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Neyman-Pearson Optimal and Suboptimal Detection for Signals in General Clutter Mixture Distributions

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ABSTRACT

A general framework is introduced that enables the construction of the Neyman-Pearson optimal detector for a single general target model embedded within complex clutter whose amplitude distribution forms a mixture. This enables the determination of asymptotic decision rules for high resolution high grazing angle detection of targets in spiky sea clutter. In addition, suboptimal approximations are derived and analysed.

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Executive Summary

This research supports the contribution of Electronic Warfare and Radar Division's Maritime Radars Group's work for Task 07/040 (support to AIR 7000). The research is aimed at giving the Group an understanding of what radar detectors would be suitable for detecting small maritime targets embedded in high resolution spiky sea clutter, from an airborne maritime surveillance platform.

This report introduces a general framework which enables the determination of the Neyman-Pearson optimal radar detector for single targets embedded within an arbitrary clutter mixture model. Such clutter models have found applicability in the modelling of high resolution and high grazing angle sea clutter. In particular, the KK-Distribution, which is a mixture of two K-Distributions, has been used to improve the fit of theoretical distribution functions to the corresponding empirical distributions functions. This has proven critical in the case of modelling high grazing angle clutter obtained from a horizontally polarised radar. The research presented here shows how the optimal decision rule can be derived for any mixture of amplitude distributions as the clutter model, with a target model that consists of independent components. This extends recent work on the determination of optimal decision rules for Swerling I targets in high grazing angle clutter, with the latter modelled as a KK-Distribution. The importance of the Neyman-Pearson optimal detector is that it is the one which, for a fixed false alarm rate, provides the maximum detectability of targets in clutter. Hence it will outperform any other decision rule.

Due to the formulation of the Neyman-Pearson Lemma, the optimal detector will depend on the target signal to clutter ratio. This is a serious shortcoming in terms of practical applications of detectors. This is because it will be difficult to assess the signal strength in spiky sea clutter without invoking an auxiliary detection scheme. Since the key requirement for AIR 7000 is detection of small targets, we can use this restriction to obtain suboptimal approximations. This is done using a novel approach of constructing upper bounds on the likelihood function, and using this as a suboptimal detector. It

has been found that such suboptimal decision rules can provide excellent performance for small target detection. This can be coupled together with a linear detector to provide a detection scheme, independent of the signal to clutter ratio, with reasonable performance in a large number of scenarios.

In a recent thesis by the author¹, the case of detection of a Swerling-type target in KK-Distributed clutter was examined extensively. It is of interest to AIR 7000 to analyse the results from this thesis for the case of limiting clutter forms. This means we are interested in the detection schemes when the K-Distributed components in the KK-mixture limit to Rayleigh as the K-Distribution's shape parameter increases. It will be shown that limiting detectors are derived easily using the general results derived in this report.

¹“Optimal and Suboptimal Radar Detection Of Targets From High Grazing Angles and in High Resolution Sea Clutter”, MSc Thesis, University of Adelaide, 2010.

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1 Introduction and Problem Formulation

1.1 Background

The purpose of this work is to present a general solution to the construction of Neyman-Pearson optimal detectors for clutter amplitude distributions that are mixtures. In particular, the analysis and results in [1] are generalised considerably. In the latter, the focus is on optimal detection schemes for the case of Swerling-type targets in high grazing angle sea clutter distributions of relevance to the modelling and analysis efforts for AIR 7000. Namely, [1] focused on the KK-Distribution, which is a mixture of two K-Distributions. It turns out that some of the results in [1] generalise to arbitrary clutter mixtures, and the Swerling I target assumption can be removed and replaced by a more general model. Notably the major result Theorem 3.3.1 in [1], which shows the intensity squared density for a Swerling I target in KK-Distributed clutter factors into a weighted sum of the same target but in two separate K-Distributed clutter cases, can be shown to apply much more generally. This enables a general theorem, for the optimal detector for certain targets in mixture clutter distributions, to be derived. This theorem shows the mixture property is preserved through the intensity function.

This discovery of the generalisation only occurred because the limiting cases associated with the KK-model are of interest to AIR 7000. In particular, since the K-Distribution limits to a Rayleigh as the shape parameter increases, it was of interest to investigate the effects of this on the optimal and suboptimal detectors in [1].

The idea of mixture distributions for clutter modelling was introduced in [2], in the context of improving the tail fit of clutter models to real high resolution radar sea clutter, obtained from high grazing angles. Modern high resolution radars are subject to severely non-Rayleigh clutter [3]. The clutter of interest is that obtained from an airborne maritime surveillance platform, operating from high altitude, and scanning the sea surface at high grazing angles. Such clutter is highly spiky and as the major targets of interest are small in a signal to clutter ratio sense, it is important to have good models for such clutter and detection schemes that are close to optimal.

The well-known K-Distribution [3, 4, 5] has been determined to be a good model

for such clutter. It fits into the class of compound Gaussian models [6], and models adaptively the two key characteristics of sea clutter. These are a component modelling the fast fluctuations of clutter, and another part accounting for underlying modulation of the clutter [6]. The short term or fast fluctuation is referred to as the speckle component, while the modulation or slow varying component is called the texture. The K-Distribution model uses a root Gamma distribution to model the slow varying component, while a bivariate Gaussian process is the fast varying component [7, 8].

It has been reported that although the K-Distribution is a good fit to such clutter obtained from a vertically polarised radar, when the polarisation is switched to horizontal, it has been observed that the K-Distribution does not provide an adequate model in the clutter tail regions [2]. Hence it was proposed that a mixture of two K-Distributions may provide a more suitable solution. An interpretation and justification for this mixture model is given in [2], which states that one mixture distribution models the Bragg/whitecap scatterers, while the second models sea spikes [1]. This new distribution is known as the KK-clutter model, and as reported in [2], it significantly improved the fitting of the modelling distributions of horizontally polarised radar clutter returns.

Mixture distributions are formulated in the amplitude domain, and in general any two amplitude distributions can be combined to produce a mixture density through a weighting factor. Means and moments can then be specified in terms of the separate mixing distributions. Additionally, the distribution function also decomposes into a weighted sum of mixed individual distribution functions. See [1] for an analysis of such distributions.

The optimal detector considered in this work is that derived via the Neyman-Pearson Lemma [9]. This states that the form of the best decision rule is based on a threshold comparison of the likelihood ratio. Best decision rule refers to the fact that this Lemma yields a statistical hypothesis test that will have the maximum power for a given test size [10]. In radar terminology this translates to a detector which, for a fixed false alarm rate, will have the maximal probability of detection in comparison to similar decision rules based upon a fixed threshold.

Before turning to a mathematical formulation of the detection problem under investigation, useful references are indicated. For an excellent exposition on mathematical

statistics, including hypothesis testing and the Neyman-Pearson Lemma, refer to [10]. An excellent account of single pulse detection is outlined in [11], which also discusses Swerling target models. Conditional probability and stochastic decompositions are a central component in the proof of key results, and so the reader is referred to [12] for probability and conditioning, while [13] is a suitable reference on probability and stochastic processes. A good reference on probability distributions and their properties is [14].

1.2 Mathematical Formulation

To put the problem of interest in a mathematical framework, we suppose the complex signal is $\mathbf{s} = |\mathbf{s}|e^{i\Theta}$, where $|\mathbf{s}|$ is the signal amplitude and $\Theta \stackrel{d}{=} R(0, 2\pi)$ is the signal phase, assumed to be uniformly distributed. Without loss of generality we assume these are independent. The clutter in the complex domain is $\mathbf{c} = |\mathbf{c}|e^{i\Phi}$, where $|\mathbf{c}|$ is the clutter amplitude and $\Phi \stackrel{d}{=} R(0, 2\pi)$ is the uniformly distributed phase. It is assumed that each of these four random variables introduced thus far are independent. The clutter models of relevance are those whose amplitude distributions form mixtures. This translates to the assumption that our clutter amplitude density can be decomposed into

$$f_{|\mathbf{c}|}(t) = (1 - k)f_{|\mathbf{c}_1|}(t) + kf_{|\mathbf{c}_2|}(t), \quad (1.1)$$

where \mathbf{c}_1 and \mathbf{c}_2 represent the mixture components in the complex domain, and $k \in [0, 1]$ is the mixture coefficient. The random variables $|\mathbf{c}_1|$ and $|\mathbf{c}_2|$ are non-negative amplitude distributions. We assume these complex random variables possess independent phase distributions on the interval $[0, 2\pi]$. For the KK-Distribution model in [2], both $|\mathbf{c}_1|$ and $|\mathbf{c}_2|$ are K-Distributions, with different scale parameters but the same shape parameter.

As in [1], we focus on single look detection. Additionally, we will focus on amplitude squared, or intensity, detection. The problem of detecting a signal within clutter can be stated as a statistical hypothesis test. If we let \mathbf{t} be the radar test value from a single scan in the complex envelope, we are interested in testing whether \mathbf{t} is just clutter, or whether it also contains a target signature. This can be formulated as

$$H_0 : \mathbf{t} = \mathbf{c} \text{ vs } H_1 : \mathbf{t} = \mathbf{s} + \mathbf{c}, \quad (1.2)$$

where H_0 is the null-hypothesis and H_1 is the alternative hypothesis. The Neyman-Pearson Lemma [10] states that the optimal test takes the form of rejecting the null hypothesis if the likelihood ratio exceeds a fixed threshold. The likelihood ratio is the ratio of the densities of the radar return under each hypothesis, and the threshold is determined through the size of the statistical test. This is the probability of rejecting the null hypothesis when it is actually true. In statistical signal processing terminology, this is the false alarm probability [11].

Since we are only examining single scan returns, the likelihood ratio is just the ratio of the radar return under each hypothesis. However, as stated previously, we will be focusing on amplitude squared detection. Hence we must construct the densities under each hypothesis for amplitude squared statistics. Let $f_0 = f_{|c|^2}$ be the density of the radar return under H_0 , and $f_1 = f_{|s+c|^2}$ be that for a return under H_1 . Then the likelihood ratio test becomes

$$L(t) := \frac{f_1(t)}{f_0(t)} > \tau, \quad (1.3)$$

where τ is a fixed threshold, which is determined by setting the false alarm rate to a fixed level. Clearly, what is required is the construction of the appropriate densities, and a test value $t = |\mathbf{t}|^2$, corresponding to an amplitude squared measurement, is then used to assess whether there is a target in the radar return. Although this is a simple formulation, the construction of these densities can be problematic. Additionally, there may not be closed form solutions for these.

The remainder of the report is organised as follows. Section 2 derives the optimal Neyman-Pearson detector for the test formulated above. Section 3 is concerned with the application of the results in Section 2 to the cases of a Swerling target in Rayleigh/K-Distributed clutter, as well as the case of Rayleigh/Rayleigh mixture. Section 4 pursues the construction of suboptimal likelihoods based upon the optimal detectors in Section 3. Finally, Section 5 examines the performance of all detectors derived in Sections 3 and 4 using receiver operating characteristic (ROC) curves. The latter is a visual measure of performance, since for a fixed false alarm probability, it plots the signal to clutter ratio against the detection probability for a given detector.

2 The Neyman-Pearson Optimal Detector

2.1 Density Under H_1

As shown in the previous Section, the key to constructing the optimal Neyman-Pearson detector is the construction of the appropriate intensity density functions. Clearly the more challenging density to construct is that of $|\mathbf{s} + \mathbf{c}|^2$. The following shows how the mixture property is preserved through the decomposition of \mathbf{c} into its residual components:

Theorem 2.1. *Let \mathbf{s} be a complex signal with independent real and imaginary components, and let \mathbf{c} be the complex clutter model, such that the latter consists of a mixture distribution in amplitude, as defined by (1.1), with uniform phase distribution on the interval $[0, 2\pi]$. Suppose further that \mathbf{s} and \mathbf{c} are statistically independent. Then the intensity density factorises as*

$$f_{|\mathbf{s}+\mathbf{c}|^2}(t) = (1 - k)f_{|\mathbf{s}+\mathbf{c}_1|^2}(t) + kf_{|\mathbf{s}+\mathbf{c}_2|^2}(t), \quad (2.1)$$

where $k \in [0, 1]$ is the mixture coefficient, and \mathbf{c}_1 and \mathbf{c}_2 are the two complex mixture distributions.

This Theorem is a major generalisation of Theorem 3.3.1 in [1]. Observe that the target model only requires independent in-phase and quadrature components, generalising the Swerling I assumption in [1]. Before presenting the proof of Theorem 2.1 two technical Lemmas are presented. The first is known as the Convolution Lemma:

Lemma 2.1. *Suppose X and Y are two real valued continuous independent random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the random variable $X + Y$ is well defined and has density*

$$f_{X+Y}(t) = \int_{\mathcal{D}} f_X(t - \omega) f_Y(\omega) d\omega, \quad (2.2)$$

where f_X and f_Y are the densities of X and Y respectively, and \mathcal{D} is an appropriate domain of integration.

The domain \mathcal{D} in Lemma 2.1 is determined by the range of values over which the integrand in (2.2) is defined. As an example, if both X and Y are nonnegative, then clearly

$\mathcal{D} = [0, t]$. A second useful result specifies the density of a squared random variable in terms of its own density:

Lemma 2.2. *Let X be a continuous random variable with density f_X . Then the random variable X^2 has density*

$$f_{X^2}(t) = \frac{1}{2}t^{-\frac{1}{2}} \left(f_X(\sqrt{t}) + f_X(-\sqrt{t}) \right). \quad (2.3)$$

Both Lemmas 2.1 and 2.2 are proven easily by considering the distribution function of the random variables of interest, and then simplifying and differentiating. The references [12] and [13] can be consulted for clarification of this. Note that in the case where a random variable is nonnegative, the second term in the right hand side in (2.3) is identically zero.

The proof of Theorem 2.1 is now presented. The key to this is to decompose the random variable $|\mathbf{s} + \mathbf{c}|^2$ into independent components and apply Lemma 2.1 repeatedly. From the definitions of the signal and clutter in the complex domain, we can write

$$\begin{aligned} |\mathbf{s} + \mathbf{c}|^2 &= |\mathbf{s}|^2 + |\mathbf{c}|^2 + 2|\mathbf{s}||\mathbf{c}|\cos(\Theta - \Phi) \\ &= (|\mathbf{c}| + |\mathbf{s}|\cos(\Theta - \Phi))^2 + (|\mathbf{s}|\sin(\Theta - \Phi))^2, \end{aligned} \quad (2.4)$$

where the first equality follows by the definition of the complex modulus and applying trigonometric identities, and the second follows by completing the square. For brevity, define two random variables $Z_1 = |\mathbf{s}|\cos(\Theta - \Phi)$ and $Z_2 = |\mathbf{s}|\sin(\Theta - \Phi)$. Observe that these two variables are the real and imaginary components of a complex signal with amplitude $|\mathbf{s}|$ and with phase being the difference between that of \mathbf{s} and \mathbf{c} . It is claimed that the assumption imposed upon the components of the signal \mathbf{s} induce independence between Z_1 and Z_2 . To see this, observe that since both Θ and Φ are independent and uniformly distributed on the interval $[0, 2\pi]$, it is not difficult to show that the difference $\Theta - \Phi \stackrel{d}{=} R(-2\pi, 2\pi)$. Further, using statistical conditioning, the law of total probability and distribution functions, it is not difficult to show that we can write $\Theta - \Phi \equiv KR$, where $R \stackrel{d}{=} R(0, 2\pi)$ and K takes values ± 1 with equal probabilities, and is independent of R .

Hence, for a measurable set $A \subseteq [-1, 1]$,

$$\begin{aligned}
\mathbb{P}(\cos(\Theta - \Phi) \in A) &= \mathbb{P}(K = 1)\mathbb{P}(\cos(KR) \in A|K = 1) \\
&\quad + \mathbb{P}(K = -1)\mathbb{P}(\cos(KR) \in A|K = -1) \\
&= \frac{1}{2}\mathbb{P}(\cos(R) \in A) + \frac{1}{2}\mathbb{P}(\cos(R) \in A) \\
&= \mathbb{P}(\cos(R) \in A), \tag{2.5}
\end{aligned}$$

and so we see that $\cos(\Theta - \Phi) \stackrel{d}{=} \cos(R)$. By appealing to the identity $\sin(\Theta - \Phi) = \pm\sqrt{1 - \cos^2(\Theta - \Phi)}$, it can be shown that $\sin(\Theta - \Phi) \stackrel{d}{=} \sin(R)$. Hence $Z_1 \equiv |\mathbf{s}|\cos(R)$ and $Z_2 \equiv |\mathbf{s}|\sin(R)$, where $R \stackrel{d}{=} R(0, 2\pi)$ is independent of \mathbf{s} . Hence by assumption these must be independent also.

Returning to the proof of the main result, we see that an application of the convolution Lemma 2.1 yields

$$f_{|\mathbf{c}|+Z_1}(t) = \int_0^\infty f_{|\mathbf{c}|}(\omega)f_{Z_1}(t - \omega)d\omega, \tag{2.6}$$

where f_{Z_1} is the density of Z_1 . Hence, it follows with an application of Lemma 2.2 to (2.6),

$$f_{(|\mathbf{c}|+Z_1)^2}(t) = \frac{1}{2}t^{-\frac{1}{2}} \int_0^\infty f_{|\mathbf{c}|}(\omega)f_{Z_1}(\sqrt{t} - \omega)d\omega, \tag{2.7}$$

yielding the density of the first component of (2.4). A second application of the convolution Lemma 2.1, together with (2.7), generates

$$\begin{aligned}
f_{Z_2^2+(|\mathbf{c}|+Z_1)^2}(t) &= \int_{\mathcal{D}} f_{(|\mathbf{c}|+Z_1)^2}(x)f_{Z_2^2}(t - x)dx \\
&= \int_{\mathcal{D}} \int_0^\infty \frac{1}{2}x^{-\frac{1}{2}}f_{|\mathbf{c}|}(\omega)f_{Z_1}(\sqrt{x} - \omega)f_{Z_2^2}(t - x)d\omega dx, \tag{2.8}
\end{aligned}$$

where \mathcal{D} is an appropriate domain of integration, determined by where the density of the squared variable Z_2^2 in (2.8) is defined as a function of $t - x$. Since the amplitude clutter distribution is assumed to be a mixture, it follows from (2.8) that the density $f_{Z_2^2+(|\mathbf{c}|+Z_1)^2}(t)$ will also be a weighted sum of densities. The two components can be found by setting $k = 0$ and $k = 1$ respectively, and noting at these extremes the clutter model corresponds to \mathbf{c}_1 and \mathbf{c}_2 respectively. This completes the proof of Theorem 2.1, as required.

2.2 Optimal Detector

We are now in a position to state the form of the likelihood function for the Neyman-Pearson optimal detector. The following is the main result:

Theorem 2.2. *For the binary statistical test formulated above, and for the signal \mathbf{s} and clutter \mathbf{c} specified in Theorem 2.1, the Neyman-Pearson Optimal detector, in the intensity domain, has likelihood*

$$L(t) = \frac{2\sqrt{t} [(1-k)f_{|\mathbf{s}+\mathbf{c}_1|^2}(t) + kf_{|\mathbf{s}+\mathbf{c}_2|^2}(t)]}{(1-k)f_{|\mathbf{c}_1|}(\sqrt{t}) + kf_{|\mathbf{c}_2|}(\sqrt{t})}, \quad (2.9)$$

where $t = |\mathbf{t}|^2$ is the intensity test value. The optimal decision rule is $L(t) \underset{H_0}{\overset{H_1}{\geq}} \tau$, where τ is the detection threshold, which means we reject the null hypothesis when the likelihood exceeds this detection threshold, and do not reject the null hypothesis otherwise. The threshold can be determined by setting the radar false alarm probability to a prescribed level.

The proof of Theorem 2.2 follows immediately from Theorem 2.1 and an application of Lemma 2.2 to the clutter density $f_{|\mathbf{c}|}$.

In the next Section optimal detectors are given for two clutter mixture distributions of interest, which are the limiting forms of the KK-Distribution clutter model.

3 The Rayleigh/K and Rayleigh/Rayleigh Mixture Cases with Swerling I Target

The Rayleigh Distribution is the well-known limit of the K-Distribution as the shape parameter ν increases [8, 15]. It is hence of interest to consider the mixture distribution (1.1) for the case where one component is Rayleigh and the other K-Distributed. Also, the situation where both components are Rayleigh is of interest. This Section will focus on deriving the Neyman-Pearson optimal detectors for these limiting mixture clutter distributions.

3.1 Rayleigh/K Case

Recall, a random variable X has a Rayleigh distribution [11, 14] with parameter $\mu > 0$ if its density is given by

$$f_X(t) = 2\mu t e^{-\mu t^2}, \quad (3.1)$$

and we write $X \stackrel{d}{=} \text{Ray}(\mu)$. Its associated distribution function $F_X(t) = \mathbb{P}(X \leq t)$ can be shown to be

$$F_X(t) = 1 - e^{-\mu t^2}. \quad (3.2)$$

Such a distribution has mean $\mathbb{E}(X) = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}$ and variance $\mathbf{var}(X) = [1 - \frac{\pi}{4}] \frac{1}{\mu}$.

The K-Distribution [3, 4, 5] with scale parameter $c > 0$ and shape parameter $\nu > 0$ has density

$$f_K(t) = \frac{2c}{\Gamma(\nu)} \left(\frac{ct}{2}\right)^\nu K_{\nu-1}(ct), \quad (3.3)$$

where we write $K \stackrel{d}{=} K(c, \nu)$. The function $K_\nu(t)$ is the modified Bessel function of the second kind, of order ν and Γ is the Gamma function. The distribution function F_K associated with the K-Distribution can be shown to have the compact form

$$F_K(t) = \mathbb{P}(K \leq t) = 1 - \frac{(ct)^\nu K_\nu(ct)}{2^{\nu-1}\Gamma(\nu)}. \quad (3.4)$$

As reported in [2], the mean and variance of K are given by

$$\begin{aligned} \mathbb{E}(K) &= \frac{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}{c\Gamma(\nu)} \\ \mathbf{var}(K) &= \mathbb{E}(K^2) - [\mathbb{E}(K)]^2 = \frac{1}{c^2} \left[4\nu - \frac{\pi\Gamma^2(\nu + \frac{1}{2})}{\Gamma^2(\nu)} \right]. \end{aligned} \quad (3.5)$$

Throughout the following we will assume the target model is Swerling I, so that its amplitude has a Rayleigh Distribution with parameter λ , and its phase Θ is uniformly distributed on $[0, 2\pi]$. Hence $|\mathbf{s}| \stackrel{d}{=} \text{Ray}(\lambda)$. It is shown in [1] that, with the Rayleigh amplitude assumption, the complex signal \mathbf{s} has a bivariate Gaussian distribution, such that its marginal distributions are independent. Hence $X_1 = |\mathbf{s}| \cos(\Theta)$ and $X_2 = |\mathbf{s}| \sin(\Theta)$ are independent, and it is also shown in [1] that each possesses a Gaussian distribution with zero mean and variance $\frac{1}{2\lambda}$. In order to apply Theorem 2.1, we first examine the case of the Swerling I target in Rayleigh clutter, which we denote by \mathbf{cR} in the complex

domain. Suppose the clutter has a Rayleigh amplitude distribution but with a parameter μ , and with independent phase $\Phi \stackrel{d}{=} R(0, 2\pi)$. Then its components are also independent Gaussians with $Y_1 = |\mathbf{c}_R| \cos(\Phi) \stackrel{d}{=} N(0, \frac{1}{2\mu})$ and $Y_2 = |\mathbf{c}_R| \sin(\Phi) \stackrel{d}{=} N(0, \frac{1}{2\mu})$. Then we can write

$$|\mathbf{s} + \mathbf{c}_R|^2 = (X_1 + Y_1)^2 + (X_2 + Y_2)^2, \quad (3.6)$$

and since this is the sum of squares of two independent Gaussian random variables, each with zero mean and variance $\frac{1}{2\mu} + \frac{1}{2\lambda}$, it follows that $|\mathbf{s} + \mathbf{c}_R|^2 \stackrel{d}{=} \left(\frac{1}{2\mu} + \frac{1}{2\lambda}\right) \chi_2^2$, where χ_n^2 is a Chi-Squared distribution with n degrees of freedom [10, 14]. Consequently, recalling a χ_2^2 distribution is identically Exponential with mean 2, the following Lemma is thus proven:

Lemma 3.1. *Under the assumption of a Swerling I target in Rayleigh Distributed clutter, the amplitude squared density has an Exponential Distribution with mean $\frac{1}{\lambda} + \frac{1}{\mu}$. Hence*

$$f_{|\mathbf{s} + \mathbf{c}_R|^2}(t) = \frac{1}{\frac{1}{\lambda} + \frac{1}{\mu}} e^{-\frac{t}{\left(\frac{1}{\lambda} + \frac{1}{\mu}\right)}}.$$

Lemma 3.1 can be applied immediately to Theorem 2.2 for mixture distributions involving a Rayleigh component. The next result provides the amplitude squared distribution for a Swerling I target in K-Distributed clutter:

Lemma 3.2. *Under the assumption of a Swerling I target in K-Distributed clutter, where the latter has scale parameter c and shape parameter ν , the amplitude squared density is*

$$f_{|\mathbf{s} + \mathbf{c}_K|^2}(t) = \frac{c^{2\nu} \phi(t; c, \nu, \lambda)}{2^{2\nu} \lambda^{\nu-1} \Gamma(\nu)}, \quad (3.7)$$

where

$$\phi(t; c, \nu, \lambda) := \int_0^\infty \frac{z^{\nu-1}}{z+1} e^{-\frac{c^2}{4\lambda} z} e^{-\frac{\lambda t}{z+1}} dz. \quad (3.8)$$

The proof can be found in [1], which also includes a detailed discussion of the special function ϕ and its evaluation.

It is now rather straightforward to specify the Neyman-Pearson optimal detector for a Swerling I target in a mixture of Rayleigh and K-Distributed clutter. The following is the main result:

Theorem 3.1. *Suppose the Swerling target model has a Rayleigh amplitude distribution with parameter $\lambda > 0$, and that the clutter amplitude is a mixture of Rayleigh and K-Distributions. Assume the clutter parameters are μ and (c, ν) respectively. Then with a mixture parameter $k \in [0, 1]$, the likelihood ratio for amplitude squared detection is*

$$L(t) = \frac{\frac{(1-k)}{\frac{1}{\lambda} + \frac{1}{\mu}} e^{-\frac{t}{\frac{1}{\lambda} + \frac{1}{\mu}}} + \frac{kc^{2\nu} \phi(t; c, \nu, \lambda)}{2^{2\nu} \lambda^{\nu-1} \Gamma(\nu)}}{(1-k)\mu e^{-\mu t} + \frac{kc^{\nu+1} t^{\frac{\nu-1}{2}} K_{\nu-1}(c\sqrt{t})}{2^\nu \Gamma(\nu)}}. \quad (3.9)$$

3.2 Rayleigh/Rayleigh Case

Similarly, for the case of an amplitude clutter distribution consisting of two Rayleigh distributions, we can derive the following main result:

Theorem 3.2. *In the case of a mixture of two Rayleigh distributions, with parameters μ_1 and μ_2 respectively, and with a Swerling I target model with amplitude parameter λ , the optimal detector has likelihood*

$$L(t) = \frac{\frac{(1-k)}{\frac{1}{\lambda} + \frac{1}{\mu_1}} e^{-\frac{t}{\frac{1}{\lambda} + \frac{1}{\mu_1}}} + \frac{k}{\frac{1}{\lambda} + \frac{1}{\mu_2}} e^{-\frac{t}{\frac{1}{\lambda} + \frac{1}{\mu_2}}}}{(1-k)\mu_1 e^{-\mu_1 t} + k\mu_2 e^{-\mu_2 t}}. \quad (3.10)$$

Observe that the optimal detectors in Theorems 3.1 and 3.2 depend on the target parameter λ . From a practical radar detection perspective, this is a significant shortcoming, since the radar will not be aware of the target's mean. It will also be difficult to estimate this in a highly spiky clutter environment [1]. Nevertheless, it is possible to construct sub-optimal decision rules that do not depend on λ , and give good performance in a number of cases. This is examined in the next Section.

4 Suboptimal Approximations of Optimal Detectors

4.1 Signal to Clutter Ratio

The likelihoods in Theorems 3.1 and 3.2 both require knowledge of the target strength through the parameter λ , and as such are not useful in practical radar detection scenar-

ios where this parameter is unknown. Hence it is important to investigate whether this parameter can be estimated or approximated in the likelihoods of the optimal detectors. Estimation of the average target strength in a highly spiky clutter environment is extremely difficult. This is because sea clutter spikes will mask the target, and may also be confused with targets. Hence estimation of λ in the likelihoods in Theorem 3.1 and 3.2 may not be simple to perform.

An alternative approach, introduced in [1], is to construct upper bounds on the likelihood by investigating ranges over which λ may vary. This can be done, for example, by assuming we are interested in the detection of small targets in high clutter scenarios. As an example, we may assume $\lambda > 1$ and produce a corresponding bound on the likelihood. Analysis in [1] found that by using an upper bound such as this, near optimal detection can be performed for quite a reasonable range of target sizes relative to the clutter strength. The latter can be measured through the signal to clutter ratio (SCR), which for the case of a Rayleigh/K Clutter mixture and Swerling I target model, the SCR is given by

$$SCR = \frac{1}{\lambda \left(\frac{1-k}{\mu} + \frac{4k\nu}{c^2} \right)}, \quad (4.1)$$

where the target parameter is λ , μ is the Rayleigh clutter component parameter and (c, ν) is the parameter set for the K-Distributed component. As before $k \in [0, 1]$ is the mixing coefficient. For the case of a mixture of two Rayleigh distributions, the SCR is

$$SCR = \frac{1}{\lambda \left(\frac{1-k}{\mu_1} + \frac{k}{\mu_2} \right)}, \quad (4.2)$$

where the Rayleigh components have parameters μ_1 and μ_2 and k is the mixing coefficient.

4.2 Suboptimal Likelihoods

Two approximate likelihoods that follow very easily from the main results Theorems 3.1 and 3.2 are now presented. The first is for a Swerling I target in a mixture of Rayleigh and K-Distributed clutter:

Corollary 4.1. *The function*

$$M(t) = \frac{(1-k)\mu e^{-\frac{\mu t}{\mu+1}} + \frac{kc^{2\nu}\phi(t;c,\nu,1)}{2^{2\nu}\Gamma(\nu)}}{(1-k)\mu e^{-\mu t} + \frac{kc^{\nu+1}t^{\frac{\nu-1}{2}}K_{\nu-1}(c\sqrt{t})}{2^\nu\Gamma(\nu)}} \quad (4.3)$$

can be used as a suboptimal approximation for the likelihood function in Theorem 3.1. This function is an upper bound on the likelihood Theorem 3.1 for the case where the target parameter exceeds 1.

Note that the ratio of the special function $\phi(t; c, \nu, \lambda)$ and $\lambda^{\nu-1}$ is essentially replaced by unity in the construction of this bound.

For the case of a Rayleigh/Rayleigh mixture with Swerling I target, the following is the relevant result:

Corollary 4.2. *The expression*

$$N(t) = \frac{(1-k)\mu_1 e^{-\frac{\mu_1 t}{\mu_1+1}} + k\mu_2 e^{-\frac{\mu_2 t}{\mu_2+1}}}{(1-k)\mu_1 e^{-\mu_1 t} + k\mu_2 e^{-\mu_2 t}} \quad (4.4)$$

is an upper bound on the optimal likelihood in Theorem 3.2, for the case where $\lambda > 1$, and can be used as a suboptimal approximation to the likelihood in this Theorem.

The next Section will examine the performance of these suboptimal approximations relative to the performance of the optimal decision rules.

5 Receiver Operating Characteristic Curves

5.1 False Alarm Probability and Detection Threshold

Before examining detector performance, we first outline how the detection threshold τ can be determined once the false alarm probability is specified. The false alarm probability is that of rejecting the null hypothesis when it is actually true. This means the return is just clutter. The probability of false alarm is given by $P_{FA} = \mathbb{P}(L(T) > \tau | H_0)$. It can be shown that

$$P_{FA} = (1-k)e^{-\mu\tau} + \frac{k(c\sqrt{\tau})^\nu K_\nu(c\sqrt{\tau})}{2^{\nu-1}\Gamma(\nu)}, \quad (5.1)$$

for the case of a mixture of Rayleigh and K-Distributed clutter with parameter sets μ and (c, ν) respectively.

In the case of a Rayleigh/Rayleigh clutter mixture, with parameters μ_1 and μ_2 , the relationship is

$$P_{FA} = (1 - k)e^{-\mu_1\tau} + ke^{-\mu_2\tau}. \quad (5.2)$$

The relationships (5.1) and (5.2) can be derived by appealing to the distribution functions (3.2) and (3.4), and noting we are focusing on amplitude squared detection. In specific applications, these equations can be solved numerically to yield the desired detection threshold τ . Throughout the following, we set $P_{FA} = 10^{-6}$, which is a value typically used in radar detection analysis.

5.2 Detector Performance: Rayleigh/K Clutter Mixture

Our numerical analysis begins with the receiver operating characteristic (ROC) curves in figure 5.1, which is for a Rayleigh/K mixture with clutter parameters $\mu = 0.001, c = 1$ and $\nu = 5.5$ (for the left plot), and $\mu = 1, c = 20$ and $\nu = 7.5$ (for the right plot). In both cases the mixing parameter is $k = 0.1$. Each figure shows the optimal detector for this clutter mixture, as well as the suboptimal detector based upon (4.3). In addition to this, what is termed a linear detector is also included for comparative purposes. Such a detector rejects the null hypothesis if its test value exceeds a threshold. In statistical terms, if T is the test statistic, then the decision rule is to reject H_0 if $T > \kappa$, for a threshold κ determined from the false alarm probability. It is important to observe that if a likelihood function is monotonic, then a uniformly most powerful test exists [10], and the optimal detector, given by either Theorem 2.1 or 2.2 reduces to this linear detector. This is because $L(t)$ is invertible in these cases. See [1] for extensive examination of this in the case of KK-Distributed clutter.

With reference to the simulation in figure 5.1, we see the left plot shows the optimal detector is slightly better than the linear detector. The suboptimal detector matches the optimal detector up until roughly 22 dB, and then becomes significantly worse than both the linear and optimal detectors.

The second plot in figure 5.1 shows improved suboptimal detector performance up to about 26 dB, and then a detection loss is introduced in the suboptimal detector. Here we observe that the optimal detector is closely matching the linear detector. This is

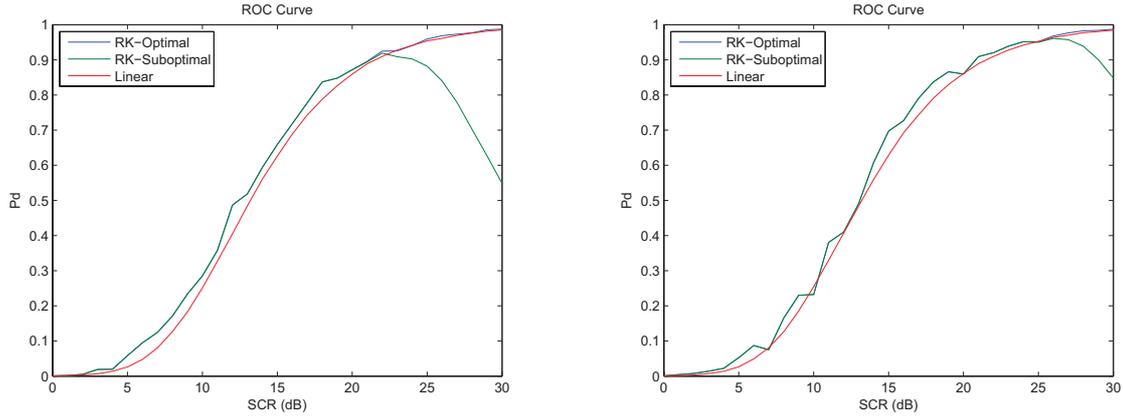


Figure 5.1: Receiver Operating Characteristic (ROC) curves for the case of a Swerling I target in a mixture of Rayleigh and K-Distributed clutter. The figure on the left is for the case where the clutter parameters are $\mu = 0.001$, $c = 1$ and $\nu = 5.5$, with mixing coefficient $k = 0.1$. The false alarm rate is 10^{-6} , and 10^5 simulations were used in the Monte Carlo estimation of each ROC point. The right figure is for the case where $\mu = 1$, $c = 20$ and $\nu = 7.5$. All other parameters as for the left figure.

because, in this case, the K-Distribution in the mixture distribution is dominant, and for these parameter in the corresponding K-Distribution optimal detector, the likelihood is monotonic [1]. Note that the reciprocal of μ is proportional to the mean of the Rayleigh component. Hence for large μ we would expect small contributions from the Rayleigh component in the clutter mixture.

In cases where the likelihood is approximately monotonic, the linear detector will be the most appropriate choice for a suboptimal detector. This is because it requires no knowledge of the target parameter.

The second set of simulations examined are in figure 5.2, which is for the case where the Rayleigh clutter has parameter $\mu = 5$, and the K-Distributed clutter has $c = 100$ and $\nu = 5.5$. The left plot uses a mixing parameter of $k = 0.01$, while the right uses $k = 0.9$. The former case implies the Rayleigh component contributes more, while the latter is for the case of K-dominance. The left plot shows the optimal and suboptimal detectors match right up to the 30 dB point, and are better than the linear detector. The right plot shows much the same, except there are periods where the optimal and suboptimal detectors are much better than the linear detector. Further simulations show that the suboptimal

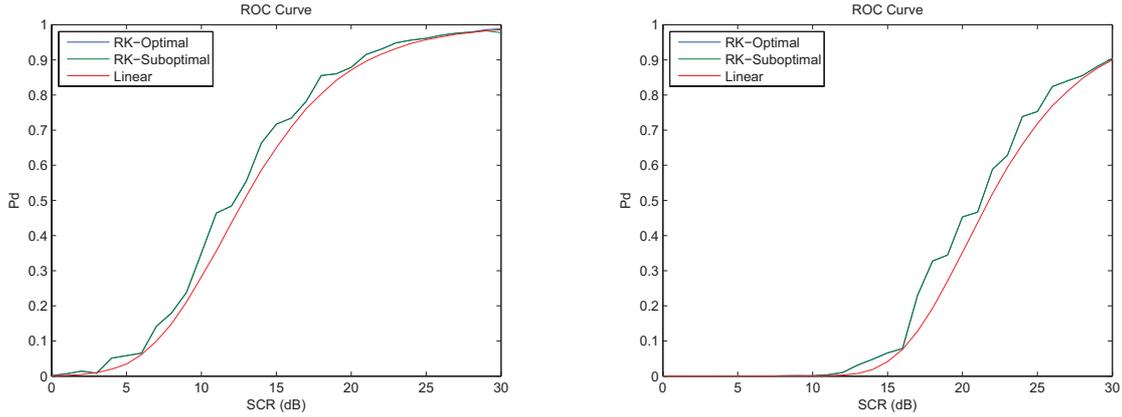


Figure 5.2: ROC curves, again for the Rayleigh/K-Distribution mixtures. Both figures use clutter parameters $\mu = 5, c = 100$ and $\nu = 5.5$. Left plot uses mixing parameter $k = 0.01$, while right uses $k = 0.9$. As for figure 5.1, the suboptimal detector is matching the optimal detector very well.

decision rule introduces a detection loss for much higher SCR.

figure 5.3 shows scenarios where the suboptimal detector can have unexpected results. In the plot on the left, the Rayleigh clutter has $\mu = 0.005$, while $c = 5$ and $\nu = 2.5$ for the K-Distribution. The mixing parameter is $k = 0.1$. We see the suboptimal detector completely fails. In this case it is clear that the linear detector is a good approximation to the optimal detector. The right plot is for $\mu = 0.001$, $c = 50$ and $\nu = 5.5$ with $k = 0.01$. Here the suboptimal performs well until about 23 dB, and then it introduces a detection loss.

5.3 Performance of Optimal Detector: Rayleigh/Rayleigh Clutter Mixture

Next we examine some ROC curves for a Rayleigh/Rayleigh mixture. Figures 5.4 and 5.5 show very similar behaviour as for the Rayleigh/K mixture. The first plot in figure 5.4 is for the case where the two Rayleigh components have parameters $\mu_1 = 10$ and $\mu_2 = 50$ respectively. The mixing parameter is $k = 0.1$. It is clear the suboptimal detector is matching the optimal detector very well, but it only introduces a minor improvement in detection probability when compared to a linear detector. The right plot is for the case

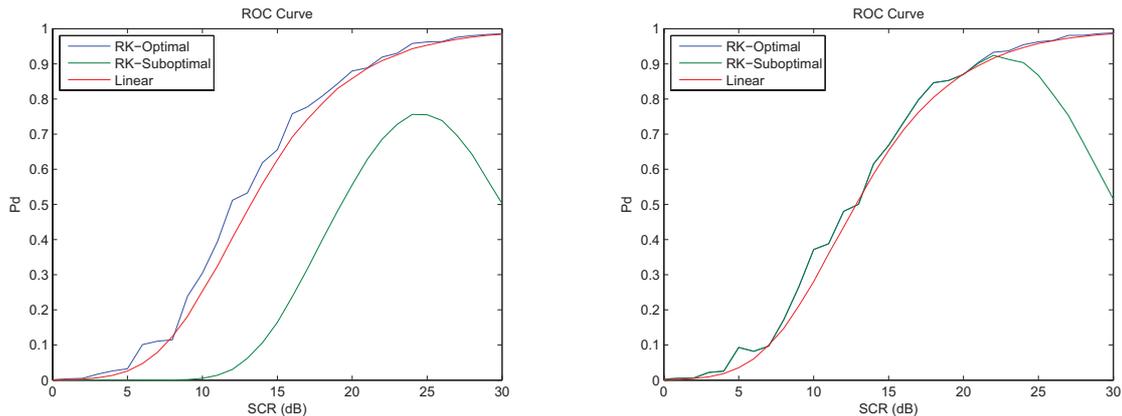


Figure 5.3: ROC curves for a Rayleigh/ K mixture, left plot is for $\mu = 0.005$, $c = 5$ and $\nu = 2.5$, with $k = 0.1$. Right plot is for $\mu = 0.001$, $c = 50$ and $\nu = 5.5$ with $k = 0.01$. The left simulation shows the failure of the suboptimal detector in this case. The right plot shows the suboptimal works very well until roughly 24 dB.

where $\mu_1 = 0.01$, $\mu_2 = 0.05$ and $k = 0.01$. The suboptimal detector has much better performance, and matches the optimal detector, until about 22 dB, then it introduces a detection loss.

In a majority of simulations considered for the case of a Rayleigh/Rayleigh mixture, it was found that the optimal detector offered only marginal improvement over the linear detector. This seemed to suggest that it may be possible that a uniformly most powerful test is close to the linear detector. Analysis of the behaviour of the derivative of the likelihood function in Theorem 3.2 can give some insight into this. It can be shown, by applying the quotient rule of differential calculus to the likelihood in Theorem 3.2, that the sign of the derivative of the likelihood $L(t)$ is determined by the function

$$\begin{aligned}
 H(t) = & e^{-[\alpha_1 + \mu_1]t} (1 - k) \mu_1 \alpha_1 [\mu_1 - \alpha_1] + e^{-[\alpha_1 + \mu_2]t} k (1 - k) \mu_2 \alpha_1 [\mu_2 - \alpha_1] \\
 & + e^{-[\alpha_2 + \mu_1]t} k (1 - k) \mu_1 \alpha_2 [\mu_1 - \alpha_2] + e^{-[\alpha_2 + \mu_2]t} k \mu_2 \alpha_2 [\mu_2 - \alpha_2], \quad (5.3)
 \end{aligned}$$

where $\alpha_1 = \frac{1}{\lambda + \frac{1}{\lambda_1}}$ and $\alpha_2 = \frac{1}{\lambda + \frac{1}{\lambda_2}}$. Note that in general $\alpha_1 < \mu_1$ and $\alpha_2 < \mu_2$, and so the first and fourth terms in (5.3) are nonnegative. Clearly the second and third terms are nonnegative when $\mu_2 > \alpha_1$ and $\mu_1 > \alpha_2$. This condition is equivalent to the requirement

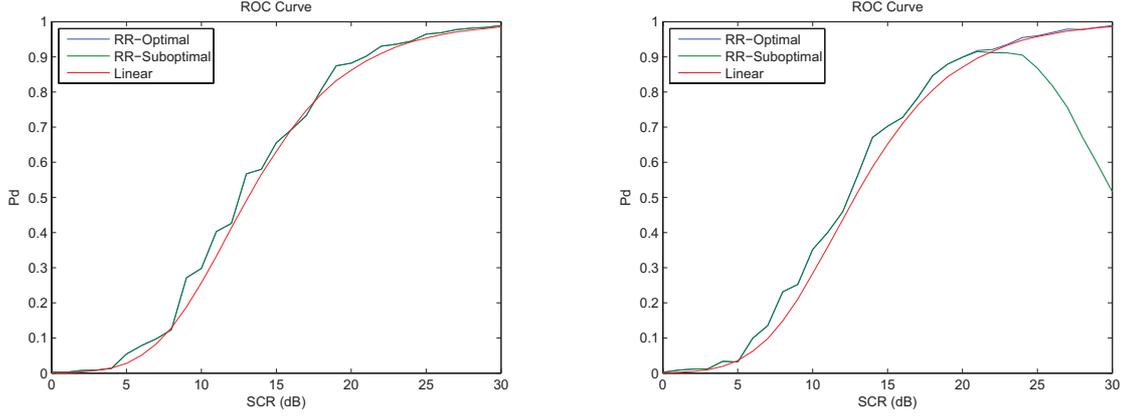


Figure 5.4: ROC curve for a Rayleigh/Rayleigh mixture, with parameters $\mu_1 = 10, \mu_2 = 50$ and mixing factor $k = 0.1$ (left figure). The right is for the case where $\mu_1 = 0.01, \mu_2 = 0.05$ and $k = 0.01$. On the left, we see the suboptimal works very well, while on the right, it introduces a detection loss at about 22 dB.

that

$$\frac{\mu_2}{\lambda} + \frac{\mu_2}{\mu_1} > 1 \text{ and } \frac{\mu_1}{\lambda} + \frac{\mu_1}{\mu_2} > 1. \tag{5.4}$$

Upon inspection we can conclude that the likelihood will be an increasing function under a number of scenarios. In the case where $\mu_1 = \mu_2$, we see that condition (5.4) will be satisfied. Others include when $\min\{\mu_1, \mu_2\} > \lambda$, when $\mu_1 = \lambda$ and μ_2 is an integral multiple of μ_1 and vice versa. Note that the mixing coefficient k can also dictate whether the likelihood is nonnegative. Clearly at its endpoints, the second and third terms are eliminated in (5.3), implying a nonnegative derivative sign.

Figure 5.5 is an example where the suboptimal detector completely fails. The simulation is for the case where the Rayleigh/Rayleigh clutter has parameters $\mu_1 = 0.1$ and $\mu_2 = 0.001$, with mixing coefficient $k = 0.4$. The left plot shows the ROC curve, while the right plot shows the likelihood function (optimal detector) for two λ values. We see the likelihood is non-monotonic initially, then becomes roughly decreasing, flattening out to zero as t increases.

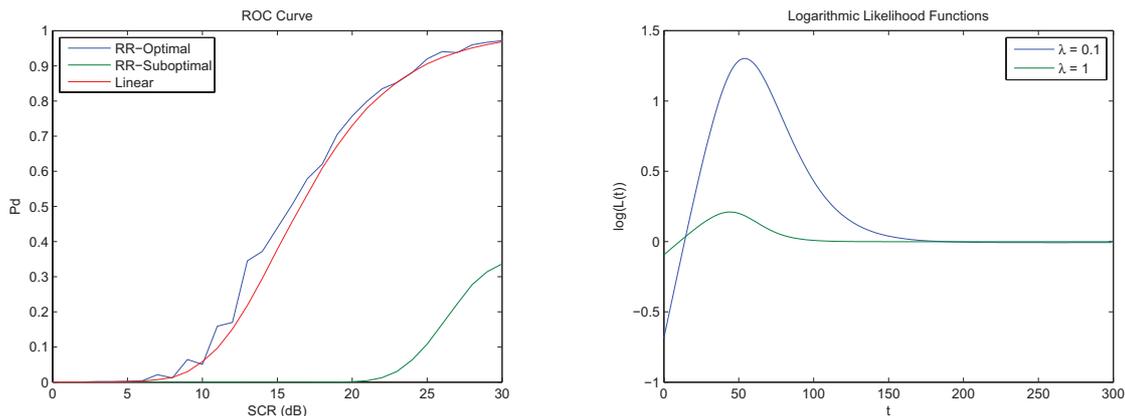


Figure 5.5: Left figure shows ROC curve for the case where the Rayleigh mixture has parameters $\mu_1 = 0.1$ and $\mu_2 = 0.001$, with mixing coefficient $k = 0.4$. The right plot shows the corresponding likelihood function, for two values of λ as given in the caption. The likelihood function becomes flat as t increases, which explains why it produces an optimal detector that is close to the linear detector for large SCR. In this situation, the suboptimal detector has terrible performance.

5.4 Guidelines on Suboptimal Detectors

The simulations presented in this Section suggest a number of strategies that can be used when designing a suboptimal detector for the target detection problem analysed here. In the case where the K-Distribution is dominant in the clutter mixture model, the likelihood function can be tested for monotonicity via numerical methods. If it is monotonic, a uniformly most powerful test exists, and it coincides with the linear decision rule. The latter requires no knowledge of the target parameter. If the likelihood does not appear to be monotonic, then one of the suboptimal decision rules can be applied, generally for small target detection. For larger target detection, the linear detector can be employed.

In the case of a Rayleigh/Rayleigh clutter mixture, the linear detector is more reliable and can be used for all cases of target size, with only a small detection loss relative to the optimal detector.

6 Conclusions and Further Research

The Neyman-Pearson optimal detector was derived for the case of detection of a complex signal with independent components embedded within complex clutter whose amplitude distribution forms a mixture model. This result generalises the analogue in [1], which is limited to the case of optimal detection of a Swerling I target in KK-Distributed clutter. The general result derived here allowed the determination of the asymptotic forms of the optimal detector in KK-Distributed clutter. Specifically, decision rules for detection of Swerling targets in Rayleigh/K and Rayleigh/Rayleigh clutter amplitude mixtures were derived.

To remove the dependence of decision rules on the signal to clutter ratio, upper bounds were used to derive suboptimal detectors. These were shown to perform well in a number of scenarios, namely for small signal strengths relative to the clutter, and a series of guidelines were proposed for their use, coupled with the option of using a linear detector.

Further work will be devoted to finding better suboptimal approximations for larger signal to clutter ratios. Additionally, the case of detection based upon a scan of several pulses is under investigation.

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