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Previously, Chang reported a new class of high-order Conservation Element Solution Element, CESE, methods for solving nonlinear hyperbolic partial differential equations. The scheme was then extended by Bilyeu, et al. to solve a 1D vector equation with an arbitrary order of convergence. This new high-order CESE method shares many favorable attributes of the original second-order CESE method, including: (i) compact mesh stencil involving only the immediate mesh nodes surrounding the node where the solution is sought, (ii) the CFL stability constraint remains the same, i.e., _ 1, as , and (iii) superb shock capturing capability without using an approximate Riemann solver. In the present extended abstract, we extend the 1D formulation to 2D. A general formulation is presented for solving the coupled equations with 4th order accuracy. To demonstrate the formulation, two linear cases are reported. The linear test cases shown are the convection equation and the linear acoustic equation. In the final paper we plan to present non-linear test cases and solve the equations on a quad mesh.

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A Generic 4thOrder 2D Unstructured Euler Solver for the CESE Method

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Previously, Chang[1] reported a new class of high-order Conservation Element Solution Element, CESE, methods for solving nonlinear hyperbolic partial differential equations. The scheme was then extended by Bilyeu, et al.[2] to solve a 1D vector equation with an arbitrary order of convergence. This new high-order CESE method shares many favorable attributes of the original second-order CESE method, including: (i) compact mesh stencil involving only the immediate mesh nodes surrounding the node where the solution is sought, (ii) the CFL stability constraint remains the same, i.e., ≤ 1, as , and (iii) superb shock capturing capability without using an approximate Riemann solver. In the present extended abstract, we extend the 1D formulation to 2D. A general formulation is presented for solving the coupled equations with 4thorder accuracy. To demonstrate the formulation, two linear cases are reported. The linear test cases shown are the convection equation and the linear acoustic equation.

In the final paper we plan to present non-linear test cases and solve the equations on a quad mesh.

I.   Introduction

In this work, we extend the 1D solver for one nonlinear hyperbolic equation to two dimensions. The new formulation is currently restricted to 4thorder but can be extended to higher orders by extending the Taylor series. To demonstrate the capabilities of the new scheme, we apply the method to solve two sets of equations: (i) the linearized acoustic equations, and (ii) a convection equation.

The original second-order CESE method[3,4] solves the hyperbolic PDEs by discretizing the space-time domain by using the conservation elements (CEs) and solution elements (SEs). The profiles of unknowns are prescribed by assumed discretization inside SEs. Aided by the approximation for the unknowns in the SEs, space-time flux conservation can be enforced over each CE. The calculation of space-time flux conservation results in the formulation for updating the unknowns in the time marching process. The special features of the CESE method include: (i) The conserved variables are represented by a Taylor series expansion in both space and time. The order of the Taylor series is also the order of the solver. (ii) The a scheme, the core scheme of the CESE method, is non-dissipative. (iii) The CESE method has the most compact mesh stencil possible involving only the immediate neighboring mesh points that surround the mesh node where the solution is sought. (iv) The method uses explicit integration in time marching. The stability criterion is CFL ≤ 1. (v) No approximate Riemann solver is used and the scheme is simple and efficient.

The space-time stencil used in this derivation is the same as that reported by Wang and Chang[4] and is repeated here for completeness. Figure 1 is a top down view of the computational domain. In this figure the solid lines represent the mesh and the dashed lines are the boundaries of the conservation element centered at $G(j)$. The squares are vertices's of the mesh, the circles are the solution points and the x are the centroid of each triangle. Figures 2a and 2b are the conservation element, CE, and solution element, SE, associated with the solution point $G(j)$. These CE is the 3D space-time element that is used to conserve flux and the SE is the domain that that the Taylor series is assumed to be valid in.

In the following derivation we make the following assumptions:

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1. The flux is a known function of the flow variables

2. Inside of a solution element the flow variables and fluxes can be expressed as a Taylor series

3. The desired order of convergences is even

4. Each even derivative has an odd derivative in each spatial direction.

5.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

6. The source term is handled via operator splitter

7. For non steady state problems the order of convergence of the initial conditions must match or exceed the desired order of accuracy of the solver.

Shown below is a 4th order Taylor expansion in two spatial derivatives

$$u^* = u + u_x \Delta x + u_y \Delta y + \left(u_{xx} \Delta x^2 + u_{xy} \Delta x \Delta y + u_{yx} \Delta y \Delta x + u_{yy} \Delta y^2 \right)/2.0 +$$
$$\left(u_{xxx} \Delta x^3 + u_{yyy} \Delta y^3 + (u_{xxy} + u_{xyx} + u_{yxx}) \Delta x^2 \Delta y + (u_{xyy} + u_{yxy} + u_{yyx}) \Delta y^2 \Delta x \right)/6.0. \tag{1}$$

If we assume that the Taylor series expansion of u_{xy} and u_{yx} are equal then

$$\frac{\partial u_{xy}}{\partial x} = \frac{\partial u_{yx}}{\partial x} \text{ and } \frac{\partial u_{xy}}{\partial y} = \frac{\partial u_{yx}}{\partial y}$$
$$\text{or} \tag{2}$$
$$u_{xyx} = u_{yxx} \text{ and } u_{xyy} = u_{yxy}.$$

Under theses assumption the Taylor series becomes

$$u^* = u + u_x \Delta x + u_y \Delta y + \left(u_{xx} \Delta x^2 + 2u_{xy} \Delta x \Delta y + u_{yy} \Delta y^2 \right)/2.0 +$$
$$\left(u_{xxx} \Delta x^3 + u_{yyy} \Delta y^3 + (u_{xxy} + 2u_{xyx}) \Delta x^2 \Delta y + (2u_{xyy} + u_{yyx}) \Delta y^2 \Delta x \right)/6.0. \tag{3}$$

This extended abstract is organized as follows. Section II derives the equations necessary to progress the conserved variables to the next time step. This section is broken up into 4 sub sections one for calculating the fluxes, temporal derivatives, even and odd derivatives of the conserved variables. Section III contains the preliminary results from the convection equation and linear acoustic equation.

II.   Derivation

The governing equation that we wish to solve is

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}^{x_j} = \mathbf{S}, \tag{4}$$

where

$$\mathbf{U} = (u_1, u_2, \ldots, u_m)^T, \mathbf{F}^x = (f_1^x, f_2^x, \ldots, f_m^x)^T, \mathbf{F}^y = (f_1^y, f_2^y, \ldots, f_m^y)^T, \mathbf{S} = (s_1, s_2, \ldots, s_m)^T, \tag{5}$$

m is the number of equations. Each u_m is expressed as a Taylor series,

$$u_m^* = \sum_{a=0}^{N} \sum_{b=0}^{N-a} \sum_{c=0}^{N-a-b} \frac{\partial^{a+b+c} u_m}{\partial x^a \partial y^b \partial t^c} \frac{1}{a!b!c!} (x - x_j)^a (y - y_j)^b (t - t^n)^c, \tag{6}$$

where N is equal to the desired order of convergence minus 1. For 4th order N is equal to 3. For future reference the power in the numerator of the ∂ will be dropped, e.g.

$$\frac{\partial^{a+b+c} u_m}{\partial x^a \partial y^b \partial t^c} \equiv \frac{\partial u_m}{\partial x^a \partial y^b \partial t^c}. \tag{7}$$

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The derivative of any Taylor series can be written as

$$\frac{\partial u_m^*}{\partial x^I \partial y^J \partial t^K} = \sum_{a=0}^{A} \sum_{b=0}^{B-a} \sum_{c=0}^{C-a-b} \frac{\partial u_m}{\partial x^{I+a} \partial y^{J+b} \partial t^{K+c}} \frac{1}{a!b!c!} (x - x_j)^a (y - y_j)^b (t - t^n)^c, \tag{8}$$

where $A = N - I - J - K$, $B = N - I - J - K$ and $C = N - I - J - K$. A similar expansion for the fluxes is also true

$$\frac{\partial f_m^*}{\partial x^I \partial y^J \partial t^K} = \sum_{a=0}^{A} \sum_{b=0}^{B-a} \sum_{c=0}^{C-a-b} \frac{\partial f_m}{\partial x^{I+a} \partial y^{J+b} \partial t^{K+c}} \frac{1}{a!b!c!} (x - x_j)^a (y - y_j)^b (t - t^n)^c. \tag{9}$$

For 2D there are a total of $3 \sum_{n=0}^{N} (N - n)(n + 1)$ unknowns per governing equation. For a 4$^{\text{th}}$ order 2D Euler equation there would be a total of 180 unknowns. In what follows we will show that the only independent variables are the spatial derivatives of the conserved variables

$$\frac{\partial u_m}{\partial x^I \partial y^J} \quad \begin{matrix} I = 0, 1, \ldots, N \\ J = 0, 1, \ldots, N - I \end{matrix} \quad m = 1, \ldots, Neq. \tag{10}$$

For a 4$^{\text{th}}$ solver the independent variables are

Table 1: Table of the independent conserved variables

Even	Odd	
u	u_x	u_y
u_{xx}	u_{xxx}	u_{xxy}
u_{xy}	u_{xyx}	u_{xyy}
u_{yy}	u_{yyx}	u_{yyy}

II.A. Fluxes

Since the fluxes are known functions of the conserved variables the derivatives of the fluxes can be found from the chain rule.

$$\frac{\partial f_m^{x^i}}{\partial Y_1} = \sum_{l}^{Neq} \frac{\partial f_m^{x^i}}{\partial u_l} \frac{\partial u_l}{\partial Y_1}, \quad Y_1 = x, y, t \tag{11}$$

$$\frac{\partial^2 f_m^{x^i}}{\partial Y_1 \partial Y_2} = \sum_{l}^{Neq} \frac{\partial f_m^{x^i}}{\partial u_l} \frac{\partial^2 u_l}{\partial Y_1 \partial Y_2} + \sum_{l,k}^{Neq} \frac{\partial^2 f_m^{x^i}}{\partial u_l \partial u_k} \frac{\partial u_l}{\partial Y_1} \frac{\partial u_k}{\partial Y_2} \quad (Y_1, Y_2) = \begin{matrix} (x, x) & (y, y) \\ (t, t) & (x, y) \\ (x, t) & (y, t) \end{matrix} \tag{12}$$

$$\frac{\partial^3 f_m^{x^i}}{\partial Y_1 \partial Y_2 \partial Y_3} = \sum_{l}^{Neq} \frac{\partial f_m^{x^i}}{\partial u_l} \frac{\partial^3 u_l}{\partial Y_1 \partial Y_2 \partial Y_3} +$$

$$\sum_{l,k}^{Neq} \frac{\partial^2 f_m^{x^i}}{\partial u_l \partial u_k} \left(\frac{\partial^2 u_l}{\partial Y_1 \partial Y_2} \frac{\partial u_k}{\partial Y_3} + \frac{\partial^2 u_l}{\partial Y_1 \partial Y_3} \frac{\partial u_k}{\partial Y_2} + \frac{\partial^2 u_l}{\partial Y_2 \partial Y_3} \frac{\partial u_k}{\partial Y_1} \right) +$$

$$\sum_{l,k,p}^{Neq} \frac{\partial^3 f_m^{x^i}}{\partial u_l \partial u_k \partial u_p} \frac{\partial u_l}{\partial Y_1} \frac{\partial u_k}{\partial Y_2} \frac{\partial u_p}{\partial Y_3} \tag{13}$$

$$(Y_1, Y_2, Y_3) = \begin{matrix} (x, x, x) & (y, y, y) & (t, t, t) & (x, y, t) \\ (x, x, y) & (y, y, t) & (x, x, t) \\ (x, y, y) & (y, t, t) & (x, t, t) \end{matrix}$$

for $x^i = x, y$. These equations will continue until all of the fluxes are found. After these equations are applied there will be $\sum_{n=0}^{N} (N - n)(n + 1)$ unknowns left per governing equation. This method will work for all higher derivatives but becomes more and more complicated as the number of derivatives increases. Another approach is to directly calculate each derivative with out using a Jacobian.

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II.B. Temporal derivatives

In this section we will use the governing equation and its derivatives to find the temporal derivatives of the conserved variables.

$$
\begin{aligned}
\frac{\partial u_m}{\partial t} &= s_m - \frac{\partial f_m^x}{\partial x} - \frac{\partial f_m^y}{\partial y}, \\
\frac{\partial}{\partial x}\left(\frac{\partial u_m}{\partial t}\right) &= \frac{\partial s_m}{\partial x} - \frac{\partial^2 f_m^x}{\partial x \partial x} - \frac{\partial^2 f_m^y}{\partial y \partial x}, \\
\frac{\partial}{\partial y}\left(\frac{\partial u_m}{\partial t}\right) &= \frac{\partial s_m}{\partial y} - \frac{\partial^2 f_m^x}{\partial x \partial y} - \frac{\partial^2 f_m^y}{\partial y \partial y}, \dots.
\end{aligned}
\tag{14}
$$

In general any temporal derivative of a conserved variable can be found using

$$
\frac{\partial u_m}{\partial x^I \partial y^J \partial t^K} = \frac{\partial s_m}{\partial x^I \partial y^J \partial t^{K-1}} - \frac{\partial f_m^x}{\partial x^{I+1} \partial y^J \partial t^{K-1}} - \frac{\partial f_m^y}{\partial x^I \partial y^{J+1} \partial t^{K-1}},
\tag{15}
$$
$$
\text{for } K = 1, 2, \dots, N; \quad J = 0, 1, \dots, N - K; \quad I = 0, 1, \dots, N - K - J.
$$

This leaves us with $\sum_{n=0}^{N}(n+1)$ unknowns per governing equation. For a 4$^{\text{th}}$order Euler solver this would be 40 unknowns, 10 per equation.

II.C. Space-Time Flux Conservation

To find the next set of conserved variables we conserve flux across both space and time. It should be noted that the geometry of the Conservation Element and Solution Elements are the same as the ones presented by Wang and Chang.[4] By using the Gauss-divergence theorem we can convert the governing equation into integral form

$$
\oint \mathbf{h_m}(x, y, t) \cdot d\mathbf{s} = \iiint s_m dV.
\tag{16}
$$

This equation will be used to find all even derivatives, e.g. $u, u_{xx}, u_{xy}, u_{yy}, u_{xxxx}, \dots$. The rest of the unknowns will be found using a central difference like procedure. Eq. (16) can be used to find u_m and its derivative will be used to find the other even derivatives

$$
\oint \frac{\partial \mathbf{h_m}}{\partial x^I y^J}(x, y, t) \cdot d\mathbf{s} = \iiint \frac{\partial s_m}{\partial x^I y^J} dV.
\tag{17}
$$

II.D. Even

The even derivatives are found by conserving the flux through the CE. To make this calculation more manageable we will separate this calculation into three steps, (1) the flux through a side face, (2) the flux through the bottom face and (3) the flux through the top face. The flux through the side and bottom faces are calculated using the neighboring solution points while the flux through the top face is calculated from the current solution point.

To begin we will calculate the flux through the side faces. Figure 3 shows the faces associated with the solution point j_r. The coordinate axis is located at cell center and the vertex (x_0, y_0) is the solution point.

First we will shift the coordinate axis to the neighboring solution point. For future reference we will denote the actual location with capital letters, (X, Y), and relative coordinates with lower case letters, (x, y). The flux can now be expressed as

$$
f_m^{x_i*} = \sum_a^A \sum_b^B \sum_c^C \frac{\partial f_m}{\partial x^a \partial y^b \partial t^c} \frac{x^a y^b (t - t^{n-1/2})^c}{a! b! c!},
\tag{18}
$$

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where $x = X - X_{j_r}$ and $y = Y - Y_{j_r}$. Next we apply the following coordinate transformation

$$
\begin{aligned}
x &= u(x_1 - x_0) + x_0 \\
y &= u(y_1 - y_0) + y_0 \\
t &= \frac{\Delta t}{2}v + t^{n-1/2} \\
0 &\leq u \leq 1, \quad 0 \leq v \leq 1.
\end{aligned}
\tag{19}
$$

The normal vector is equal to $N = [\frac{\Delta t}{2}(y_1 - y_0), -\frac{\Delta t}{2}(x_1 - x_0), 0]$. Using the coordinate transformation Eq. (18) can be rewritten as

$$
f_m^{x_i*} = \sum_a^A \sum_b^B \sum_c^C \frac{\partial f_m}{\partial x^a \partial y^b \partial t^c} \frac{(u(x_1 - x_0) + x_0)^a (u(y_1 - y_0) + y_0)^b \left(\frac{\Delta t}{2}v\right)^c}{a!b!c!}.
\tag{20}
$$

The flux through a side face is equal to

$$
\iint (f_m^{x_i*} N_i)_r \, du\, dv =
$$

$$
\sum_{i=1}^2 \sum_a^A \sum_b^B \sum_c^C \int_0^1 \int_0^1 \left(\frac{\partial f_m^{x_i}}{\partial x^a \partial y^b \partial t^c}\right)_{j_r}^{n-1/2} \frac{(u(x_1 - x_0) + x_0)^a (u(y_1 - y_0) + y_0)^b \left(\frac{\Delta t}{2}v\right)^c}{a!b!c!} N_{ri} \, du\, dv.
\tag{21}
$$

The flux through the faces associated with solution point j_r is

$$
\begin{aligned}
(F_m)_{j_r} = &\sum_{i=1}^2 \sum_a^A \sum_b^B \sum_c^C \left(\frac{\partial f_m^{x^i}}{\partial x^a \partial y^b \partial t^c}\right)_r \frac{\left(\frac{\Delta t}{2}\right)^{c+1}}{(c+1)!a!b!} \\
&\left[\left(N^{x^i}\right)_{01} \int_0^1 (u(x_1 - x_0) + x_0)^a (u(y_1 - y_0) + y_0)^b \, du + \right. \\
&\left. \left(N^{x^i}\right)_{02} \int_0^1 (u(x_2 - x_0) + x_0)^a (u(y_2 - y_0) + y_0)^b \, du\right].
\end{aligned}
\tag{22}
$$

The total flux through the side faces is $\sum_{r=1}^3 (F_m)_{j_r}$.

Next we will calculate the flux through the top and bottom surface. This requires us to integrate over an arbitrary quadrilateral. The only assumption that can be made about the quadrilateral is that it is concave. As with the side face we shift the centroid to the current solution point. An arbitrary bottom face is shown in Fig. 4. In this figure point 1 is the centroid of the adjacent CE and point 3 is the centroid of the CE and the current solution point. Points 2 and 4 are the shared vertices's of the mesh. To integrate the flux through the bottom surface we split the quadrilateral into two triangles and then provide a coordinate transformation that will convert the triangles into right triangles. The area of the triangles are

$$
A_I = \frac{1}{2} \begin{vmatrix} x_3 - x_1 & y_3 - y_1 \\ x_4 - x_1 & y_4 - y_1 \end{vmatrix} \quad A_{II} = \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}.
\tag{23}
$$

The transformation equation used for region I is

$$
u_I = u_x(x - x_1) + u_y(y - y_1) \quad v_I = v_x(x - x_1) + v_y(y - y_1).
\tag{24}
$$

To express x and y as functions of u and v we set $(u_3, v_3) = (1, 0) \quad (u_4, v_4) = (1, 1)$

$$
\begin{aligned}
x_I &= u(x_3 - x_1) + v(x_4 - x_3) + x_1 \\
y_I &= u(y_3 - y_1) + v(y_4 - y_3) + y_1,
\end{aligned}
\tag{25}
$$

where $0 \leq v \leq u$ and $0 \leq u \leq 1$. The normal is equal to $N = [0, 0, 2A_I]$. This process is repeated for region II yielding

$$
\begin{aligned}
x_{II} &= u(x_2 - x_1) + v(x_3 - x_2) + x_1 \\
y_{II} &= u(y_2 - y_1) + v(y_3 - y_2) + y_1,
\end{aligned}
\tag{26}
$$

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where $0 \leq v \leq u$ and $0 \leq u \leq 1$. The normal is equal to $N = [0, 0, 2A_{II}]$. Since the first two terms in the normal are zero the flux through the bottom surface is

$$(U_m)_{j_r}^{n-1/2} = 2 \iint u_I^* A_I + u_{II}^* A_{II} \, du \, dv =$$

$$2 \int_0^1 \int_0^u \sum_a^A \sum_b^B \frac{1}{a!b!} \left(\frac{\partial u_m}{\partial x^a \partial y^b} \right)_{j_r}^{n-1/2} \Big[A_I \left(u(x_3 - x_1) + v(x_4 - x_3) + x_1 \right)^a \left(u(y_3 - y_1) + v(y_4 - y_3) + y_1 \right)^b +$$

$$A_{II} \left(u(x_2 - x_1) + v(x_3 - x_2) + x_1 \right)^a \left(u(y_2 - y_1) + v(y_3 - y_2) + y_1 \right)^b \Big] \, du \, dv.$$

$$(27)$$

The total flux through the bottom faces is $\sum_{r=1}^3 (U_m)_{j_r}^{n-1/2}$. The flux through the top face is found in a like manner. Since the bottom and top face have the exact same shape we use the same diagram for both the top and the bottom. The difference is that the corresponding solution point is located at point 3. Which means that $(x_3, y_3) = (0, 0)$. Using this the flux through the top face is

$$(U_m)_{j_r}^n = 2 \iint u_I^* A_I + u_{II}^* A_{II} \, du \, dv =$$

$$2 \int_0^1 \int_0^u \sum_a^A \sum_b^B \frac{1}{a!b!} \left(\frac{\partial u_m}{\partial x^a \partial y^b} \right)_j^n \Big[A_I \left(u(x_3 - x_1) + v(x_4 - x_3) + x_1 \right)^a \left(u(y_3 - y_1) + v(y_4 - y_3) + y_1 \right)^b +$$

$$A_{II} \left(u(x_2 - x_1) + v(x_3 - x_2) + x_1 \right)^a \left(u(y_2 - y_1) + v(y_3 - y_2) + y_1 \right)^b \Big] \, du \, dv.$$

$$(28)$$

The total flux through the top face is equal to $\sum_{r=1}^3 (U_m)_{j_r}^n$. These equations are combined to progress the even derivatives to the next time step

$$\sum_{r=1}^3 (U_m)_{j_r}^n = \iiint s_m \, dV - \sum_{r=1}^3 \left((U_m)_{j_r}^{n-1/2} + (F_m)_{j_r} \right).$$

$$(29)$$

Since the geometry of the CE and SE are the same as in the second order version we can use the same simplifications. This results in the first derivatives u_x, u_y canceling out. This allows us to solve for u_m explicitly

$$(u_m)_j^n = \frac{1}{A_{tot}} \left(\iiint s_m \, dV - \sum_{r=1}^3 \left(\left(\tilde{U}_m \right)_{j_r}^n + (U_m)_{j_r}^{n-1/2} + (F_m)_{j_r} \right) \right).$$

$$(30)$$

where A_{tot} is the total area of the CE and

$$\left(\tilde{U}_m \right)_{j_r}^n =$$

$$2 \int_0^1 \int_0^u \sum_a^A \sum_b^B \frac{1}{a!b!} \left(\frac{\partial u_m}{\partial x^a \partial y^b} \right)_j^n \Big[A_I \left(u(x_3 - x_1) + v(x_4 - x_3) + x_1 \right)^a \left(u(y_3 - y_1) + v(y_4 - y_3) + y_1 \right)^b +$$

$$A_{II} \left(u(x_2 - x_1) + v(x_3 - x_2) + x_1 \right)^a \left(u(y_2 - y_1) + v(y_3 - y_2) + y_1 \right)^b \Big] \, du \, dv \, (a, b) \neq \begin{array}{c} (0, 0), \\ (1, 0), \\ (0, 1) \end{array} .$$

$$(31)$$

None of the integrals in Eqs (22), (27), and (28) have been evaluated. All of these integrations can be found using multiple methods. The easiest method is to use a program capable of symbolic integration. But they could also be evaluated by hand using $\int f \, dg = fg - \int g \, df$ multiple times. It is also possible to use the binomial theorem to generate a formula that could be used for any combination of a and b. In our investigation we evaluated the integrals using a symbolic math package.

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II.E. Odd

The odd derivatives are evaluated using a central difference like approach. The stencil used for the odd derivatives is shown in Fig. 5. In this figure the black dots are the vertices's and the circles are the solution points. The crosses represent the shifted locations of the solutions points. A shifted point is used to reduce the dissipation as the CFL number approaches zero. The location of the crosses is calculated by

$$
\left(x_{j'_r}, y_{j'_r}\right) = \left[\tau(x_{j_r} - x_j) + x_j, \tau(y_{j_r} - y_j) + y_j\right],
\tag{32}
$$

where τ is the absolute value of the local cfl number. We take a Taylor expansion from j to any j'_r, $r = 1, 2, 3$ as

$$
u_m^*(j_r, n) = \sum_{a=0}^{A}\sum_{b=0}^{B}\left(\frac{\partial u_m}{\partial x^a \partial y^b}\right)_j^n \frac{1}{a!b!}\left(X'_{j_r} - X_j\right)^a \left(Y'_{j_r} - Y_j\right)^b.
\tag{33}
$$

We will shift the coordinate axis so that point j is at the origin. The Taylor expansion now becomes.

$$
u_m^*(j_r, n) = \sum_{a=0}^{A}\sum_{b=0}^{B}\left(\frac{\partial u_m}{\partial x^a \partial y^b}\right)_j^n \frac{1}{a!b!}\left(x'_{j_r}\right)^a \left(y'_{j_r}\right)^b.
\tag{34}
$$

In order to find the two unknowns we will construct two equations.

$$
\begin{bmatrix} x'_1 & y'_1 \\ x'_2 & y'_2 \end{bmatrix}\begin{bmatrix} u_{mx}^{(1)} \\ u_{my}^{(1)} \end{bmatrix} = \begin{bmatrix} (u_{mj_1}^*)_j^n - \sum_{a=0}^{A}\sum_{b=0}^{B}\left(\frac{\partial u_m}{\partial x^a \partial y^b}\right)_j^n \frac{1}{a!b!}x'^{a}_{1}y'^{b}_{1} \\ (u_{mj_2}^*)_j^n - \sum_{a=0}^{A}\sum_{b=0}^{B}\left(\frac{\partial u_m}{\partial x^a \partial y^b}\right)_j^n \frac{1}{a!b!}x'^{a}_{2}y'^{b}_{2} \end{bmatrix}
\tag{35}
$$
$$
\text{for } (a, b) \neq (0, 1), (1, 0).
$$

Since $u_{mj_r}^*$ can not be evaluated yet we approximate it with

$$
\left(u_{j_r}^*\right)_j^n \approx \sum_{a=0}^{A}\sum_{b=0}^{B}\sum_{c=0}^{C}\left(\frac{\partial u_m}{\partial x^a \partial y^b \partial t^c}\right)_{j_r}^{n-\frac{1}{2}}\frac{1}{a!b!c!}\left(x'_{j_r} - x_{j_r}\right)^a \left(y'_{j_r} - y_{j_r}\right)^b \left(\frac{\Delta t}{2}\right)^c.
\tag{36}
$$

Eqs. (35) and (36) are used to find one possible solution to u_{mx} and u_{my} using information from two of the three surrounding solution points. This will be repeated using all possible combinations of adjacent solution points, e.g. (j_1, j_2), (j_2, j_3), (j_3, j_1). These solutions are then weighted using this formula.

$$
u_{mx} = \omega(u_{mx}^{(1)}, u_{mx}^{(2)}, u_{mx}^{(3)}, \alpha) \quad u_{my} = \omega(u_{my}^{(1)}, u_{my}^{(2)}, u_{my}^{(3)}, \alpha)
\tag{37}
$$

where the weighting function, ω as defined by Wang and Chang[4]

$$
\omega = \frac{(\theta_{m2}\theta_{m3})^{\alpha}u_{mx_i}^{(1)} + (\theta_{m3}\theta_{m1})^{\alpha}u_{mx_i}^{(2)} + (\theta_{m1}\theta_{m2})^{\alpha}u_{mx_i}^{(3)}}{(\theta_{m1}\theta_{m2})^{\alpha} + (\theta_{m2}\theta_{m3})^{\alpha} + (\theta_{m3}\theta_{m1})^{\alpha}},
\tag{38}
$$

where $x_i = x, y$ and

$$
\theta_{mr} = \sqrt{\left(u_{mx}^{(r)}\right)^2 + \left(u_{my}^{(r)}\right)^2}.
\tag{39}
$$

Equations (35) and (36) are used to progress the odd derivatives to the next time step. Since the odd derivatives are dependent on the next lower derivative it must be calculated before hand.

III.   Preliminary Results

To date we have simulated the convection and linear acoustic equations. In both cases a 4$^{\text{th}}$order Taylor series was used which should lead to a convergence rate of 4. To determine the convergence rates we used the l_2 defined as

$$l_2 = \sqrt{\int \varepsilon^2 dA,} \tag{40}$$

where

$$\varepsilon = u_{numerical} - u_{analytical}. \tag{41}$$

Since the Taylor series coefficients are constant Eq. 40 is simplified to

$$l_2 = \sqrt{\sum_i \varepsilon_i^2 A_i,}$$

where A_i is the area of cell i.

Each convergence test was done on two different types of meshes. The first is a uniform mesh that is created by splitting a quad mesh into 4 triangles. The triangles are created by splitting the quad through the diagonals. The second mesh is an unstructured mesh generated using an advancing front algorithm. Both meshes were created using Cubit. For the uniform mesh the QTri mesh was used. For the nonuniform mesh the TriAdvance was used. The TriAdvance algorithm first tires the Delaunay algorithm, then the TriAdvance then finally the QTri method.

III.A.   Advection Equation

The governing equation is

$$\frac{\partial u}{\partial t} + a_x \frac{\partial u}{\partial x} + a_y \frac{\partial u}{\partial y} = 0.$$

Under periodic boundary conditions and a domain of $-\pi \le x \le \pi$, $-\pi \le y \le \pi$ u is equal to

$$u = \sin(a_x x + a_y y + a_t t),$$

where $a_x = a_y = 1$ and $a_t = -\sqrt{a_x^2 + a_y^2}$. In this test case we let the simulation run for a time of 15 at an approximate CFL number of 1.0.

When calculating the convergence rates the characteristic length was the square root of the average area triangle and the square root of the largest area triangle for the uniform and nonuniform mesh respectively.

cells	h	l_2	O(l_2)	cells	h	l_2	O(l_2)
400	9.870E-02	2.158E-01	-	232	1.690E-01	2.929E-01	-
1600	2.467E-02	6.726E-03	5.00	926	4.255E-02	9.727E-03	4.94
3600	1.097E-02	8.762E-04	5.03	2068	1.909E-02	1.315E-03	4.99
6400	6.169E-03	2.065E-04	5.02	3694	1.068E-02	3.131E-04	4.94
10000	3.948E-03	6.735E-05	5.02	5774	6.835E-03	1.022E-04	5.01
(a) Structured grid				(b) Unstructured grid			

Table 2: Convergecne rates for the advection equation.

As seen in Table 2 the convergence rates are 5[th]order instead of 4[th]order as expected. We plan on investigating this in more detail before the final paper is submitted.

III.B.   Linear Acoustic

The governing equation is

$$\frac{\partial \rho}{\partial t} + \rho_0 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_x}{\partial x} \right) = 0$$

$$\frac{\partial v_x}{\partial t} + \frac{a_0^2}{\rho_0} \frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial v_y}{\partial t} + \frac{a_0^2}{\rho_0} \frac{\partial \rho}{\partial y} = 0,$$

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where ρ, v_x, v_y, a_0, and ρ_0 are respectively the density, x and y component of the velocity, free stream speed of sound and free stream density. Under periodic boundary conditions, and a domain $-2\pi \leq x \leq 2\pi$, $-2\pi \leq y \leq 2\pi$ an analytical solution is

$$\rho = \rho' \sin(a_x x + a_y y + a_t t)$$

$$V_x = -a_0^2 \frac{\rho'}{\rho_0} \frac{a_x}{a_t} \sin(a_x x + a_y y + a_t t)$$

$$V_y = -a_0^2 \frac{\rho'}{\rho_0} \frac{a_y}{a_t} \sin(a_x x + a_y y + a_t t),$$

where ρ' is the amplitude of the perturbation. In the test case we set $a_x = a_y = 1$, $\rho' = 0.01$, $a_0 = rho_0 = 1.0$. We let the case run for a time of 18 at an approximate CFL number of 1.0.

The method used to find the characteristic length is that same as the one used for the advection equation. As seen in Table 3 the convergence rates are also higher than expected. It should also be noted that for

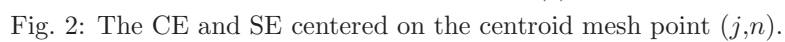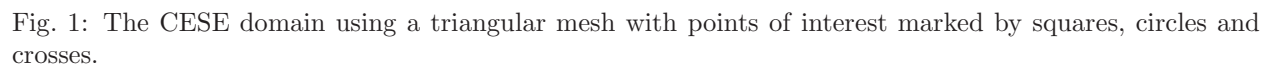| cells	h	l_2	$O(l_2)$	cells	h	l_2	$O(l_2)$
1600	9.870E-02	4.189E-03	-	926	1.705E-01	4.330E-03	-
3600	4.386E-02	6.585E-04	4.56	2068	7.636E-02	7.193E-04	4.47
6400	2.467E-02	1.621E-04	4.87	3694	4.275E-02	1.756E-04	4.86
10000	1.579E-02	5.361E-05	4.96	5774	2.735E-02	5.849E-05	4.92
(a) Structured grid				(b) Unstructured grid			

Table 3: Convergecne rates for the acoustic equation.

similar values of h the unstructured mesh tends to have a smaller error than the structured mesh.

IV.   Conclusion

In this extended abstract we presented a derivation for a 2D 4[th]order solver on an unstructured mesh using the CESE scheme. We demonstrated higher order convergence using the advection and linear acoustic equations.

In the final paper we will also include solutions of the Euler Equation, provide a computational efficiency report and provide an explanation as to why we were able to achieve a convergence rate higher than expected.
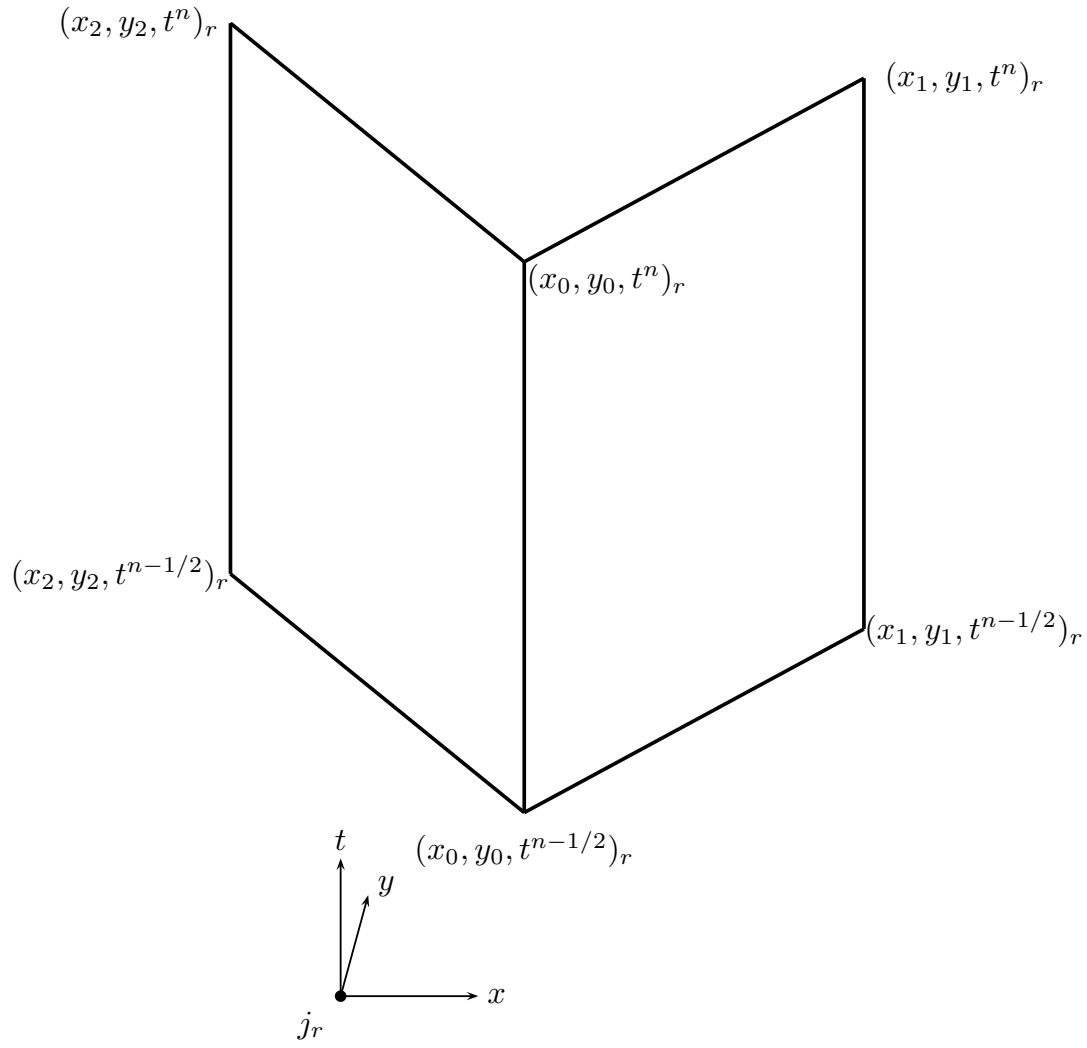
Fig. 1: The CESE domain using a triangular mesh with points of interest marked by squares, circles and crosses.

(a) Conservation Element

(b) Solution Element

Fig. 2: The CE and SE centered on the centroid mesh point (j,n).

V. Figures

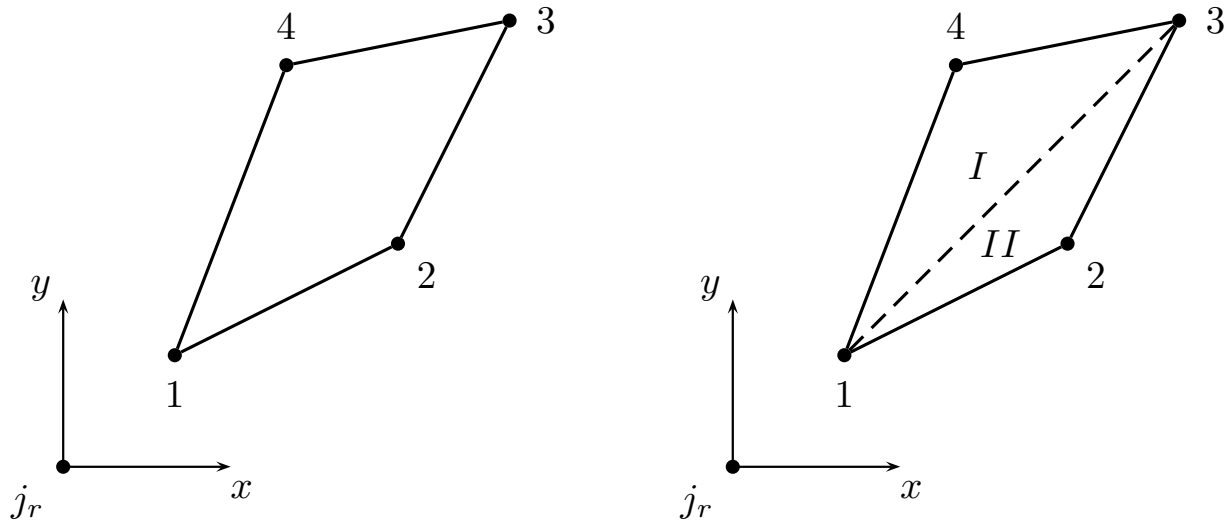Fig. 3: The side faces associated with a solution point
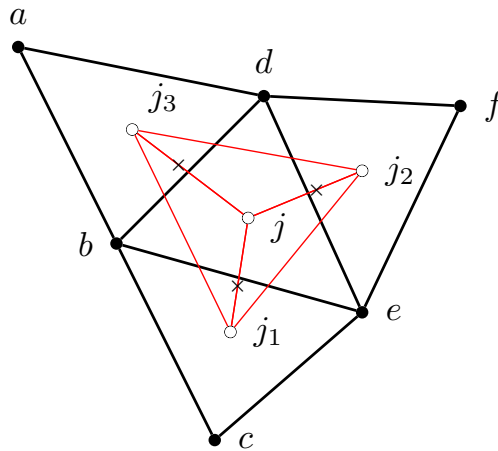
Fig. 4: The bottom face

Fig. 5: The stencils used for the odd derivatives

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[3]Chang, S., "The Method of Space-Time Conservation Element and Solution Element – A New Approach for Solving the Navier-Stokes and Euler Equations," *Journal of Computational Physics*, Vol. 119, No. 2, July 1995, pp. 295–324.

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