \otimes \overline{\mathbf{T}}_k^{l'*} \right\rangle \delta_{kk'} \delta_{ll'} \overline{\overline{\mathbf{I}}} \tag{37}$$

On substituting this in (36) we get,

$$\left\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \right\rangle_v = \left\langle \mathbf{E}_j \right\rangle_v \otimes \left\langle \mathbf{E}_j^* \right\rangle_v + \sum_{k=1}^{N} \rho_k \left\langle \overline{\mathbf{G}}_{jk} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{jk}^* \right\rangle_v \left\langle \overline{\mathbf{T}}_k \otimes \overline{\mathbf{T}}_k^* \right\rangle \left\langle \mathbf{E}_k \otimes \mathbf{E}_k^* \right\rangle_v \tag{38}$$

Next we average (38) w.r.t. the surface fluctuations to get

$$\left\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \right\rangle_{vs} = \left\langle \left\langle \mathbf{E}_j \right\rangle_v \otimes \left\langle \mathbf{E}_j^* \right\rangle_v \right\rangle_s + \sum_{k=1}^{N} \rho_k \left\langle \left\langle \overline{\mathbf{G}}_{jk} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{jk}^* \right\rangle_v \right\rangle_s \left\langle \overline{\mathbf{T}}_k \otimes \overline{\mathbf{T}}_k^* \right\rangle \left\langle \mathbf{E}_k \otimes \mathbf{E}_k^* \right\rangle_{vs} \tag{39}$$

where we have used the following approximation

$$\left\langle \left\langle \overline{\mathbf{G}}_{jk} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{jk}^* \right\rangle_v \left\langle \overline{\mathbf{T}}_k \otimes \overline{\mathbf{T}}_k^* \right\rangle \left\langle \mathbf{E}_k \otimes \mathbf{E}_k^* \right\rangle_v \right\rangle_s \approx \left\langle \left\langle \overline{\mathbf{G}}_{jk} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{jk}^* \right\rangle_v \right\rangle_s \left\langle \overline{\mathbf{T}}_k \otimes \overline{\mathbf{T}}_k^* \right\rangle \left\langle \mathbf{E}_k \otimes \mathbf{E}_k^* \right\rangle_{vs} \tag{40}$$

We call this the 'weak surface correlation' approximation, which we will see later to be an important condition embedded in the RT approach to our problem.

As it stands this equation is very difficult to solve either analytically or numerically. Besides, one important goal for us is to investigate the conditions needed for employing the radiative transfer approach for our problem. With these in mind we introduce Wigner transforms. Note that (39) is an equation for the coherence function which is a 'space-space' quantity. On the other hand the RT equation, as we saw earlier, is an equation for the specific intensity which is a 'phase-space' quantity. Wigner transforms serve as a bridge to link these two quantities [35,8,16,22].

We introduce Wigner transforms of waves and Green's functions as

$$\varepsilon_m\left(\frac{\mathbf{r}+\mathbf{r}'}{2}, \mathbf{k}\right) = \int \left\langle \mathbf{E}_m(\mathbf{r}) \right\rangle_v \otimes \left\langle \mathbf{E}_m^*(\mathbf{r}') \right\rangle_{vs} \exp\left[-ik \cdot (\mathbf{r}-\mathbf{r}')\right] d(\mathbf{r}-\mathbf{r}') \tag{41a}$$

$$\varepsilon_m^s\left(\frac{\mathbf{r}+\mathbf{r}'}{2}, \mathbf{k}\right) = \int \left\langle \left\langle \mathbf{E}_m(\mathbf{r}) \right\rangle_v \otimes \left\langle \mathbf{E}_m^*(\mathbf{r}') \right\rangle \right\rangle_s \exp\left[-ik \cdot (\mathbf{r}-\mathbf{r}')\right] d(\mathbf{r}-\mathbf{r}') \tag{41b}$$

$$\mathcal{G}_{mn}\left(\frac{\mathbf{r}+\mathbf{r}'}{2}, \mathbf{k} \,|\, \frac{\mathbf{r}_1+\mathbf{r}_1'}{2}, 1\right) = \int d(\mathbf{r}-\mathbf{r}') \int d(\mathbf{r}_1-\mathbf{r}_1') \mathrm{e}^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \mathrm{e}^{i\mathbf{l}\cdot(\mathbf{r}_1-\mathbf{r}_1')} \left\langle \left\langle \overline{\mathbf{G}}_{mn}(\mathbf{r},\mathbf{r}_1) \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{mn}^*(\mathbf{r}',\mathbf{r}_1') \right\rangle_v \right\rangle_s \tag{42}$$

In terms of these transforms (39) becomes

$$\mathscr{E}_m(\mathbf{r},\mathbf{k}) = \mathscr{E}_m^s(\mathbf{r},\mathbf{k}) + \frac{4}{(2\pi)^4}\sum_{n=1}^{N}\rho_n \int_{\Omega_n} d\mathbf{r}' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\, \mathscr{G}_{mn}(\mathbf{r},\mathbf{k}\,|\,\mathbf{r}',\boldsymbol{\alpha})\, \mathsf{T}_n(\mathbf{r}',\boldsymbol{\alpha}\,|\,\mathbf{r}'',\beta)\, \mathscr{E}_n(\mathbf{r}'',\beta) \tag{43}$$

with

$$\mathsf{T}_n(\mathbf{R}_1,\boldsymbol{\alpha}\,|\,\mathbf{R}_2,\boldsymbol{\beta}) = \int d\mathbf{r}_1 \int d\mathbf{r}_2 \exp\{-i\boldsymbol{\alpha}\cdot\mathbf{r}_1 + i\boldsymbol{\beta}\cdot\mathbf{r}_2\} \left\langle \overline{\mathbf{T}}\left(\mathbf{R}_1 + \tfrac{\mathbf{r}_1}{2}, \mathbf{R}_2 + \tfrac{\mathbf{r}_2}{2}\right) \otimes \overline{\mathbf{T}}^*\left(\mathbf{R}_1 - \tfrac{\mathbf{r}_1}{2}, \mathbf{R}_2 - \tfrac{\mathbf{r}_2}{2}\right)\right\rangle \tag{44}$$

where $\overline{\mathbf{T}}$ is the element transition operator in *n*-th layer.

The fact that our problem has translational invariance in azimuth implies the following:

$$\mathscr{E}_m(\mathbf{r},\mathbf{k}) = \mathscr{E}_m(z,\mathbf{k}) \tag{45a}$$

$$\mathscr{G}_{mn}(\mathbf{r},\mathbf{k}\,|\,\mathbf{r}',\mathbf{l}) = \mathscr{G}_{mn}(z,\mathbf{k}\,|\,z',\mathbf{l};\mathbf{r}_\perp - \mathbf{r}'_\perp) \tag{45b}$$

Using these relations in (43) we have

$$\mathscr{E}_m(z,\mathbf{k}) = \mathscr{E}_m^s(z,\mathbf{k}) + \frac{4}{(2\pi)^4}\sum_{n=1}^{N}\rho_n \int_{z_n}^{z_{n-1}} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\, \mathscr{G}_{mn}(z,\mathbf{k}\,|\,z',\boldsymbol{\alpha};0)\, \mathscr{F}_n(\boldsymbol{\alpha},\beta)\, \mathscr{E}_n(z',\beta) \tag{46}$$

where

$$\mathscr{G}_{mn}(z,\mathbf{k}\,|\,z',\boldsymbol{\alpha};0) = \int \mathscr{G}_{mn}(z,\mathbf{k}\,|\,z',\boldsymbol{\alpha};\mathbf{r}_\perp - \mathbf{r}'_\perp)\, d(\mathbf{r}_\perp - \mathbf{r}'_\perp) \tag{47}$$

$$\mathscr{F}_n(\boldsymbol{\alpha},\beta) = \overline{\mathbf{f}}(\boldsymbol{\alpha},\beta) \otimes \overline{\mathbf{f}}^*(\boldsymbol{\alpha},\beta) \tag{48}$$

$\overline{\mathbf{f}}$ is the element scattering matrix in the *n*-th layer. Since the medium is assumed to be sparse inter-particle scattering takes place in the far-field zone of each other. It is based on this fact that we have transitioned from T_n to $\mathscr{F}_n$. Also we have employed the on-shell approximation to T_n.

To proceed further we need to evaluate $\mathscr{G}_{mn}$. Further we need to relate this system with that of RT, which involves the boundary conditions at the interfaces. Therefore we need to identify the coherence functions corresponding to up- and down-going wave functions. To facilitate this we decompose $\langle G_{mn}\rangle_v$ into its components,

$$\left\langle \overline{\mathbf{G}}_{mn} \right\rangle_v = \delta_{mn}\overline{\mathbf{G}}_m^0 + \overline{\mathbf{G}}_{mn}^{uu} + \overline{\mathbf{G}}_{mn}^{ud} + \overline{\mathbf{G}}_{mn}^{du} + \overline{\mathbf{G}}_{mn}^{dd} \tag{49}$$

where the first term is the singular part of the Green's function. The superscripts *u* and *d* indicate up- and down-going elements of the waves. The other components are due to reflections from boundaries. These are formally constructed using the concept of surface scattering operators as follows [33],

$$\left\langle \overline{\mathbf{G}}_{mn}^{ab}(\mathbf{r},\mathbf{r}') \right\rangle_v^{\mu\nu} = \frac{1}{(2\pi)^4} \int d\mathbf{k}_\perp \int d\mathbf{k}'_\perp \left\{ S_{mn}^{ab}(\mathbf{k}_\perp,\mathbf{k}'_\perp) \right\}^{\mu\nu} \exp\left\{ i\mathbf{k}_\perp\cdot\mathbf{r} + iaq_m^\mu(\mathbf{k}_\perp)z \right\} \exp\left\{ -i\mathbf{k}'_\perp\cdot\mathbf{r}' - ibq_n^\nu(\mathbf{k}'_\perp)z' \right\} \tag{50}$$

where $\overline{\mathbf{S}}_{mn}^{ab}$ is the surface scattering operator. The superscripts *a* and *b* on *S* are used to indicate whether the waves are up-going or down-going. In the exponents, *a,b*=1 if the waves are up-going. We let *a, b*=-1 if the waves are down-going. The *z*-component of the mean propagation constants in the *n*-th layer is denoted as q_n. We recall that $\mathscr{F}_{mn}$ is the Wigner transform of $\left\langle \left\langle \overline{\mathbf{G}}_{mn} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{mn}^* \right\rangle_v \right\rangle_s$. The superscripts μ,ν stand for polarization, either *h* or *v*. When we use (34) to perform the Wigner transform we ignore all cross terms. In other words, we make the following approximation,

$$\mathscr{G}_{mn} = \delta_{mn}\mathscr{G}_m^0 + \mathscr{G}_{mn}^{uu} + \mathscr{G}_{mn}^{ud} + \mathscr{G}_{mn}^{du} + \mathscr{G}_{mn}^{dd} \tag{51}$$

where $\mathscr{G}_{mn}^{ab}$ is the Wigner transform of $\left\langle \left\langle \overline{\mathbf{G}}_{mn}^{ab} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{mn}^{ab*} \right\rangle_v \right\rangle_s$. This approximation reminds us of the concept of incoherent addition of intensities associated with the phenomenological radiative transfer theory.

With the introduction of this representation for $\mathscr{G}_{mn}$ in (43) we can trace up- and down-going waves to obtain the following equations for the coherence function:

$$\begin{aligned} \mathscr{E}_m^u(z,\mathbf{k}) = \mathscr{E}_m^{su}(z,\mathbf{k}) &+ \frac{4}{(2\pi)^4}\rho_m \int_{z_m}^{z} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\, \mathscr{G}_m^{>}(z,\mathbf{k}\,|\,z',\boldsymbol{\alpha};0)\Upsilon_m^{ua}(\boldsymbol{\alpha},\beta)\mathscr{E}_m^a(z',\beta) \\ &+ \frac{4}{(2\pi)^4}\sum_{n=1}^{N}\rho_n \int_{z_n}^{z_{n-1}} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\, \mathscr{G}_{mn}^{ua}(z,\mathbf{k}\,|\,z',\boldsymbol{\alpha};0)\Upsilon_m^{ab}(\boldsymbol{\alpha},\beta)\mathscr{E}_n^b(z',\beta) \end{aligned} \tag{52a}$$

$$\mathscr{E}_m^d(z,\mathbf{k}) = \mathscr{E}_m^{sd}(z,\mathbf{k})$$

$$+\frac{4}{(2\pi)^4}\rho_m\int_z^{z_{m-1}}dz'\int d\boldsymbol{\alpha}\int d\boldsymbol{\beta}\mathscr{G}_m^{<}\left(z,\mathbf{k}\,|\,z',\boldsymbol{\alpha};0\right)\Upsilon_m^{da}(\boldsymbol{\alpha},\beta)\mathscr{E}_m^{a}(z',\beta)$$
$$+\frac{4}{(2\pi)^4}\sum_{n=1}^{N}\rho_n\int_{z_n}^{z_{n-1}}dz'\int d\boldsymbol{\alpha}\int d\boldsymbol{\beta}\mathscr{G}_{mn}^{da}\left(z,\mathbf{k}\,|\,z',\boldsymbol{\alpha};0\right)\Upsilon_n^{ab}(\boldsymbol{\alpha},\beta)\mathscr{E}_n^{b}(z',\beta) \tag{52b}$$

Note that summation over $a,b=\{u,d\}$ is implied in the above equations. The first term in these equations, ε^{sa}, represents the contribution due exclusively to surface scattering, and has the following form:

$$\left\{\varepsilon_m^{sa}(z,\mathbf{k})\right\}^{\mu\nu}=2\pi\delta\left\{k_z-\frac{1}{2}a\left[q_m^{\mu}(\mathbf{k}_\perp)+q_m^{\nu*}(\mathbf{k}_\perp)\right]\right\}\exp\left[ia\left(q_m^{\mu}-q_m^{\nu*}\right)z\right]$$
$$\left\langle\left\{\Sigma_m^{a}\right\}^{\mu\mu'}\left\{\Sigma_m^{a*}\right\}^{\nu\nu'}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})\right\rangle_s E_{\mu' i}E_{\nu' i}^{*} \tag{53}$$

where Σ_m^a is the amplitude of the up-going wave in the m-th layer after volumetric averaging is performed. This means that it is a random function of surface fluctuations. When we substitute (53) and the expressions for $\Im_{mn}$ in (52) we find that

$$\left\{\mathscr{E}_m^{a}(z,\mathbf{k})\right\}_{\mu\nu}=2\pi\delta\left\{k_z-\frac{1}{2}a\left[q_m^{\mu}(\mathbf{k}_\perp)+q_m^{\nu*}(\mathbf{k}_\perp)\right]\right\}\exp\left[ia\left(q_m^{\mu}-q_m^{\nu*}\right)z\right]\left\{\mathscr{E}_m^{a}(z,\mathbf{k}_\perp)\right\}_{\mu\nu} \tag{54}$$

On substituting this in (52) and differentiating w.r.t. z we obtain the following transport equations:

$$\left\{\frac{d}{dz}-i\left[q_u(\mathbf{k}_\perp)-q_\nu^{*}(\mathbf{k}_\perp)\right]\right\}\mathscr{E}_{\mu\nu}^{u}(z,\mathbf{k}_\perp)=$$
$$=4\rho_m\int d\boldsymbol{\alpha}_\perp S_\mu^{>}\left(\boldsymbol{\alpha}_\perp\right)\otimes S_\nu^{>*}\left(\boldsymbol{\alpha}_\perp\right)$$
$$\mathscr{F}_{\mu\nu;\mu'\nu'}^{ua}\left\{\mathbf{k}_\perp,\frac{1}{2}\left[q_\mu(\mathbf{k}_\perp)+q_\nu^{*}(\mathbf{k}_\perp)\right];\boldsymbol{\alpha}_\perp,\frac{1}{2}a\left[q_{\mu'}(\boldsymbol{\alpha}_\perp)+q_{\nu'}^{*}(\boldsymbol{\alpha}_\perp)\right]\right\}\mathscr{E}_{\mu'\nu'}^{a}(z,\boldsymbol{\alpha}_\perp) \tag{55a}$$

$$\left\{-\frac{d}{dz}-i\left[q_u(\mathbf{k}_\perp)-q_\nu^{*}(\mathbf{k}_\perp)\right]\right\}\varepsilon_{\mu\nu}^{d}(z,\mathbf{k}_\perp)=$$
$$=4\rho_m\int d\boldsymbol{\alpha}_\perp S_\mu^{<}\left(\boldsymbol{\alpha}_\perp\right)\otimes S_\nu^{<*}\left(\boldsymbol{\alpha}_\perp\right)$$
$$\mathscr{F}_{\mu\nu;\mu'\nu'}^{ua}\left\{\mathbf{k}_\perp,-\frac{1}{2}\left[q_\mu(\mathbf{k}_\perp)+q_\nu^{*}(\mathbf{k}_\perp)\right];\boldsymbol{\alpha}_\perp,\frac{1}{2}a\left[q_{\mu'}(\boldsymbol{\alpha}_\perp)+q_{\nu'}^{*}(\boldsymbol{\alpha}_\perp)\right]\right\}\mathscr{E}_{\mu'\nu'}^{a}(z,\boldsymbol{\alpha}_\perp) \tag{55b}$$

where summation over a is implied. When the superscript a corresponds to u the value of a in the argument of γ_m take the value +1; on the other hand when the superscript a

corresponds to d the value of a in the argument of γ_m take the value -1. Since all quantities in (55) correspond to the same layer m we have dropped the subscript m in γ and ε to avoid cumbersome notations. To obtain appropriate boundary conditions we have to go back to the integral equation representations for $\varepsilon^{u}_{\mu\nu}$ and $\varepsilon^{d}_{\mu\nu}$, examine their behavior at the interfaces, and seek a relation between them. After considerable effort we managed to arrive at the following boundary conditions. At the *(m-1)*-th interface we have

$$\mathcal{E}^{d}_{m}(z_{m-1},\mathbf{k}_\perp)=\int\left\langle\ddot{\Re}_{m-1,m}(\mathbf{k}_\perp,\mathbf{k}'_\perp)\right\rangle\mathcal{E}^{u}_{m}(z_{m-1},\mathbf{k}'_\perp)d\mathbf{k}'_\perp \tag{56}$$

with $\ddot{\Re}=\dddot{\overline{\mathbf{R}}}\otimes\dddot{\overline{\mathbf{R}}}^*$ where $\dddot{\overline{\mathbf{R}}}_{m-1,m}$ is the stack reflection matrix (not the local reflection matrix) for a wave incident from below on the *(m-1)*-th interface. Similarly

$$\mathcal{E}^{u}_{m}(z_{m-1},\mathbf{k}_\perp)=\int\left\langle\ddot{\Re}_{m+1,m}(\mathbf{k}_\perp,\mathbf{k}'_\perp)\right\rangle\mathcal{E}^{d}_{m}(z_{m-1},\mathbf{k}'_\perp)d\mathbf{k}'_\perp \tag{57}$$

where $\ddot{\Re}_{m+1,m}$ is the tensor product of stack reflection matrix for a wave incident from above on the *(m-1)*-th interface. We were able to obtain the boundary conditions only after imposing certain approximations such as the one given below. Consider the following identity:

$$\overline{\mathbf{S}}^{du}_{mm}=\overline{\mathbf{D}}_m\dddot{\overline{\mathbf{R}}}_{m-1,m}\left\{\overline{\mathbf{S}}^{>}_{m}+\overline{\mathbf{S}}^{uu}_{mm}\right\}\overline{\mathbf{D}}_m \tag{58}$$

where $\overline{\mathbf{D}}_m=\mathrm{diag}\left\{\exp(iq_h d_m),\exp(iq_v d_m)\right\}$. Notice that this is an operator relation where all elements are operators. Taking the tensor product of (58) with its complex conjugate we have

$$\begin{aligned}\overline{\mathbf{S}}^{du}_{mm}\otimes\overline{\mathbf{S}}^{du*}_{mm}=(\overline{\mathbf{D}}_m\otimes\overline{\mathbf{D}}^*_m)&\left(\dddot{\overline{\mathbf{R}}}_{m-1,m}\otimes\dddot{\overline{\mathbf{R}}}^*_{m-1,m}\right)\\&\left[\left\{\overline{\mathbf{S}}^{>}_{m}+\overline{\mathbf{S}}^{uu}_{mm}\right\}\otimes\left\{\overline{\mathbf{S}}^{>}_{m}+\overline{\mathbf{S}}^{uu}_{mm}\right\}^*\right]\left(\overline{\mathbf{D}}_m\otimes\overline{\mathbf{D}}^*_m\right)\end{aligned} \tag{59}$$

Next we average (59) w.r.t. surface fluctuations and get

$$\begin{aligned}\left\langle\overline{\mathbf{S}}^{du}_{mm}\otimes\overline{\mathbf{S}}^{du*}_{mm}\right\rangle\approx(\overline{\mathbf{D}}_m\otimes\overline{\mathbf{D}}^*_m)&\left\langle\dddot{\overline{\mathbf{R}}}_{m-1,m}\otimes\dddot{\overline{\mathbf{R}}}^*_{m-1,m}\right\rangle\\&\left\langle\left\{\overline{\mathbf{S}}^{>}_{m}+\overline{\mathbf{S}}^{uu}_{mm}\right\}\otimes\left\{\overline{\mathbf{S}}^{>}_{m}+\overline{\mathbf{S}}^{uu}_{mm}\right\}^*\right\rangle\left(\overline{\mathbf{D}}_m\otimes\overline{\mathbf{D}}^*_m\right)\end{aligned} \tag{60}$$

where we have approximated that the two tensor products in the middle are weakly correlated. A further approximation that we impose is given as follows:

$$\left\langle\left\{\overline{\mathbf{S}}^{>}_{m}+\overline{\mathbf{S}}^{uu}_{mm}\right\}\otimes\left\{\overline{\mathbf{S}}^{>}_{m}+\overline{\mathbf{S}}^{uu}_{mm}\right\}^*\right\rangle\approx\overline{\mathbf{S}}^{>}_{m}\otimes\overline{\mathbf{S}}^{>*}_{m}+\left\langle\overline{\mathbf{S}}^{uu}_{mm}\otimes\overline{\mathbf{S}}^{uu*}_{mm}\right\rangle \tag{61}$$

These are the kinds of approximations required to arrive at our boundary conditions.

7. Transition to Radiative Transfer

Next we have to transition from this transport equation (55) to the phenomenological radiative transfer equation discussed earlier. To accomplish this we have to link the key quantities of waves and radiative transfer, viz., coherence function and specific intensity. The relation between them is obtained by computing the energy density using the two concepts. Thus we have

$$\frac{1}{2}\varepsilon\left\langle E_{\mu}(\mathbf{r})E^{*}_{\nu}(\mathbf{r})\right\rangle = \sqrt{\mu\varepsilon}\int I_{\mu\nu}(\mathbf{r},\hat{s})d\Omega_{s} \tag{62}$$

Wigner transform provides us following relation:

$$\left\langle E_{\mu}(\mathbf{r})E^{*}_{\nu}(\mathbf{r})\right\rangle = \frac{1}{(2\pi)^{2}}\int \mathcal{E}_{\mu\nu}(z,\mathbf{k}_{\perp})d\mathbf{k}_{\perp} \tag{63}$$

From (62) and (63) we relate I to ε as

$$I_{\mu\nu}(z,\hat{s}) = \frac{1}{2\eta}\frac{k'^{2}}{(2\pi)^{2}}\cos\theta\varepsilon_{\mu\nu}(z,\mathbf{k}_{\perp}) \tag{64}$$

Now we can transition to the phenomenological RT equations. Using the relation between $\mathcal{E}$ and I we change the integration variable to solid angle and arrive at the following equation,

$$\left\{\cos\theta\frac{d}{dz}+\eta_{ij}\right\}I_{j}^{u}(z,\hat{s}) = \int\left\{P_{ij}^{uu}\left(\Omega,\Omega'\right)I_{j}^{u}(z,\hat{s}') + P_{ij}^{ud}\left(\Omega,\Omega'\right)I_{j}^{d}(z,\hat{s}')\right\}d\Omega' \tag{65a}$$

$$\left\{-\cos\theta\frac{d}{dz}+\eta_{ij}\right\}I_{j}^{d}(z,\hat{s}) = \int\left\{P_{ij}^{du}\left(\Omega,\Omega'\right)I_{j}^{u}(z,\hat{s}') + P_{ij}^{dd}\left(\Omega,\Omega'\right)I_{j}^{d}(z,\hat{s}')\right\}d\Omega' \tag{65b}$$

where $\bar{\bar{\eta}}$ is the extinction matrix and $\bar{\bar{\mathbf{P}}}$ is the phase matrix. Implicit summation over subscript *j* is assumed in (65). To facilitate comparison with the results of Ulaby et al. [30], and Lam and Ishimaru [12] we have used a modified version of Stokes vector [10]. Instead of the standard form $\{I,Q,U,V\}$ we use $\{(I+Q)/2,(I-Q)/2,U,V\}$. The subscript of I denotes the element number of our modified Stokes vector. Although the structure of this equation is identical to that of the RT (equation (2)), the elements of the phase matrix and the extinction matrices are not the same. The primary reason is because of the differences in the real part of the mean propagation constants of horizontally and vertically polarized waves. On assuming that $q_{h}' = q_{v}' = k_{mz}'$ we obtain the following expressions for the extinction and phase matrices:

$$\bar{\bar{\eta}} = -\cos\theta\ \mathrm{diag}\left\{2q_{v}'',2q_{h}'',q_{v}''+q_{h}'',q_{v}''+q_{h}''\right\} \tag{66}$$

$$P_{ij}^{ab}(\Omega,\Omega') = \mathscr{P}_{ij}\left\{\mathbf{k}_\perp, ak\cos\theta; \mathbf{k}'_\perp, bk\cos\theta'\right\} \tag{67}$$

where

$$\begin{array}{llll}
\mathscr{P}_{11} = \left\langle |f_{vv}|^2 \right\rangle & \mathscr{P}_{12} = \left\langle |f_{vh}|^2 \right\rangle & \mathscr{P}_{13} = \left\langle \mathrm{Re}(f_{vv} f_{vh}^*) \right\rangle & \mathscr{P}_{14} = -\left\langle \mathrm{Im}(f_{vv} f_{vh}^*) \right\rangle \\
\mathscr{P}_{21} = \left\langle |f_{hv}|^2 \right\rangle & \mathscr{P}_{22} = \left\langle |f_{hh}|^2 \right\rangle & \mathscr{P}_{23} = \left\langle \mathrm{Re}(f_{hv} f_{hh}^*) \right\rangle & \mathscr{P}_{24} = -\left\langle \mathrm{Im}(f_{hv} f_{hh}^*) \right\rangle \\
\mathscr{P}_{31} = 2\left\langle \mathrm{Re}(f_{vv} f_{hv}^*) \right\rangle & \mathscr{P}_{32} = 2\left\langle \mathrm{Re}(f_{vh} f_{hh}^*) \right\rangle & \mathscr{P}_{33} = 2\left\langle \mathrm{Re}(f_{vv} f_{hh}^*) \right\rangle & \mathscr{P}_{34} = -\left\langle \mathrm{Im}(f_{vv} f_{hh}^*) \right\rangle \\
 & & & \qquad +\left\langle \mathrm{Re}(f_{vh} f_{hv}^*) \right\rangle \\
\mathscr{P}_{41} = 2\left\langle \mathrm{Im}(f_{vv} f_{hv}^*) \right\rangle & \mathscr{P}_{42} = 2\left\langle \mathrm{Im}(f_{vh} f_{hh}^*) \right\rangle & \mathscr{P}_{43} = \left\langle \mathrm{Im}(f_{vv} f_{hh}^*) \right\rangle & \mathscr{P}_{44} = \left\langle \mathrm{Re}(f_{vv} f_{hh}^*) \right\rangle \\
 & & \qquad +\left\langle \mathrm{Im}(f_{vh} f_{hv}^*) \right\rangle & \qquad -\left\langle \mathrm{Re}(f_{vh} f_{hv}^*) \right\rangle
\end{array} \tag{68}$$

We have suppressed the arguments for brevity. For instance,
$\mathscr{P}_{13}\left\{\mathbf{k}_\perp, ak\cos\theta; \mathbf{k}'_\perp, bk\cos\theta'\right\}$

$$= \left\langle \mathrm{Re}\left\{ f_{vv}\left(\mathbf{k}_\perp, ak\cos\theta; \mathbf{k}'_\perp, bk\cos\theta'\right) f_{vh}^*\left(\mathbf{k}_\perp, ak\cos\theta; \mathbf{k}'_\perp, bk\cos\theta'\right)\right\}\right\rangle \tag{69}$$

Note that these transport equations (65) are identical to those of classical RT equations (2) that we described in Section 2. Thanks to our statistical wave approach we now have explicit expressions for the extinction matrix and phase matrix in terms of the statistical parameters of the problem. Let us now next turn our attention to the boundary conditions (BC). In our wave approach we obtained BCs in terms of `stack' reflection matrix $\dddot{\overline{\mathbf{R}}}$, whereas in the RT approach the BCs are given in terms of the local interface reflection matrices. We can readily reconcile with this apparent difference. Note that the BC in the wave approach forms a closed system whereas in the RT approach it is open (linked to adjacent layer intensities). Let us take a look at the BC at the (*m*-1)-th interface. $\dddot{\overline{\mathbf{R}}}_{m-2,m}$ can be expressed in terms of $\overline{\mathbf{R}}_{m-2,m-1}$ as follows,

$$\dddot{\overline{\mathbf{R}}}_{m-1,m} = \overline{\mathbf{R}}_{m-1,m} + \overline{\mathbf{T}}_{m,m-1}\left[\overline{\mathbf{I}} - \dddot{\overline{\mathbf{R}}}_{m-2,m-1}\overline{\mathbf{D}}_{m-1}\overline{\mathbf{R}}_{m,m-1}\right]^{-1}\dddot{\overline{\mathbf{R}}}_{m-2,m-1}\overline{\mathbf{D}}_{m-1}\overline{\mathbf{T}}_{m-1,m} \tag{70}$$

This is the relation between the stack reflection coefficients of adjacent interfaces. The $\overline{\mathbf{R}}$ and $\overline{\mathbf{T}}$ are local (single interface) reflection and transmission matrices at the *(m-1)*-th interface. On operating $\mathbf{E}_m^u$ with (70) we get

$$\mathbf{E}_m^d = \overline{\mathbf{R}}_{m-1,m}\mathbf{E}_m^u + \overline{\mathbf{T}}_{m,m-1}\mathbf{E}_{m-1}^d \tag{71}$$

Notice that this boundary condition now involves only local interface Fresnel coefficients. Take the tensor product of (71) with its complex conjugate and average w.r.t. surface fluctuations. Employing the Wigner transform operator on this, we obtain a boundary condition at the (m-1)-th interface similar to that of the RT system. However, the reflection and transmission matrices used in the RT system correspond to unperturbed medium as opposed to the average medium as in the case of the wave approach.

Similarly we write $\dddot{\mathbf{R}}_{m+1,m}$ in terms of $\dddot{\mathbf{R}}_{m+2,m+1}$ and hence obtain the BC at the m-th interface as

$$\mathbf{E}_m^u = \overline{\mathbf{R}}_{m+1,m}\mathbf{E}_m^d + \overline{\mathbf{T}}_{m,m+1}\mathbf{E}_{m+1}^u \tag{72}$$

Take the tensor product of (71) with its complex conjugate and average w.r.t. surface fluctuations. Employing the Wigner transform operator on this, we obtain boundary condition at the m-th interface identical to (3) (after making the approximation as before).

8. Discussion

Now that we have made the transition from statistical wave theory to radiative transfer theory it is instructive to itemize the assumptions implicitly involved in the RT approach:

1. Quasi-stationary field approximation
2. Sparse distribution of scatterers

These are the well-known conditions necessary for the unbounded random medium problem. However, if the medium is bounded we need to impose additional conditions. We found that the extinction coefficients calculated in the wave approach and the RT approach are different and only after applying further approximations can they be made to agree with each other. The following additional condition is required for our bounded random medium problem:

3. Layer thickness must be of the same order or greater than the corresponding mean free path.

When the interfaces are randomly rough we need the following additional conditions.

4. Weak surface correlation approximation.
5. All fluctuations of the problem are statistically independent of each other.

Recently Mishchenko et al. [17] (hereafter referred to as MTL for brevity) derived the vector radiative transfer equation (VRTE) for a bounded discrete random medium using a rigorous microphysical approach. This enabled them to identify the following assumptions embedded in the VRTE.

1. Scattering medium is illuminated by a plane wave.
2. Each particle is located in the far field zone of all other particles and the observation point is also located in the far field zones of all the particles forming the scattering medium.
3. Neglect all scattering paths going through a particle two or more times (Twersky approximation).
4. Assume that the scattering system is ergodic and averaging over time can be replaced by averaging over particle positions and states.
5. Assume that (i) the position and state of each particle are statistically independent of each other and those of all other particles and (ii) spatial distribution of the particles throughout the medium is random and statistically uniform.
6. Assume that the scattering medium is convex.
7. Assume that the number of particles N forming the scattering medium is very large.
8. Ignore all the diagrams with crossing connections in the diagrammatic expansion of the coherency dyadic.

Below we make a connection between the two by considering each condition derived by MTL and relating it to ours. We will denote the condition numbers derived by MTL as MTL # and those obtained in this paper as SM #

- **MTL 1**:- We also have a plane electromagnetic wave illuminating our system, although as pointed out by MTL it can be a quasi-plane wave.

- **MTL 2**:- We also have implicitly employed the far field approximation. It is embedded in SM 1 and SM 2.
- **MTL 3**:- This is embedded in SM 2. Although not explicitly stated, the scattering processes as mentioned in MTL 3 are avoided. This is identical to the Foldy's approximation use in this paper.
- **MTL 4**:- In our paper we have restricted our attention to the time-independent problem and hence did not encounter the issue of ergodicity.
- **MTL 5**:- We have also employed this assumption although we did not itemize this as a condition.
- **MTL 6**:- In our problem we have distinct scattering boundaries and the character of the waves exiting or entering them are explicitly contained in the boundary conditions. Hence convexity of the scattering medium is not a necessary condition for us.
- **MTL 7**:- This condition is embedded in SM 3.
- **MTL 8**:- This condition is embedded in SM 2. Under sparse medium conditions we only take into consideration the leading term of the intensity operator.

Since the problem we considered in this report involves scattering boundaries we have some additional conditions beyond those of MTL. There are a few more remarks that we would like to make before closing. (a) In RT theory the medium is assumed to be sparse and hence the “refraction effects” of the fluctuations are ignored. Thus in the boundary conditions we should use the background medium parameters rather than the effective medium parameters as derived in our statistical wave theory. (b) To arrive at (50) from (49) we have ignored the contribution of evanescent modes.

To summarize, we have enquired into the assumptions involved in adopting the radiative transfer approach to scattering from layered random media with rough interfaces. To facilitate this enquiry we adopted a wave approach to this problem and derived the governing equations for the first and second moments of the wave fields. We employed Wigner transforms and transitioned to the system corresponding to that of radiative transfer approach. In this process we found that there are more conditions implicitly involved in the RT approach to this problem than it is widely believed to be sufficient. With the recent development of fast and efficient algorithms for scattering computations and the enormous increase in computer resources it is now feasible to take an entirely numerical approach to this problem without imposing any approximations.

In spite of such developments, to keep the size of the problem manageable only special cases have been studied thus far [9,23,21,25]. Hence it is very much of relevance, interest and convenience to apply the RT approach to these problems. However, one should keep in mind the assumptions involved in such an approach. Otherwise interpretations of results based on RT theory can be misleading.

9. References

[1] G. Arsar, editor. *Theory and Applications of Optical Remote Sensing*. John Wiley and Sons, New York, 1989.
[2] Yu. N. Barabanenkov. Multiple scattering of waves by ensemble of particles and the theory of radiative transfer. *Sov. Phys. Uspekhi*, 18:673–689, 1975.
[3] Yu. N. Barabanenkov, A. G. Vinogradov, Yu. A. Kravtsov, and V. I. Tatarskii. Application of the theory of multiple scattering of waves to the derivation of the radiation transfer equation for a statistically inhomogeneous medium. *Izv. VUZ Radiofiz.*, 15:1852–1860, 1972.
[4] F. G. Bass and I. M. Fuks. *Wave Scattering from Statistically Rough Surfaces*. Pergamon, Oxford, 1979.
[5] P. Beckmann and A. Spizzichino. *The Scattering of Electromagnetic Waves from Rough Surfaces*. Artech House, Norwood, Massachusetts, 1987.
[6] D.R. Huffman C.F. Bohren. *Absorption and scattering of light by small particles*. John Wiley, New York, 2004.
[7] S. Chandrasekar. *Radiative Transfer*. Dover, New York, 1960.
[8] A. T. Friberg. Energy transport in optical systems with partially coherent light. *Applied Optics*, 25:4547–4556, 1986.
[9] H. Giovannini, M. Saillard, and A. Sentenac. Numerical study of scattering from rough inhomogeneous films. *J. Opt. Soc. Am. A*, 15:1182–1191, 1998.
[10] A. Ishimaru. *Wave Propagation and Scattering in Random Media*. IEEE Press, New York, 1997.
[11] M. Kerker, editor. *Selected Papers on Light Scattering*. SPIE, Bellingham, 1988.
[12] C. M. Lam and A. Ishimaru. Muller matrix representation for a slab of random medium with discrete particles and random rough surfaces. *Waves Random Media*, 3:111–125, 1993.
[13] M. Lax. Multiple scattering of waves. *Rev. Mod. Phys.*, 23:287–310, 1951.
[14] J. Lenoble. *Atmospheric Radiative Transfer*. A. Deepak Publishing, Hampton, Virginia, 1993.
[15] S. Liang. *Quantitative Remote Sensing of Land Surfaces*. Wiley-Interscience, New York, 2003.
[16] E. W. Marchand and E.Wolf. Radiometry with sources in any state of coherence. *J. Opt. Soc. Am.*, 64:1219–1226, 1974.
[17] A.A. Lacis M.I. Mishchenko, L.D. Travis. *Scattering, absorption, and emission of light by small particles*. Cambridge University Press, Cambridge, 2002.
[18] C.D. Mobley. *Light and Water: Radiative Transfer in Natural Waters*. Academic Press, San Diego, 1994.
[19] M.F. Modest. *Radiative Heat Transfer*. McGraw Hill, New York, 1993.
[20] H.H. Natsuyama, S. Ueno, and A.P. Wang. *Terrestrial Radiative Transfer: Modelling, Computation and Data Analysis*. Springer-Verlag, New York, 1998.
[21] K. Pak, L. Tsang, L. Li, and C. H. Chan. Combined random rough surface and volume scattering based on Monte-Carlo simulation of Maxwell equations. *Radio Sci.*, 23:331–338, 1993.
[22] H. M. Pederson and J. J. Stamnes. Radiometric theory of spatial coherence in free space propagation. *J. Opt. Soc. Am. A*, 17:1413–1420, 2000.

[23] G. Pelosi and K. Coccioli. A finite element approach for scattering from inhomogeneous media with a rough interface. *Waves Random Media*, 7:119–127, 1997.
[24] E. Saatdjian. *Transport Phenomena: Equations and Numerical Solutions*. John Wiley, New York, 2000.
[25] K. Sarabandi, O. Yisok, and F.T. Ulaby. A numerical simulation of scattering from one-dimensional inhomogeneous dielectric rough surfaces. *IEEE Trans. Geosci. Remote Sens.*, 34:425–432, 1996.
[26] H. Sato and M. C. Fehler. *Seismic Wave Propagation and Scattering in the Heterogeneous Earth*. Springer-Verlag, New York, 1998.
[27] V. V. Sobolev. *A Treatise on Radiative Transfer*. Van Nostrand, Princeton, New Jersey, 1963.
[28] G.E. Thomas and K. Stamnes. *Radiative Transfer in the Atmosphere and Ocean*. Cambridge University Press, Cambridge, 1999.
[29] L. Tsang, J. A. Kong, and R. T. Shin. *Theory of Microwave Remote Sensing*. John Wiley, New York, 1985.
[30] F. T. Ulaby, R. K. Moore, and A. K. Fung. *Microwave Remote Sensing: Active and Passive*, volume 3. Artech House, Norwood, Massachusetts, 1986.
[31] H.C. van de Hulst. *Light Scattering by Small Particles*. Dover Publications, New York, 1981.
[32] V.V. Varadan V.K. Varadan, editor. *Acoustic, Electromagnetic, and Elastic Wave Scattering - Focus on the T-matrix Approach*. Pergamon, New York, 1980.
[33] A. G. Voronovich. *Wave Scattering from Rough Surfaces*. Springer-Verlag, Berlin, 2nd edition, 1999.
[34] P.C. Waterman. Matrix formulation of electromagnetic scattering. *Proc. IEEE*, 53:805–811, 1965.
[35] K. Yoshimori. Radiometry and coherence in a nonstationary optical field. *J. Opt. Soc. Am. A*, 15:2730–2734, 1998.