

Weakly Nonlinear Harmonic Acoustic Waves in Classical Thermoviscous Fluids: A Perturbation Analysis

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Abstract—Using regular perturbation analysis, we investigate the propagation of a time-harmonic acoustic signal, generated by a sinusoidal boundary condition, in a half-space filled with a classical thermoviscous fluid. It is assumed that the flow is described by a recently introduced, weakly nonlinear partial differential equation (PDE) that, unlike earlier models, exhibits a Hamiltonian structure in the lossless limit.

I. INTRODUCTION

In their recent article on acoustic propagation in classical thermoviscous fluids (i.e., Newtonian fluids for which the heat flux vector obeys Fourier's law), Rasmussen et al. [1] presented the latest addition to the family of equations used to describe acoustic phenomena under the finite-amplitude approximation. Assuming propagation in 1D along the x -axis, this weakly nonlinear PDE, which we shall refer to as the Rasmussen–Sørensen–Gaididei–Christiansen (or RSGC) equation, assumes the form

$$(c_0^2 - \phi_t)\phi_{xx} - \phi_{tt} + \delta\phi_{txx} = \partial_t[(\phi_x)^2 + c_0^{-2}(\beta - \frac{3}{2})(\phi_t)^2]. \quad (1)$$

Here, $\phi = \phi(x, t)$ is the velocity potential, where $u = \phi_x$ and the velocity vector is of the form $\mathbf{v} = (u(x, t), 0, 0)$; the positive constants c_0 and δ denote the sound speed in the undisturbed fluid and the diffusivity of sound [2], respectively; and $\beta (> 1)$, the coefficient of nonlinearity¹, is given by $\beta = 1 + B/(2A)$, where the ratio B/A is known as the nonlinearity parameter [3].

Unlike its better known counterpart Kuznetsov's equation [4], which in the present context takes the form

$$c_0^2\phi_{xx} - \phi_{tt} + \delta\phi_{txx} = \partial_t[(\phi_x)^2 + c_0^{-2}(\beta - 1)(\phi_t)^2], \quad (2)$$

(1) admits a Hamiltonian structure in the limit $\delta \rightarrow 0$ (i.e., the lossless limit), as do the Euler equations to which this limiting case corresponds. Of course, because so much is known about Hamiltonian systems, and the fact that they are amiable to treatment by an array of analytical methods, a Hamiltonian structure is a highly desirable property for one's mathematical model to possess. On the other hand, Kuznetsov's equation and its variants, which have been the subject of intense study for almost forty years, exhibit a number of interesting features;

¹In the case of gases, the coefficient of nonlinearity can also be expressed as $\beta = (\gamma + 1)/2$, where $\gamma (> 1)$ denotes the adiabatic index [3].

see, e.g., [5]. As a case in point, the well known Burgers' equation is readily derivable from (2) by assuming, in part, unidirectional plane wave flow; see [4], [5].

While clear differences exist between (1) and (2), it is important to point out that both are derived from the usual mass, momentum (i.e., Navier–Stokes), and energy conservation equations, augmented with the non-isentropic, Taylor series-based equation of state [6]

$$P \approx A \left[s + \frac{B}{2A}s^2 - \kappa c_0^{-2}(\gamma - 1)\nabla \cdot \mathbf{v} \right], \quad (3)$$

for a classical thermoviscous fluid undergoing irrotational, compressible flow. As noted in [1], however, the lossless version (1) is also derivable from a variational approach, whereas the lossless version of (2) is not. Here, P is the acoustic (or relative) pressure, s is the condensation, and the positive constant κ denotes the thermal diffusivity. It should also be noted that, in spite of the fact that there is no exact, general equation of state for liquids, (3) is applicable to both gases and liquids. The derivations of (1) and (2) are based on both the assumption that the relative perturbations about the equilibrium state are small, but finite, and the fact that (3) is an analytically tractable expression for P that, while approximate, is still able to capture the essential nonlinear physics involved.

Another PDE that we shall encounter in our investigation is known as Stokes' equation [7].

$$c_0^2\phi_{xx} - \phi_{tt} + \delta\phi_{txx} = 0, \quad (4)$$

which is the linearized version of both (1) and (2). It is of interest to note that this equation, which is readily derived from the linearized mass, momentum, and energy conservation equations by neglecting the s^2 term in (3), also describes the transverse vibrations of an internally damped string [8] and certain magnetohydrodynamic (or MHD) flows [9].

In the present article, we carry out a perturbation analysis of (1) and (2) in the half-space $x > 0$ subject to the boundary conditions (BC)s

$$\phi_x(0, t) = U_0 \sin(\Omega t), \quad \phi_x(\infty, t) = 0 \quad (t > 0). \quad (5)$$

In the boundary-value problem (BVP) we consider, which can be regarded as the weakly nonlinear acoustic version of

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14. ABSTRACT Using regular perturbation analysis, we investigate the propagation of a time-harmonic acoustic signal, generated by a sinusoidal boundary condition, in a half-space filled with a classical thermoviscous fluid. It is assumed that the flow is described by a recently introduced, weakly nonlinear partial differential equation (PDE) that, unlike earlier models, exhibits a Hamiltonian structure in the lossless limit.					
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Stokes' second problem, the positive constants U_0 and Ω denote the amplitude and angular frequency, respectively, of the time-harmonic excitation applied at the boundary $x = 0$. Here, it should be noted that similar perturbation studies involving some of the various other PDEs of nonlinear acoustics can be found in the literature; see, e.g., [10]–[13]. In particular, the analytical approach taken here closely follows those of Zabolotskaya et al. [12] and Jordan [13], who considered the modified Burgers' equation and the Darcy–Jordan poroacoustic model, respectively.

The primary aim of this investigation is to compare the RSGC equation with Kuznetsov's equation in the context of the above-mentioned BVP. That is, we seek to understand how the differences in these two PDEs are manifested in their perturbation solutions. In particular, we compare/contrast the lowest-order correction terms in both the low- and high-frequency regimes. And, of course, the exact solution of (4), which is also the zeroth-order term in both perturbation solutions, is derived and analyzed.

To this end, the present article is arranged as follows. In Sect. II, the two-term perturbation solutions of these equations, subject to the above BCs, are determined. In Sect. III, low- and high-frequency asymptotic expressions are derived and numerical results are presented. Finally, in Sect. IV, a summary is given and conclusions are stated

II. STOKES' SECOND PROBLEM: NONLINEAR ACOUSTIC VERSION

Re-expressing (1), (2), and the BCs given in (5) in terms of the following nondimensional variables: $\bar{\phi} = \phi/(LU_0)$, $\bar{u} = u/U_0$, $\bar{x} = x/L$, and $\bar{t} = t(c_0/L)$, where the positive constant L is a characteristic length, and switching to a more compact form of notation, our BVP becomes

$$\square\phi + \lambda\phi_{txx} = \epsilon\{\sigma\phi_t\phi_{xx} + [(\phi_x)^2 + f(\sigma)(\phi_t)^2]_t\},$$

$$(x, t) \in (0, \infty) \times (0, \infty); \quad (6a)$$

$$\phi_x(0, t) = \sin(\omega t), \quad \phi_x(\infty, t) = 0, \quad (t > 0); \quad (6b)$$

where $\square^2 := \partial_{xx} - \partial_{tt}$ denotes the 1D d'Alembertian operator, $\epsilon = U_0/c_0$ is the Mach number, $\omega = \Omega L/c_0$, and λ denotes the dimensionless diffusivity of sound. In addition, the index

$$\sigma := \begin{cases} 1, & \text{RSGC equation,} \\ 0, & \text{Kuznetsov's equation,} \end{cases} \quad (7)$$

and the function $f(\sigma) = -(\sigma - B/A)/2$ have been introduced here for convenience; $0 < \epsilon \ll 1$ follows from the weakly nonlinear approximation; we observe that $\lambda = 1/\text{Re}_d$, where Re_d denotes the Reynolds number based on δ ; and all superposed bars have been suppressed but remain understood.

With ϵ as the natural perturbation parameter, and recalling that both (1) and (2) were derived under the finite amplitude approximation, we assume a regular expansion [11], [14], [15] correct only through $\mathcal{O}(\epsilon)$; specifically, subject to $\epsilon|\phi_1| \ll |\phi_0|$, we set $\phi = \phi^{(1)}$, where

$$\phi^{(1)} := \phi_0 + \epsilon\phi_1. \quad (8)$$

Substituting (8) into BVP (6) yields, after expanding and equating like powers of ϵ , the following sequence of linear PDEs:

$$\begin{aligned} & \square^2\phi_n + \lambda\partial_{txx}\phi_n \\ & = \begin{cases} 0, & n = 0, \\ \sigma(\partial_t\phi_0)\partial_{xx}\phi_0 + [f(\sigma)(\partial_t\phi_0)^2 + (\partial_x\phi_0)^2]_t, & n = 1, \end{cases} \end{aligned} \quad (9)$$

which are to be solved subject to

$$\partial_x\phi_n(0, t) = \begin{cases} \sin(\omega t), & n = 0, \\ 0, & n = 1, \end{cases} \quad \partial_x\phi_n(\infty, t) = 0, \quad (10)$$

successively for $n = 0, 1$.

Using first the substitution $\phi_0(x, t) = \frac{1}{2}iX_0(x)\exp(-i\omega t) + \text{c.c.}$, where ‘‘c.c.’’ denotes the complex conjugate of the preceding term, the zeroth order solution, which, as noted earlier, is the exact solution of Stokes' equation in the setting of BVP (6), is easily found to be

$$\phi_0 = -\frac{1}{2}iP_0(\omega)\exp[-(\alpha_0 - i\beta_0)x]\exp(-i\omega t) + \text{c.c.}, \quad (11)$$

where $P_0(\omega) = (\alpha_0 - i\beta_0)^{-1}$ and the corresponding attenuation coefficient and wavenumber are respectively given by

$$\alpha_0 = \omega\sqrt{\frac{-1 + \sqrt{1 + \lambda^2\omega^2}}{2(1 + \lambda^2\omega^2)}}, \quad \beta_0 = \omega\sqrt{\frac{1 + \sqrt{1 + \lambda^2\omega^2}}{2(1 + \lambda^2\omega^2)}}. \quad (12)$$

In turn, the first perturbation of ϕ_0 is determined by first setting $\phi_1(x, t) = \frac{1}{2}iX_1(x)\exp(-2i\omega t) + \text{c.c.}$ and then solving the resulting inhomogeneous ODE

$$X_1''(1 - 2i\lambda\omega) + 4\omega^2X_1 = \omega[(1 + \frac{1}{2}\sigma)(\alpha_0 - i\beta_0)^2 - \omega^2f(\sigma)]X_0^2. \quad (13)$$

where a prime denotes d/dx . Omitting the details, it is readily show that

$$\phi_1 = \frac{1}{2}iP_1(\omega)\{2\exp[-(\alpha_1 - i\beta_1)x] - (\alpha_1 - i\beta_1) \times P_0(\omega)\exp[-2(\alpha_0 - i\beta_0)x]\}e^{-2i\omega t} + \text{c.c.}, \quad (14)$$

where

$$P_1(\omega) = \frac{(\lambda\omega + i)(\alpha_0 - i\beta_0)[1 + \frac{1}{2}\sigma - \omega^2P_0^2(\omega)f(\sigma)]}{4\lambda\omega^2(\alpha_1 - i\beta_1)}, \quad (15)$$

$$\alpha_1 = 2\omega\sqrt{\frac{-1 + \sqrt{1 + 4\lambda^2\omega^2}}{2(1 + 4\lambda^2\omega^2)}}, \quad (16)$$

and

$$\beta_1 = 2\omega\sqrt{\frac{1 + \sqrt{1 + 4\lambda^2\omega^2}}{2(1 + 4\lambda^2\omega^2)}}. \quad (17)$$

Finally, using the defining relation $u = \partial_x\phi$, we find that

$$u^{(1)}(x, t) := u_0(x, t) + \epsilon u_1(x, t), \quad (18)$$

where

$$u_0 = \partial_x\phi_0 = -\exp(-\alpha_0x)\sin(\beta_0x - \omega t) \quad (19)$$

and

$$u_1 = \partial_x \phi_1 = R_1(\omega) [e^{-\alpha_1 x} \sin(\beta_1 x - 2\omega t + \psi_1) - e^{-2\alpha_0 x} \sin(2\beta_0 x - 2\omega t + \psi_1)]. \quad (20)$$

Here,

$$R_1(\omega) = |A_1 + iB_1| \quad \text{and} \quad \psi_1 = \text{Arg}(A_1 + iB_1), \quad (21)$$

where the real and imaginary parts of $A_1 + iB_1 = 2P_1(\omega)(\alpha_1 - i\beta_1)$ are given by

$$A_1 = \frac{(1 + \frac{1}{2}\sigma)(\alpha_0 \lambda \omega + \beta_0) - f(\sigma)(\alpha_0 \lambda \omega - \beta_0) \sqrt{1 + \lambda^2 \omega^2}}{2\lambda \omega^2}$$

and

$$B_1 = \frac{(1 + \frac{1}{2}\sigma)(\alpha_0 - \beta_0 \lambda \omega) - f(\sigma)(\alpha_0 + \beta_0 \lambda \omega) \sqrt{1 + \lambda^2 \omega^2}}{2\lambda \omega^2},$$

and where $|\cdot|$ and $\text{Arg}(\cdot)$ denote the modulus and the principal value of the argument, respectively, of a complex quantity.

III. ANALYTICAL AND NUMERICAL RESULTS

A. Phase speeds and penetration depths

Along with the attenuation coefficients and wave numbers, two other quantities are of importance in the study of waves produced by harmonic excitations. Specifically, the phase speeds and penetration depths, which are respectively given here by

$$V_n = \frac{\omega}{\beta_n} \quad \text{and} \quad d_n = \frac{1}{\alpha_n} \quad (n = 0, 1). \quad (22)$$

These quantities are used to characterize the structure and behavior of the wave field over space and time. (For a discussion of the physical significance of these quantities, see [16], [17], respectively.)

In the next two subsection, we derive low- and high-frequency asymptotic expressions for $\alpha_{0,1}$, $\beta_{0,1}$, and the quantities given in (22).

B. Low-frequency results

Suppose that $\omega \ll 1/\lambda$. Then, using the binomial theorem, it can be established that the attenuation coefficients and wave numbers admit the small- ω approximations

$$\begin{aligned} \alpha_n &\approx \frac{1}{2}\lambda(n+1)^2\omega^2 \left[1 - \frac{1}{8}(n+1)^2(16 + \lambda^2)\omega^2\right], \\ \beta_n &\approx (n+1)\omega \left[1 - \frac{1}{8}(n+1)^2(16 - \lambda^2)\omega^2\right], \end{aligned} \quad (23)$$

as $\omega \rightarrow 0$, for $n = 0, 1$. Thus, from (22) and (23) it follows that

$$\begin{aligned} V_n &\approx (n+1)^{-1} \left[1 + \frac{1}{8}(n+1)^2(16 - \lambda^2)\omega^2\right], \\ d_n &\approx \frac{2}{\lambda(n+1)^2\omega^2} \left[1 + \frac{1}{8}(n+1)^2(16 + \lambda^2)\omega^2\right], \end{aligned} \quad (24)$$

as $\omega \rightarrow 0$, for $n = 0, 1$.

C. High-frequency results

Once again employing the binomial theorem, but now under the assumption $\omega \gg 1/\lambda$, the corresponding large- ω approximations are found to be

$$\begin{aligned} \alpha_n &\approx \sqrt{\frac{\lambda(n+1)\omega}{8}} \left[1 - \frac{1}{2\lambda(n+1)\omega}\right], \\ \beta_n &\approx \sqrt{\frac{\lambda(n+1)\omega}{8}} \left[1 + \frac{1}{2\lambda(n+1)\omega}\right], \end{aligned} \quad (25)$$

as $\omega \rightarrow \infty$, for $n = 0, 1$. Thus, from (22) and (25) we find that

$$\begin{aligned} V_n &\approx 2\sqrt{\frac{2\omega}{\lambda(n+1)}} \left[1 - \frac{1}{2\lambda(n+1)\omega}\right], \\ d_n &\approx 2\sqrt{\frac{2}{\lambda(n+1)\omega}} \left[1 + \frac{1}{2\lambda(n+1)\omega}\right], \end{aligned} \quad (26)$$

as $\omega \rightarrow \infty$, for $n = 0, 1$.

D. Numerical results

Fig. 1 was generated based on the solution of (13) using the software package *Mathematica* (version 5.2). The value of β taken corresponds to seawater at 20°C and 3.5% salinity [3], while those of ϵ and λ were chosen based primarily on the need to produce clear, informative graphs over the frequency range considered. In Fig. 1 we see that the difference

$$\Delta_{\max}(\omega) := \max_{x>0}\{|X'_1|\}_{\sigma=0} - \max_{x>0}\{|X'_1|\}_{\sigma=1} \quad (27)$$

clearly *increases* as ω is increased. This behavior appears to be due, at least in part, to the fact that the coefficient R_1 depends on both ω and σ . On the other hand, we see that as ω is decreased, the two curves approach each other and start to coalesce, eventually becoming indistinguishable. Note, however, that decreasing ω from 5.0 to 0.1 has also caused both quantities on the right-hand side of (27) to grow so large that, for the Mach number value taken, the primary assumption upon which our perturbation solutions are based is violated (see Fig. 1(c,d)).

IV. SUMMARY AND CONCLUSIONS

In this very brief study, we have carried out a regular perturbation analysis of the RSGC equation in the context of the acoustic version of Stokes' second problem. We have presented low- and high-frequency expressions, examined the linearized problem, and compared the RSGC equation with the classical Kuznetsov's equation using both analytical and numerical methods. Based on an analysis of these findings, we report the following:

- 1) The presence of the thermoviscous (i.e., damping) term $\delta\phi_{txx}$ in (1) and (2) was found to have a stabilizing effect in the sense that, while the $\mathcal{O}(\epsilon)$ term in the lossless (i.e., $\lambda \rightarrow 0$) case is secular (see [11, eqn. (14)]), the same term in (18) is *not*.
- 2) The solutions of both (1) and (2) assume a diffusive character for large- ω ; see Sect. III C. Consequently,

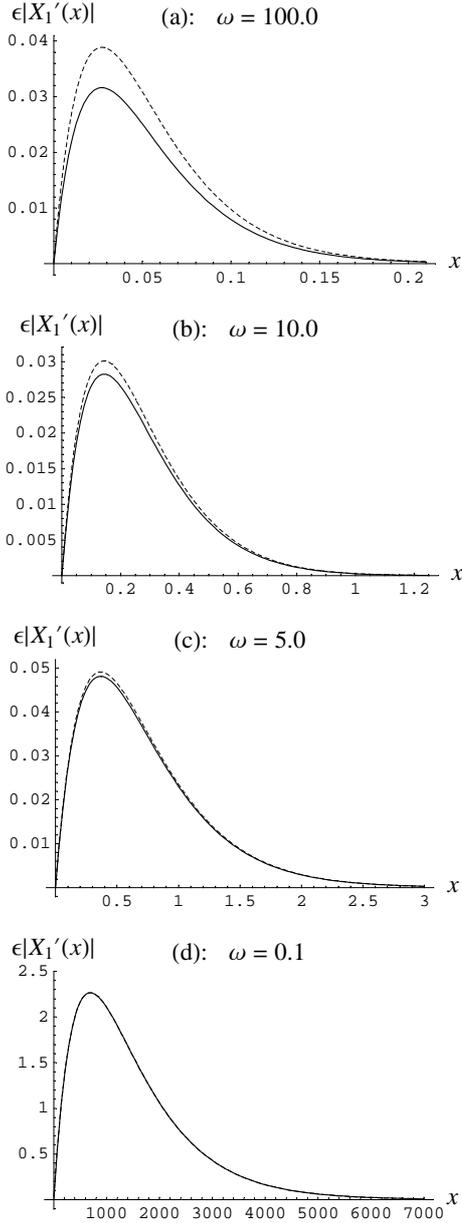


Fig. 1. $\epsilon|X_1'|$ vs. x for $\epsilon = 0.05$, $\lambda = 0.1$, and $\beta = 3.625$. Solid: $\sigma = 1$. Broken: $\sigma = 0$.

these PDEs predict that information contained in high-frequency harmonic signals is, practically speaking, rapidly and irreversibly lost, due to the “smoothing property” of the diffusion equation, as such signals propagate through classical thermoviscous fluids.

- 3) Since $V_{0,1}$ are strictly increasing functions of ω , both (1) and (2) predict that classical thermoviscous fluids exhibit *anomalous dispersion* [16], with respect to harmonic acoustic signals, where it should be noted that $1 > V_0 > V_1 > 0$ for $\omega > 0$. (Recall: $V_{0,1}$ are independent of σ .)
- 4) The only notable difference between (1) and (2) occurs

at higher frequencies. Specifically, $\Delta_{\max}(\omega)$ increases with increasing ω (at least over the frequency range considered in Fig. 1). Otherwise, the behaviors of (1) and (2) in the setting of BVP (6) appear to be all but identical. For example, our numerical simulations also suggest that, for ω sufficiently small,

$$0 \leq |X_1'|_{\sigma=0} - |X_1'|_{\sigma=1} \ll 1 \quad (\forall x > 0). \quad (28)$$

(This, of course, also suggests that Δ_{\max} is strictly non-negative.)

- 5) Fig. 1(b–d) clearly illustrates that, below a certain frequency, whose value likely depends on λ and/or β , both $\max_{x>0}\{|X_1'|_{\sigma=0}\}$ and $\max_{x>0}\{|X_1'|_{\sigma=1}\}$ start to increase (possibly without bound) as ω is further decreased. Thus, within this (low-)frequency range, there is an ω -dependent upper bound that ϵ must satisfy; otherwise, the primary assumption upon which both perturbation solutions are based is violated (see Fig. 1(d)).

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