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Derivation of Grad's Thirteen Regularized Moment Equations Using a Hermite Polynomial Representation of Velocity Distribution Function (Preprint)

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ABSTRACT

This paper derives the transport equations for rarefied gases from the Bhatnagar-Gross-Krook (BGK) model kinetic equation using Hermite polynomial representation of the velocity distribution function. We apply the Champmen-Enskog method to Grad's thirteen moment equations to derive a closure of Grad's 13 moment equations, extending them to third order of the Knudsen number. The velocity distribution function for the resulting 13 regularized moment equations is presented.

I. INTRODUCTION

One of the hardest problems in computational fluid dynamics is the modeling of medium rarefied gases with the Knudsen number in the range 0.005 – 1. In this case, the gas is rarefied to such a degree that using the Navier-Stokes-Fourier equations is questionable, but it is not sufficiently rarefied that using Direct Simulation Monte Carlo (DSMC) methods is effective. It should be stressed that in recent years computational fluid dynamics for Knudsen numbers in this range has become more and more important for many practical applications, ranging from the modeling of reentry of space vehicles into the atmosphere to the modeling of microscale flows and heat transfer in microchannels. One of the ways to cover this range of Knudsen numbers is to use thirteen (or more) moment equations instead of the Navier-Stokes-Fourier 5 equations. In 1949 Grad derived 13 moment equations corresponding to the second order of the Knudsen number [1, 2]. It is worth noting that the Navier-Stokes-Fourier equations are to first order of the Knudsen number. Unfortunately, Grad's moment equations sometimes produce unphysical solutions; for example, they fail to describe smooth shock structures for Mach numbers above a critical value [3]. In 2004 Struchtrup [4] regularized Grad's equations,

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extending them to third order of the Knudsen number. This was a very important step in rarefied gas dynamics. This step's significance can be compared to the extension of Euler's gas dynamics equations to the Fourier-Navier-Stokes gas dynamics equations. The author has developed a new closure method that is principally different from the well-known Chapmen-Enskog method [5, 6] (that was used to derive a closure of Euler's gas dynamics equations), in which he has not used the Hermite polynomial representation of the velocity distribution function. It should be noted that the Struchtrup closure method is very complicated, and this is probably one of the reasons that his method and equations are difficult to comprehend.

In the present paper we suggest a new closure for Grad's thirteen equations by using a Hermite polynomial approximation for the monatomic gas velocity distribution function, and applying the Chapman-Enskog regularization method to Grad's velocity distribution function that corresponds to his 13 moment equation [5, 6]. In our paper, the collision term is assumed to be in the BGK form. The integral representation for the 13 moments of the Boltzmann equation and the Hermite polynomial approximation of the velocity distribution function are obtained in Sections II and III respectively. Grad's regularized 13 moment equations are derived in Section IV, and conclusions are presented in Section V.

II. GENERAL EQUATION FOR 13 MOMENTS

The phase density of a monatomic ideal gas is described by the Boltzmann equation,

$$\frac{\partial(n\cdot f)}{\partial t}+V_i\cdot\frac{\partial(n\cdot f)}{\partial x_i}=St(n\cdot f)\,, \tag{1}$$

where

$$\int_{\vec{V}} f\cdot d^3\vec{V}=1\,. \tag{2}$$

Here n is the number density of gas molecules, f is the velocity distribution function, $V_i=(V_x,V_y,V_z)$ is the particle velocity, $x_i=(x,y,z)$ are the coordinates of a particle, $\int_{\vec{V}} d^3\vec{V}$ means the integration over the entire velocity space, and $St(n\cdot f)$ is the collision term that accounts for the change in the velocity distribution function due to collisions. Here we assume elastic collisions.

Let us introduce ρ as the mass density of gas molecules,

$$\rho=m\cdot n\cdot\int_{\vec{V}} f\cdot d^3\vec{V}\,, \tag{3}$$

$u_i = (u_x, u_y, u_z)$ as the flow velocity of gas molecules,

$$u_i = \int_{\vec{V}} f \cdot V_i \cdot d^3\vec{V} \; , \tag{4}$$

V_T as the thermal velocity,

$$\frac{3}{2} \cdot \frac{\rho \cdot V_T^2}{2} = \frac{\rho}{2} \cdot \int_{\vec{V}} f \cdot \left((V_x - u_x)^2 + (V_x - u_y)^2 + (V_z - u_z)^2 \right) \cdot d^3\vec{V} \; , \tag{5}$$

$q_i = \left(q_x, q_y, q_z \right)$ as the heat flux,

$$q_i = \frac{\rho}{2} \cdot \int_{\vec{V}} f(\vec{V}) \cdot (V_i - u_i) \cdot [(V_x - u_x)^2 + (V_y - u_y)^2 + (V_z - u_z)^2] \cdot d^3\vec{V} \; , \tag{6}$$

$\sigma_{ij} = (\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz})$ as the components of the stress tensor,

$$\sigma_{ij} = \rho \cdot \int_{\vec{V}} f \cdot \left((V_i - u_i) \cdot (V_j - u_j) - \delta_{ij} \cdot \frac{V_T^2}{2} \right) \cdot d^3\vec{V} \; , \tag{7}$$

were m is the mass of a particle. As one can see, the total number of moments of the particle distribution function introduced here is 13. Let us obtain general equations for these 13 moments using the Boltzmann equation, Eq. (1). First we consider equations for ρ, u_x, u_y, u_z, and V_T^2. Since the number of colliding particles, their total momentum, and their total energy are conserved in collisions, it follows that

$$n \cdot \int_{\vec{V}} St(f) \cdot d^3\vec{V} = 0 \; , \tag{8}$$

$$m \cdot n \cdot \int_{\vec{V}} V_i \cdot St(f) \cdot d^3\vec{V} = 0 \; , \tag{9}$$

$$\frac{m}{2} \cdot n \cdot \int_{\vec{V}} V^2 \cdot St(f) \cdot d^3\vec{V} = 0 \; . \tag{10}$$

We obtain from Eq. (1) the following moment equations that correspond to mass, momentum and energy conservation laws respectively,

$$\frac{\partial f}{\partial t}\left(m \cdot n \cdot \int_{\vec{V}} f \cdot d^3\vec{V} \right) + \frac{\partial f}{\partial x_i}\left(m \cdot n \cdot \int_{\vec{V}} f \cdot V_i \cdot d^3\vec{V} \right) = 0 \; , \tag{11}$$

$$\frac{\partial f}{\partial t}\left(m\cdot n\cdot\int_{\vec V} f\cdot V_k\cdot d^3\vec V\right)+\frac{\partial f}{\partial x_i}\left(m\cdot n\cdot\int_{\vec V} f\cdot V_i\cdot V_k\cdot d^3\vec V\right)=0\,, \tag{12}$$

$$\frac{\partial f}{\partial t}\left(\frac{m\cdot n}{2}\cdot\int_{\vec V} f\cdot V^2\cdot d^3\vec V\right)+\frac{\partial f}{\partial x_i}\left(\frac{m\cdot n}{2}\cdot\int_{\vec V} f\cdot V^2\cdot V_i\cdot d^3\vec V\right)=0\,, \tag{13}$$

where $V^2=V_x^2+V_y^2+V_z^2$, with indexes $i,k=x,y,z$. After tedious algebra Eq. (11) – (13) can be presented in the following form [7]

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_i}\left(\rho\cdot u_i\right)=0\,, \tag{14}$$

$$\rho\cdot\frac{\partial u_i}{\partial t}+(\rho\cdot u_j)\cdot\frac{\partial u_i}{\partial x_j}+\frac{\partial}{\partial x_i}\left(\frac{\rho\cdot V_T^2}{2}\right)+\frac{\partial\sigma_{ji}}{\partial x_j}=0\,, \tag{15}$$

$$\frac{3}{4}\cdot\rho\cdot\frac{\partial V_T^2}{\partial t}+\frac{3}{2}\cdot\rho\cdot u_i\cdot\frac{\partial}{\partial x_i}\left(\frac{V_T^2}{2}\right)+\frac{1}{2}\cdot\rho\cdot V_T^2\cdot\frac{\partial u_i}{\partial x_i}+\frac{\partial q_i}{\partial x_i}+\sigma_{ij}\cdot\frac{\partial u_i}{\partial x_j}=0\,. \tag{16}$$

We derive the general moment equations for q_i and σ_{xy} and σ_{xx} in Appendix A. They can be written as:

$$\begin{aligned}&\frac{\partial q_i}{\partial t}+\left(\frac{5}{2}\cdot\frac{\rho\cdot V_T^2}{2}\right)\cdot\frac{\partial u_i}{\partial t}+\sigma_{ik}\cdot\frac{\partial u_k}{\partial t}+\frac{\partial}{\partial x_k}\left(u_k\cdot q_i\right)+q_k\cdot\frac{\partial u_i}{\partial x_k}+\left(u_k\cdot\frac{5}{2}\cdot\frac{\rho\cdot V_T^2}{2}\right)\cdot\frac{\partial u_i}{\partial x_k}+\left(u_k\cdot\sigma_{ij}\right)\cdot\frac{\partial u_j}{\partial x_k}+\\&+\frac{\partial}{\partial x_k}\left(\frac{\rho}{2}\cdot\int_{\vec V}(V_i-u_i)\cdot(\vec V-\vec u)^2\cdot(V_k-u_k)\cdot f\cdot d^3\vec V\right)+\left(\rho\cdot\int_{\vec V}(V_i-u_i)\cdot(V_k-u_k)\cdot(V_j-u_j)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_j}{\partial x_k}=\\&=\frac{\rho}{2}\cdot\int_{\vec V}(V_i-u_i)\cdot(\vec V-\vec u)^2\cdot St(f)\cdot d^3\vec V,\end{aligned} \tag{17}$$

$$\begin{aligned}&\frac{\partial\sigma_{xy}}{\partial t}+\sigma_{ky}\cdot\frac{\partial u_x}{\partial x_k}+\sigma_{kx}\cdot\frac{\partial u_y}{\partial x_k}+\frac{\partial}{\partial x_k}\left(u_k\cdot\sigma_{xy}\right)+\rho\cdot V_T^2\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+\\&+\frac{\partial}{\partial x_k}\left(\rho\cdot\int_{\vec V}(V_k-u_k)\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot f\cdot d^3\vec V\right)=\rho\cdot\int_{\vec V}(V_x-u_x)\cdot(V_y-u_y)\cdot St(f)\cdot d^3\vec V\,.\end{aligned} \tag{18}$$

$$\begin{aligned}&\frac{\partial\sigma_{xx}}{\partial t}-\frac{1}{3}\cdot\rho\cdot V_T^2\cdot\frac{\partial u_i}{\partial x_i}-\frac{2}{3}\cdot\frac{\partial q_i}{\partial x_i}-\frac{2}{3}\cdot\sigma_{ij}\cdot\frac{\partial u_i}{\partial x_i}+\frac{\partial}{\partial x_k}\left(u_k\cdot\sigma_{xx}\right)+2\cdot\sigma_{xk}\cdot\frac{\partial u_x}{\partial x_k}+\rho\cdot V_T^2\cdot\frac{\partial u_x}{\partial x}+\\&+\frac{\partial}{\partial x_k}\left(\rho\cdot\int_{\vec V}\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot(V_k-u_k)\cdot f\cdot d^3\vec V\right)=\rho\cdot\int_{\vec V}\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot St(f)\cdot d^3\vec V\,.\end{aligned} \tag{19}$$

where $\sigma_{ij} = \sigma_{ji}$, $(\vec{V} - \vec{u})^2 = (V_x - u_x)^2 + (V_y - u_y)^2 + (V_z - u_z)^2$, and

$$\sigma_{zz} = \rho \cdot \int_{\vec{V}} f \cdot (V_z - u_z)^2 \cdot d^3\vec{V} - \frac{\rho \cdot V_T^2}{2} = -\sigma_{yy} - \sigma_{zz} . \tag{20}$$

Changing the order of indexes (x, y, z) to (x, z, y) in Eq. (18) we obtain an equation for σ_{xz},

$$\begin{aligned} &\frac{\partial \sigma_{xz}}{\partial t} + \sigma_{kz} \cdot \frac{\partial u_x}{\partial x_k} + \sigma_{kx} \cdot \frac{\partial u_z}{\partial x_k} + \frac{\partial}{\partial x_k}\left(u_k \cdot \sigma_{xz}\right) + \rho \cdot V_T^2 \cdot \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \\ &\quad + \frac{\partial}{\partial x_k}\left(\rho \cdot \int_{\vec{V}} (V_k - u_k) \cdot (V_x - u_x) \cdot (V_z - u_z) \cdot f \cdot d^3\vec{V}\right) = \rho \cdot \int_{\vec{V}} (V_x - u_x) \cdot (V_z - u_z) \cdot St(f) \cdot d^3\vec{V} . \end{aligned} \tag{21}$$

and, proceeding similarly with (z, y, x) we obtain an equation for σ_{yz},

$$\begin{aligned} &\frac{\partial \sigma_{zy}}{\partial t} + \sigma_{ky} \cdot \frac{\partial u_z}{\partial x_k} + \sigma_{kz} \cdot \frac{\partial u_y}{\partial x_k} + \frac{\partial}{\partial x_k}\left(u_k \cdot \sigma_{zy}\right) + \rho \cdot V_T^2 \cdot \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_z}{\partial z}\right) + \\ &\quad + \frac{\partial}{\partial x_k}\left(\rho \cdot \int_{\vec{V}} (V_k - u_k) \cdot (V_z - u_x) \cdot (V_y - u_y) \cdot f \cdot d^3\vec{V}\right) = \rho \cdot \int_{\vec{V}} (V_z - u_z) \cdot (V_y - u_y) \cdot St(f) \cdot d^3\vec{V} . \end{aligned} \tag{22}$$

Changing the order of indexes (x, y, z) to (y, x, z) in Eq. (19) we obtain an equation for σ_{yy}

$$\begin{aligned} &\frac{\partial \sigma_{yy}}{\partial t} - \frac{1}{3} \cdot \rho \cdot V_T^2 \cdot \frac{\partial u_i}{\partial x_i} - \frac{2}{3} \cdot \frac{\partial q_i}{\partial x_i} - \frac{2}{3} \cdot \sigma_{ij} \cdot \frac{\partial u_i}{\partial x_i} + \frac{\partial}{\partial x_k}\left(u_k \cdot \sigma_{yy}\right) + 2 \cdot \sigma_{yk} \cdot \frac{\partial u_y}{\partial x_k} + \rho \cdot V_T^2 \cdot \frac{\partial u_y}{\partial y} + \\ &\quad + \frac{\partial}{\partial x_k}\left(\rho \cdot \int_{\vec{V}} \left((V_y - u_y)^2 - \frac{V_T^2}{2}\right) \cdot (V_k - u_k) \cdot f \cdot d^3\vec{V}\right) = \rho \cdot \int_{\vec{V}} \left((V_y - u_y)^2 - \frac{V_T^2}{2}\right) \cdot St(f) \cdot d^3\vec{V} . \end{aligned} \tag{23}$$

It is worth noting that there are no collision terms in the moment equations for ρ, u_i, and V_T^2, Eqs. (14) – (16), while the equations for heat flux and stress tensor, Eqs. (17) - (19) and (21) – (23) do include collision terms. Since collision terms can produce new moments, the set of these 13 moment equations in general is not self-contained. However, in the case of Maxwell molecules and the BGK approximation of the collision term,

$$St(f) = \frac{f_M - f}{\tau} , \tag{24}$$

where τ is a collision time depending on time and coordinates and f_M is the Maxwellian velocity distribution function,

$$f_M = \left(\frac{1}{\pi \cdot V_T^2} \right)^{3/2} \cdot \exp\left(- \frac{(V_x - u_x)^2 + (V_y - u_y)^2 + (V_z - u_z)^2}{V_T^2} \right), \tag{25}$$

the collision terms do not produce any new moments. In other words, these approximations of the collision term *do not mix the moments*. This is a key point for any theory of moment approximation of the Boltzmann equation. In this paper we will use the BGK approximation of the collision term; the case of Maxwellian molecules can be described in a similar way.

III. HERMITE POLYNOMIAL APPROXIMATION OF FUNCTION DISTRIBUTION

Following Grad [1, 2] we assume here a Hermite polynomial approximation of the velocity distribution function where Hermite polynomials are described as follows:

$$H_N(\chi) = (-1)^N \cdot \exp\left(\chi^2\right) \cdot \frac{d^N}{d\chi^N} \left[\exp\left(-\chi^2\right) \right], \tag{26}$$

$$\int_{-\infty}^{+\infty} H_N(\chi) \cdot H_M(\chi) \cdot \exp\left(-\chi^2\right) \cdot d\chi = \begin{cases} 0 \ \text{ if } N \neq M \\ \sqrt{\pi} \cdot N! \cdot 2^N \ \text{ if } N = M \end{cases} . \tag{27}$$

The velocity distribution function can be described as a combination of three-dimensional Hermite polynomials that correspond to x, y, z directions of the velocity. For our purposes we need only the following set of Hermite polynomials:

$$H_0 = 1, \qquad H_{1i}(\chi_i) = 2 \cdot \chi_i, \qquad H_{2i}(\chi_i) = 4 \cdot \chi_i^2 - 2, \tag{28}$$

$$H_{3i}(\chi) = 8 \cdot \chi_i^3 - 12 \cdot \chi_i, \qquad H_{4i}(\chi) = 16 \cdot \chi_i^4 - 48 \cdot \chi_i^2 + 12, \tag{29}$$

where $i = x, y, z$ and

$$\chi_i = \frac{V_i - u_i}{V_T} . \tag{30}$$

We represent the velocity distribution via the 29 Hermite polynomials

$$f_H = f_M \cdot \sum_{k=1}^{29} \Lambda_k \cdot \hat{H}_k \left(\chi_x, \chi_y, \chi_z \right), \tag{31}$$

where f_M is a Maxwellian function, Eq. (25), coefficients Λ depend on coordinates and time, and Hermite polynomials $\hat{H}$ are

$$\hat{H}_1 = H_0 = 1, \tag{32}$$

$$\hat{H}_2 = \frac{1}{2} \cdot H_{1x} = \frac{V_x - u_x}{V_T}, \quad \hat{H}_3 = \frac{1}{2} \cdot H_{1y} = \frac{V_y - u_y}{V_T}, \quad \hat{H}_4 = \frac{1}{2} \cdot H_{1y} = \frac{V_y - u_y}{V_T} , \tag{33}$$

$$\hat{H}_5 = \frac{1}{4} \cdot H_{2x} = \left(\frac{V_x - u_x}{V_T} \right)^2 - \frac{1}{2}, \tag{34}$$

$$\hat{H}_6 = \frac{1}{4} \cdot H_{1x} \cdot H_{1y} = \left(\frac{V_x - u_x}{V_T} \right) \cdot \left(\frac{V_y - u_y}{V_T} \right), \tag{35}$$

$$\hat{H}_7 = \frac{1}{4} \cdot H_{1x} \cdot H_{1z} = \left(\frac{V_x - u_x}{V_T} \right) \cdot \left(\frac{V_z - u_z}{V_T} \right), \tag{36}$$

$$\hat{H}_8 = \frac{1}{4} \cdot H_{2y} = \left(\frac{V_y - u_y}{V_T} \right)^2 - \frac{1}{2}, \tag{37}$$

$$\hat{H}_9 = \frac{1}{4} \cdot H_{1y} \cdot H_{1z} = \left(\frac{V_y - u_y}{V_T} \right) \cdot \left(\frac{V_z - u_z}{V_T} \right) , \tag{38}$$

$$\hat{H}_{10} = \frac{1}{4} \cdot H_{2z} = \left(\frac{V_z - u_z}{V_T} \right)^2 - \frac{1}{2}, \tag{39}$$

$$\hat{H}_{11} = \frac{1}{8} \cdot \left(H_{3x} + H_{1x} \cdot H_{2y} + H_{1x} \cdot H_{2z} \right) = \left(\frac{V_x - u_x}{V_T} \right) \cdot \left[\left(\frac{V_x - u_x}{V_T} \right)^2 + \left(\frac{V_y - u_y}{V_T} \right)^2 + \left(\frac{V_z - u_z}{V_T} \right)^2 - \frac{5}{2} \right], \tag{40}$$

$$\hat{H}_{12} = \frac{1}{8} \cdot \left(H_{3y} + H_{1y} \cdot H_{2x} + H_{1y} \cdot H_{2z} \right) = \left(\frac{V_y - u_y}{V_T} \right) \cdot \left[\left(\frac{V_y - u_y}{V_T} \right)^2 + \left(\frac{V_x - u_x}{V_T} \right)^2 + \left(\frac{V_z - u_z}{V_T} \right)^2 - \frac{5}{2} \right], \tag{41}$$

$$\hat{H}_{13} = \frac{1}{8} \cdot \left(H_{3z} + H_{1z} \cdot H_{2x} + H_{1z} \cdot H_{2y} \right) = \left(\frac{V_z - u_z}{V_T} \right) \cdot \left[\left(\frac{V_z - u_z}{V_T} \right)^2 + \left(\frac{V_x - u_x}{V_T} \right)^2 + \left(\frac{V_y - u_y}{V_T} \right)^2 - \frac{5}{2} \right]. \tag{42}$$

and

$$\hat{H}_{14} = H_{4x}, \qquad \hat{H}_{15} = H_{4y}, \qquad \hat{H}_{16} = H_{4z}, \tag{43}$$

$$\hat{H}_{17} = H_{2x} \cdot H_{2y}, \qquad \hat{H}_{18} = H_{2x} \cdot H_{2z}, \qquad \hat{H}_{19} = H_{2y} \cdot H_{2z}, \tag{44}$$

$$\hat{H}_{20} = H_{3x} \cdot H_{1y}, \qquad \hat{H}_{21} = H_{3x} \cdot H_{1z}, \qquad \hat{H}_{22} = H_{3y} \cdot H_{1x}, \qquad \hat{H}_{23} = H_{3y} \cdot H_{1z},$$
$$\hat{H}_{24} = H_{3z} \cdot H_{1x}, \qquad \hat{H}_{25} = H_{3z} \cdot H_{1y}, \tag{45}$$

$$\hat{H}_{26} = H_{1x} \cdot H_{1y} \cdot H_{2z}, \qquad \hat{H}_{27} = H_{1y} \cdot H_{1z} \cdot H_{2x}, \qquad \hat{H}_{28} = H_{1z} \cdot H_{1x} \cdot H_{2z}, \tag{46}$$

$$\hat{H}_{29} = H_{1x} \cdot H_{1y} \cdot H_{1z}. \tag{47}$$

It should be stressed that all $\hat{H}$ polynomials are orthogonal, i.e.,

$$\int_{-\infty}^{+\infty} \hat{H}_i \cdot \hat{H}_j \cdot \exp\left(-\chi_x^2 - \chi_y^2 - \chi_z^2 \right) \cdot d\chi = 0, \text{ where } i \neq j. \tag{48}$$

Since the velocity distribution function f_H has to satisfy the conditions given by Eqs. (2), (4), and (5), we obtain that

$$\Lambda_1 = 1, \qquad \Lambda_2 = \Lambda_3 = \Lambda_4 = 0 \qquad \text{and} \qquad \Lambda_5 + \Lambda_8 + \Lambda_{10} = 0. \tag{49}$$

Thus, the particle distribution function $n \cdot f_H$ has 29 variables, ρ, u_x, u_y, u_z, V_T^2, Λ_5, Λ_6, Λ_7, Λ_8, Λ_9, Λ_{11} - Λ_{29}. Substituting the velocity distribution function f_H, Eq. (31), for f in Eqs. (6) and (7), we obtain relationships between q_i, σ_{ij} and Λ:

$$\Lambda_5 = \frac{2 \cdot \sigma_{xx}}{\rho \cdot V_T^2}, \quad \Lambda_6 = \frac{4 \cdot \sigma_{xy}}{\rho \cdot V_T^2}, \quad \Lambda_7 = \frac{4 \cdot \sigma_{xz}}{\rho \cdot V_T^2}, \quad \Lambda_8 = \frac{2 \cdot \sigma_{yy}}{\rho \cdot V_T^2}, \quad \Lambda_9 = \frac{4 \cdot \sigma_{yz}}{\rho \cdot V_T^2},$$
$$\Lambda_{11} = \frac{8 \cdot q_x}{5 \cdot \rho \cdot V_T^3}, \quad \Lambda_{12} = \frac{8 \cdot q_y}{5 \cdot \rho \cdot V_T^3}, \quad \Lambda_{13} = \frac{8 \cdot q_z}{5 \cdot \rho \cdot V_T^3}. \tag{50}$$

It is worth noting that the truncated velocity distribution function f_H that consists of the first nine nonzero Hermite polynomials has the form of the Champmen-Enskog and Grad's velocity distribution functions [1, 2, 7]. In the next sections it will be shown why we have selected this representation of velocity distribution function.

IV. A CLOSURE OF GRAD'S 13 MOMENT EQUATIONS.

Let us rewrite the equation for the heat flux, Eq. (17), and stress tensor, Eqs. (18) – (19), (21) – (23) for the case of the BGK collision term, Eq. (24):

$$\begin{aligned}
&\frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k \cdot q_i\right) + q_k \cdot \frac{\partial u_i}{\partial x_k} - \left(\frac{5}{2} \cdot \frac{V_T^2}{2}\right) \cdot \left(\frac{\partial}{\partial x_i}\left(\frac{\rho \cdot V_T^2}{2}\right) + \frac{\partial \sigma_{ji}}{\partial x_j}\right) - \frac{\sigma_{ik}}{\rho} \cdot \left(\frac{\partial}{\partial x_k}\left(\frac{\rho \cdot V_T^2}{2}\right) + \frac{\partial \sigma_{jk}}{\partial x_j}\right) + \\
&\quad + \frac{\partial}{\partial x_k}\left(\frac{\rho}{2} \cdot V_T^4 \cdot \int_{\vec{V}} \frac{(V_i - u_i)}{V_T} \cdot \frac{(\vec{V} - \vec{u})^2}{V_T^2} \cdot \frac{(V_k - u_k)}{V_T} \cdot f \cdot d^3\vec{V}\right) + \\
&\qquad + \left(\rho \cdot V_T^3 \cdot \int_{\vec{V}} \frac{(V_i - u_i)}{V_T} \cdot \frac{(V_k - u_k)}{V_T} \cdot \frac{(V_j - u_j)}{V_T} \cdot f \cdot d^3\vec{V}\right) \cdot \frac{\partial u_j}{\partial x_k} = -\frac{q_i}{\tau},
\end{aligned} \tag{51}$$

$$\begin{aligned}
&\frac{\partial \sigma_{xy}}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k \cdot \sigma_{xy}\right) + \sigma_{ky} \cdot \frac{\partial u_x}{\partial x_k} + \sigma_{kx} \cdot \frac{\partial u_y}{\partial x_k} + \frac{\rho \cdot V_T^2}{2} \cdot \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \\
&\quad + \frac{\partial}{\partial x_k}\left[\rho \cdot V_T^3 \cdot \int_{\vec{V}} \frac{(V_k - u_k)}{V_T} \cdot \frac{(V_x - u_x)}{V_T} \cdot \frac{(V_y - u_y)}{V_T} \cdot f \cdot d^3\vec{V}\right] = -\frac{\sigma_{xy}}{\tau},
\end{aligned} \tag{52}$$

$$\frac{\partial \sigma_{xz}}{\partial t}+\frac{\partial}{\partial x_k}\left(u_k\cdot\sigma_{xz}\right)+\sigma_{kz}\cdot\frac{\partial u_x}{\partial x_k}+\left(\sigma_{kx}\right)\cdot\frac{\partial u_z}{\partial x_k}+\frac{\rho\cdot V_T^2}{2}\cdot\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)= \\ +\frac{\partial}{\partial x_k}\left[\rho\cdot V_T^3\cdot\int_{\vec{V}}\frac{(V_k-u_k)}{V_T}\cdot\frac{(V_x-u_x)}{V_T}\cdot\frac{(V_z-u_z)}{V_T}\cdot f\cdot d^3\vec{V}\right]=-\frac{\sigma_{xz}}{\tau}\ , \tag{53}$$

$$\frac{\partial \sigma_{zy}}{\partial t}+\frac{\partial}{\partial x_k}\left(u_k\cdot\sigma_{zy}\right)+\sigma_{ky}\cdot\frac{\partial u_z}{\partial x_k}+\sigma_{kz}\cdot\frac{\partial u_y}{\partial x_k}+\frac{\rho\cdot V_T^2}{2}\cdot\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right)+ \\ +\frac{\partial}{\partial x_k}\left[\rho\cdot V_T^3\cdot\int_{\vec{V}}\frac{(V_k-u_k)}{V_T}\cdot\frac{(V_z-u_z)}{V_T}\cdot\frac{(V_y-u_y)}{V_T}\cdot f\cdot d^3\vec{V}\right]=-\frac{\sigma_{zy}}{\tau}\ , \tag{54}$$

$$\frac{\partial \sigma_{xx}}{\partial t}-\frac{2}{3}\cdot\frac{\partial q_i}{\partial x_i}+\frac{\partial}{\partial x_k}\left(u_k\cdot\sigma_{xx}\right)+2\cdot\sigma_{xk}\cdot\frac{\partial u_x}{\partial x_k}-\frac{2}{3}\cdot\sigma_{ij}\cdot\frac{\partial u_i}{\partial x_j}+\rho\cdot V_T^2\cdot\left(\frac{\partial u_x}{\partial x}-\frac{1}{3}\cdot\frac{\partial u_i}{\partial x_i}\right)+ \\ +\frac{\partial}{\partial x_k}\left(\rho\cdot V_T^3\cdot\int_{\vec{V}}\left(\frac{(V_x-u_x)^2}{V_T^2}-\frac{1}{2}\right)\cdot\frac{(V_k-u_k)}{V_T}\cdot f\cdot d^3\vec{V}\right)=-\frac{\sigma_{xx}}{\tau}\ , \tag{55}$$

$$\frac{\partial \sigma_{yy}}{\partial t}-\frac{2}{3}\cdot\frac{\partial q_i}{\partial x_i}+\frac{\partial}{\partial x_k}\left(u_k\cdot\sigma_{yy}\right)+2\cdot\sigma_{yk}\cdot\frac{\partial u_y}{\partial x_k}-\frac{2}{3}\cdot\sigma_{ij}\cdot\frac{\partial u_i}{\partial x_i}+\rho\cdot V_T^2\cdot\left(\frac{\partial u_y}{\partial y}--\frac{1}{3}\cdot\frac{\partial u_i}{\partial x_i}\right)+ \\ +\frac{\partial}{\partial x_k}\left(\rho\cdot V_T^3\cdot\int_{\vec{V}}\left(\frac{(V_y-u_y)^2}{V_T^2}-\frac{1}{2}\right)\cdot\frac{(V_k-u_k)}{V_T}\cdot f\cdot d^3\vec{V}\right)=-\frac{\sigma_{yy}}{\tau}, \tag{56}$$

where indexes $i,k=x,y,z$. To complete this system of equations, a velocity distribution function has to be chosen. Grad [1, 2] has suggested the following velocity distribution function:

$$f_{GRAD}(\vec{V})=f_M\left[1+q_k\cdot\frac{8}{5\cdot\rho\cdot V_T^3}\cdot\frac{(V_k-u_k)}{V_T}\cdot\left(\frac{(\vec{V}-\vec{u})^2}{V_T^2}-\frac{5}{2}\right)+\sigma_{ij}\cdot\frac{2}{\rho\cdot V_T^2}\cdot\left(\frac{(V_i-u_i)\cdot(V_i-u_i)}{V_T^2}-\delta_{ij}\cdot\frac{1}{2}\right)\right] \tag{57}$$

and obtained his set of equations that later has been called "Grad's 13 moment equations." As one can see Grad's velocity distribution function can be rewritten in terms of Hermite polynomials as

$$f_{GRAD}=f_M\cdot\left[\hat{H}_0+\frac{8\cdot q_x}{5\cdot\rho\cdot V_T^3}\cdot\hat{H}_{11}+\frac{8\cdot q_y}{5\cdot\rho\cdot V_T^3}\cdot\hat{H}_{12}+\frac{8\cdot q_z}{5\cdot\rho\cdot V_T^3}\cdot\hat{H}_{13}+\frac{2\cdot\sigma_{xx}}{\rho\cdot V_T^2}\cdot\hat{H}_5+\frac{4\cdot\sigma_{xy}}{\rho\cdot V_T^2}\cdot\hat{H}_6+\right. \\ \left.+\frac{4\cdot\sigma_{xz}}{\rho\cdot V_T^2}\cdot\hat{H}_7+\frac{2\cdot\sigma_{yy}}{\rho\cdot V_T^2}\cdot\hat{H}_8+\frac{4\cdot\sigma_{xz}}{\rho\cdot V_T^2}\cdot\hat{H}_9-\frac{2\cdot(\sigma_{xx}+\sigma_{yy})}{\rho\cdot V_T^2}\cdot\hat{H}_{10}\right], \tag{58}$$

where we have taken into account that $\sigma_{ij} = \sigma_{ji}$ and used Eq. (20). Grad's equations [1,2] for q_x, σ_{xx}, and σ_{xy} can be presented in the following form:

$$\begin{aligned}
&\frac{\partial q_x}{\partial t} + \frac{\partial}{\partial x_k}(u_k \cdot q_x) + q_k \cdot \frac{\partial u_x}{\partial x_k} - \left(\frac{5}{8}\cdot V_T^2\right)\cdot \frac{\partial}{\partial x}\left(\rho \cdot V_T^2\right) - \left(\frac{5}{4}\cdot V_T^2\right)\cdot \frac{\partial \sigma_{jx}}{\partial x_j} - \left(\frac{1}{2}\cdot \frac{\sigma_{xk}}{\rho}\right)\cdot \frac{\partial}{\partial x_k}\left(\rho \cdot V_T^2\right) - \frac{\sigma_{xk}}{\rho}\cdot \frac{\partial \sigma_{jk}}{\partial x_j} + \\
&\quad + \frac{\partial}{\partial x}\left(\frac{5}{8}\cdot \rho \cdot V_T^4 + \frac{7}{4}\cdot V_T^2 \cdot \sigma_{xx}\right) + \frac{\partial}{\partial y}\left(\frac{7}{4}\cdot V_T^2 \cdot \sigma_{xy}\right) + \frac{\partial}{\partial z}\left(\frac{7}{4}\cdot V_T^2 \cdot \sigma_{xz}\right) + \left(\frac{6}{5}\cdot q_x\right)\cdot \frac{\partial u_x}{\partial x} + \left(\frac{2}{5}\cdot q_y\right)\cdot \frac{\partial u_y}{\partial x} \\
&\quad + \left(\frac{2}{5}\cdot q_y\right)\cdot \frac{\partial u_x}{\partial y} + \left(\frac{2}{5}\cdot q_z\right)\cdot \frac{\partial u_z}{\partial x} + \left(\frac{2}{5}\cdot q_z\right)\cdot \frac{\partial u_x}{\partial z} + \left(\frac{2}{5}\cdot q_x\right)\cdot \frac{\partial u_y}{\partial y} + \left(\frac{2}{5}\cdot q_x\right)\cdot \frac{\partial u_z}{\partial z} = -\frac{q_x}{\tau} ,
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial t} + \frac{\partial}{\partial x_k}(u_k \cdot \sigma_{xx}) + (2\cdot \sigma_{xx})\cdot \frac{\partial u_x}{\partial x} + (2\cdot \sigma_{yx})\cdot \frac{\partial u_x}{\partial y} + (2\cdot \sigma_{zx})\cdot \frac{\partial u_x}{\partial z} - \left(\frac{2}{3}\cdot \sigma_{kl}\right)\cdot \frac{\partial u_l}{\partial x_k} + \\
&\quad + \left(\frac{2\cdot \rho \cdot V_T^2}{3}\right)\cdot \frac{\partial u_x}{\partial x_x} - \left(\frac{\rho \cdot V_T^2}{3}\right)\cdot \frac{\partial u_y}{\partial y} - \left(\frac{\rho \cdot V_T^2}{3}\right)\cdot \frac{\partial u_z}{\partial z} + \frac{8}{15}\cdot \frac{\partial q_x}{\partial x} - \frac{4}{15}\cdot \frac{\partial q_y}{\partial y} - \frac{4}{15}\cdot \frac{\partial q_z}{\partial z} = -\frac{\sigma_{xx}}{\tau} ,
\end{aligned} \tag{60}$$

$$\frac{\partial \sigma_{xy}}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k \cdot \sigma_{ixy}\right) + \sigma_{ky} \cdot \frac{\partial u_x}{\partial x_k} + \left(\frac{\rho \cdot V_T^2}{2}\right)\cdot \frac{\partial u_x}{\partial y} + \sigma_{kx} \cdot \frac{\partial u_y}{\partial x_k} + \left(\frac{\rho \cdot V_T^2}{2}\right)\cdot \frac{\partial u_y}{\partial x} + \frac{2}{5}\cdot \frac{\partial q_y}{\partial x} + \frac{2}{5}\cdot \frac{\partial q_x}{\partial y} = -\frac{\sigma_{xy}}{\tau} . \tag{61}$$

Changing the order of indexes (x, y, z) to (y, x, z) and (z, y, x) in Eq. (59) we obtain the corresponding equations for q_y and q_z; likewise, using *(y,x,z)* in Eq. (60) we obtain an equation for σ_{yy}; and finally, using (x, z, y) and (y, z, x) in Eq. (61) we obtain equations for σ_{xz} and σ_{yz} respectively. Unfortunately, as it has been mentioned in Introduction, Grad's moment equations sometimes produce unphysical solutions and, therefore need to be regularized.

Let us obtain a set of the 13 regularized Grad's moment equations using the Hermite polynomial approximation of the velocity distribution function and the Chapmen-Enskog closure method, while assuming that the collision term is in BGK form, Eq. (24).

First let us represent the polynomials in Eq. (51) with index $i = x$ and in Eqs. (52), Eq. (55) in Hyrmite forms:

$$\chi_x^2 \cdot \left(\chi_x^2 + \chi_y^2 + \chi_z^2\right) = \frac{H_{4x}}{16} + \frac{H_{2x}\cdot H_{2y}}{4} + \frac{H_{2x}\cdot H_{2z}}{4} + \frac{7\cdot H_{2x}}{4} + \frac{H_{2z}}{2} + \frac{H_{2y}}{2} + \frac{H_{2z}}{2} + \frac{5\cdot H_0}{4} , \tag{62}$$

$$\chi_x \cdot \chi_y \cdot \left(\chi_x^2 + \chi_y^2 + \chi_z^2\right) = \frac{H_{3x}\cdot H_{1y}}{16} + \frac{H_{3y}\cdot H_{1x}}{16} + \frac{H_{1x}\cdot H_{1y}\cdot H_{2z}}{16} + \frac{7\cdot H_{1x}\cdot H_{1y}}{8} , \tag{63}$$

$$\chi_x \cdot \chi_z \cdot \left(\chi_x^2 + \chi_y^2 + \chi_z^2\right) = \frac{H_{3x} \cdot H_{1z}}{16} + \frac{H_{3z} \cdot H_{1x}}{16} + \frac{H_{1x} \cdot H_{1z} \cdot H_{2y}}{16} + \frac{7 \cdot H_{1x} \cdot H_{1z}}{8}, \tag{64}$$

$$\chi_x^3 = \frac{H_{3x}}{8} + \frac{3 \cdot H_{1x}}{4}, \tag{65}$$

$$\chi_x^2 \cdot \chi_y = \frac{H_{2x} \cdot H_{1y}}{8} + \frac{1 \cdot H_{1y}}{8}, \tag{66}$$

$$\chi_x^2 \cdot \chi_z = \frac{H_{2x} \cdot H_{1z}}{8} + \frac{1 \cdot H_{1z}}{8}, \tag{67}$$

$$\chi_x \cdot \chi_y \cdot \chi_z = \frac{H_{1x} \cdot H_{1y} \cdot H_{1z}}{8}, \tag{68}$$

$$\chi_x \cdot \chi_y^2 = \frac{H_{2y} \cdot H_{1x}}{8} + \frac{1 \cdot H_{1x}}{8}, \tag{69}$$

$$\left(\chi_x^2 - \frac{1}{2}\right) \cdot \chi_y = \frac{H_{2x} \cdot H_{1y}}{8}, \tag{70}$$

$$\left(\chi_x^2 - \frac{1}{2}\right) \cdot \chi_x = \frac{H_{3x}}{8} + \frac{H_{1x}}{2}, \tag{71}$$

$$\left(\chi_x^2 - \frac{1}{2}\right) \cdot \chi_z = \frac{H_{2x} \cdot H_{1z}}{8}, \tag{72}$$

where variables χ are given by Eq. (29). As one can see, the polynomials presented in: Eqs. (62) – (68) are included in q_x; Eqs. (66), (68) and (69) in σ_{xy}; and Eq. (70) – (72) in σ_{xx}. The Hermite representations of polynomials in Eq. 51 with index $i = y, z$, and in Eqs. (53), (54), (56) for q_y, q_z, σ_{xz}, σ_{yz}, and σ_{yy} respectively is obtained from Eqs. (62) – (72) by the proper rotation of the indexes. Thus, the complete list of Hermite polynomials that represents polynomials in Eqs. (51) – (56) is

$$H_{4x}, \qquad H_{4y}, \qquad H_{4z}, \tag{73}$$

$$H_{2x} \cdot H_{2y}, \qquad H_{2x} \cdot H_{2z}, \qquad H_{2y} \cdot H_{2z}, \tag{74}$$

$$H_{3x} \cdot H_{1y}, \qquad H_{3x} \cdot H_{1z}, \qquad H_{3y} \cdot H_{1x}, \qquad H_{3y} \cdot H_{1z}, \qquad H_{3z} \cdot H_{1x}, \qquad H_{3z} \cdot H_{1y}, \tag{75}$$

$$H_{1x} \cdot H_{1y} \cdot H_{2z}, \quad H_{1x} \cdot H_{1z} \cdot H_{2y}, \quad H_{1z} \cdot H_{1y} \cdot H_{2x}, \tag{76}$$

$$H_{1x} \cdot H_{1y} \cdot H_{1z}, \tag{77}$$

$$H_{3x}, \qquad H_{3y}, \qquad H_{3z}, \tag{78}$$

$$H_{2x} \cdot H_{1y}, \qquad H_{2x} \cdot H_{1z}, \qquad H_{2y} \cdot H_{1x}, \qquad H_{2y} \cdot H_{1z}, \qquad H_{2z} \cdot H_{1x}, \qquad H_{2z} \cdot H_{1y}, \tag{79}$$

$$H_{2x}, \qquad H_{2y}, \qquad H_{2z}, \tag{80}$$

$$H_{1x} \cdot H_{1y}, \qquad H_{1x} \cdot H_{1z}, \qquad H_{1y} \cdot H_{1z}, \tag{81}$$

$$H_{1x}, \qquad H_{1y}, \qquad H_{1z}, \tag{82}$$

$$H_0 . \tag{83}$$

Now, following the Chapman-Enskog method [5, 6], let us write the velocity distribution function as

$$f = f_{GRAD} + \tau \cdot f_M \cdot f_1 \quad . \tag{84}$$

where f_1 has a Hermite polynomial form. Since the velocity distribution function $\tau \cdot f_M \cdot f_1$ does not have to contribute into the previously obtained 13 moments ρ, u_i, and V_T^2, q_i, and σ_{ij}, it follows that the Hermite polynomials included in Grad's velocity distribution function, Eq. (58), have to be excluded from f_1. But as the velocity distribution function $\tau \cdot f_M \cdot f_1$ has to contribute into the integrals in Eqs. (51) – (56), we obtain that f_1, as follows from Eqs. (62) – (72), has to be a combination only of the Hermite polynomials presented in Eqs. (73) – (77). Subsequently, we obtain that

$$f_1 = \sum_{i=14}^{29} \Lambda \cdot \hat{H}_i , \tag{85}$$

where functions $\hat{H}_i$ are shown in Eqs. (43) – (47). Thus, we have shown that the chosen set of Hermite polynomials, Eq. (32) – (47) has good physical sense for representing the velocity distribution function.

Let us introduce the following set of M-moments:

$$M_{4i} = V_T^4 \cdot \rho \cdot \int_{\vec{V}} f \cdot H_{4i} \cdot d^3\vec{V} \,, \qquad i = (x, y, z) \,, \tag{86}$$

$$M_{2i2j} = V_T^4 \cdot \rho \cdot \int_{\vec{V}} f \cdot H_{2i} \cdot H_{2j} \cdot d^3\vec{V} \,, \qquad ij = (xy, xz, yz) \,, \tag{87}$$

$$M_{3i1j} = V_T^4 \cdot \rho \cdot \int_{\vec{V}} f \cdot H_{3i} \cdot H_{1j} \cdot d^3\vec{V} \,, \qquad ij = (xy, xz, yx, yz, zx, zy) \,, \tag{88}$$

$$M_{1i1j2k} = V_T^4 \cdot \rho \cdot \int_{\vec{V}} f \cdot H_{1i} \cdot H_{1j} \cdot H_{2k} \cdot d^3\vec{V} \,, \qquad ijk = (xyz, yzx, zxy) \,, \tag{89}$$

$$M_{1i1j1k} = V_T^3 \cdot \rho \cdot \int_{\vec{V}} f \cdot H_{1i} \cdot H_{1j} \cdot H_{1k} \cdot d^3\vec{V} \,, \qquad ijk = (xyz) \,, \tag{90}$$

which correspond to Hermite polynomials in Eqs. (73) – (77). Substituting them into Eq. (51) with index $i = x$ and into Eqs. (52) and (55) we obtain the following equations for q_x, σ_{xy}, and σ_{xx}:

$$\begin{aligned}
&\frac{\partial q_x}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k \cdot q_x\right) + q_k \cdot \frac{\partial u_x}{\partial x_k} - \left(\frac{5}{2}\cdot\frac{V_T^2}{2}\right)\cdot\left(\frac{\partial}{\partial x}\left(\frac{\rho \cdot V_T^2}{2}\right) + \frac{\partial \sigma_{jx}}{\partial x_j}\right) - \frac{\sigma_{xk}}{\rho}\cdot\left(\frac{\partial}{\partial x_k}\left(\frac{\rho\cdot V_T^2}{2}\right) + \frac{\partial \sigma_{jk}}{\partial x_j}\right) + \\
&\quad + \frac{\partial}{\partial x}\left(\frac{5}{8}\cdot\rho\cdot V_T^4 + \frac{7}{4}\cdot V_T^2\cdot\sigma_{xx}\right) + \frac{\partial}{\partial y}\left(\frac{7}{4}\cdot V_T^2\cdot\sigma_{xy}\right) + \frac{\partial}{\partial z}\left(\frac{7}{4}\cdot V_T^2\cdot\sigma_{xz}\right) + \left(\frac{6}{5}\cdot q_x\right)\cdot\frac{\partial u_x}{\partial x} + \left(\frac{2}{5}\cdot q_y\right)\cdot\frac{\partial u_y}{\partial x} + \\
&\quad + \left(\frac{2}{5}\cdot q_y\right)\cdot\frac{\partial u_x}{\partial y} + \left(\frac{2}{5}\cdot q_z\right)\cdot\frac{\partial u_z}{\partial x} + \left(\frac{2}{5}\cdot q_z\right)\cdot\frac{\partial u_x}{\partial z} + \left(\frac{2}{5}\cdot q_x\right)\cdot\frac{\partial u_y}{\partial y} + \left(\frac{2}{5}\cdot q_x\right)\cdot\frac{\partial u_z}{\partial z} + \\
&\quad + \frac{\partial}{\partial x}\left(\frac{1}{32}\cdot M_{4x} + \frac{1}{8}\cdot M_{2x2y} + \frac{1}{8}\cdot M_{2x2z}\right) + \frac{\partial}{\partial y}\left(\frac{1}{32}\cdot M_{3x1y} + \frac{1}{32}\cdot M_{1x3y} + \frac{1}{32}\cdot M_{1x1y2z}\right) + \\
&\quad + \frac{\partial}{\partial z}\left(\frac{1}{32}\cdot M_{3x1z} + \frac{1}{32}\cdot M_{1x3z} + \frac{1}{32}\cdot M_{1x1z2y}\right) + \left(\frac{1}{8}\cdot M_{1x1y1z}\right)\cdot\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = -\frac{q_x}{\tau} \,,
\end{aligned} \tag{91}$$

$$\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k \cdot \sigma_{xx}\right) + \left(2\cdot\sigma_{xx}\right)\cdot\frac{\partial u_x}{\partial x} + \left(2\cdot\sigma_{yx}\right)\cdot\frac{\partial u_x}{\partial y} + \left(2\cdot\sigma_{zx}\right)\cdot\frac{\partial u_x}{\partial z} - \left(\frac{2}{3}\cdot\sigma_{kl}\right)\cdot\frac{\partial u_l}{\partial x_k} + \\
&\quad + \left(\frac{2\cdot\rho\cdot V_T^2}{3}\right)\cdot\frac{\partial u_x}{\partial x_x} - \left(\frac{\rho\cdot V_T^2}{3}\right)\cdot\frac{\partial u_y}{\partial y} - \left(\frac{\rho\cdot V_T^2}{3}\right)\cdot\frac{\partial u_z}{\partial z} + \frac{8}{15}\cdot\frac{\partial q_x}{\partial x} - \frac{4}{15}\cdot\frac{\partial q_y}{\partial y} - \frac{4}{15}\cdot\frac{\partial q_z}{\partial z} = -\frac{\sigma_{xx}}{\tau} \,,
\end{aligned} \tag{92}$$

$$\frac{\partial \sigma_{xy}}{\partial t}+\frac{\partial}{\partial x_k}\left(u_k \cdot \sigma_{xy}\right)+\sigma_{ky}\cdot\frac{\partial u_x}{\partial x_k}+\left(\frac{\rho \cdot V_T^2}{2}\right)\cdot\frac{\partial u_x}{\partial y}+\sigma_{kx}\cdot\frac{\partial u_y}{\partial x_k}+\left(\frac{\rho \cdot V_T^2}{2}\right)\cdot\frac{\partial u_y}{\partial x}+\frac{2}{5}\cdot\frac{\partial q_y}{\partial x}+$$
$$+\frac{2}{5}\cdot\frac{\partial q_x}{\partial y}+\frac{\partial}{\partial z}\left(\frac{1}{8}\cdot M_{1x1y1z}\right)=-\frac{\sigma_{xy}}{\tau}\ . \tag{93}$$

As one can see there is no contribution of new M-moments in the equation for σ_{xy}. The equations for q_y, q_z, σ_{yy}, σ_{xz}, σ_{yz} can be obtained by a proper rotations of indexes in Eqs. (91) – (93). Thus, to obtain equations for the heat flux (q_x, q_y, q_z) and the stress tensor $(\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz})$ we have to derive equations for M-moments.

Let us obtain equations for M_{1x1y1z}. Multiplying the Boltzmann Equation, Eq. (1), by

$$m\cdot 8\cdot (V_x-u_x)\cdot (V_y-u_y)\cdot (V_z-u_z) \tag{94}$$

we obtain

$$8\cdot m\cdot\left[(V_x-u_x)\cdot(V_y-u_y)\cdot(V_z-u_z)\right]\cdot\frac{\partial(n\cdot f)}{\partial t}+8\cdot m\cdot\left[(V_x-u_x)\cdot(V_y-u_y)\cdot(V_z-u_z)\right]\cdot V_k\cdot\frac{\partial(n\cdot f)}{\partial x_k}=$$
$$=8\cdot m\cdot\left[(V_x-u_x)\cdot(V_y-u_y)\cdot(V_z-u_z)\right]\cdot St(n\cdot f)\ . \tag{95}$$

Transferring the terms in front of the derivatives inside of the derivatives brackets in Eq. (95), we obtain

$$\frac{\partial}{\partial t}\left(8\cdot\rho\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot(V_z-u_z)\cdot f\right)+\left(8\cdot\rho\cdot(V_y-u_y)\cdot(V_z-u_z)\cdot f\right)\cdot\frac{\partial u_x}{\partial t}+$$
$$+\left(8\cdot\rho\cdot(V_x-u_x)\cdot(V_z-u_z)\cdot f\right)\cdot\frac{\partial u_y}{\partial t}+\left(8\cdot\rho\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot f\right)\cdot\frac{\partial u_z}{\partial t}+$$
$$+\frac{\partial}{\partial x_k}\left(8\cdot\rho\cdot(V_k-u_k)\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot(V_z-u_z)\cdot f\right)+\frac{\partial}{\partial x_k}\left(u_k\cdot 8\cdot\rho\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot(V_z-u_z)\cdot f\right)+$$
$$+\left(8\cdot\rho\cdot(V_k-u_k)\cdot(V_y-u_y)\cdot(V_z-u_z)\cdot f\right)\cdot\frac{\partial u_x}{\partial x_k}+\left(u_k\cdot 8\cdot\rho\cdot(V_y-u_y)\cdot(V_z-u_z)\cdot f\right)\cdot\frac{\partial u_x}{\partial x_k}+$$
$$+\left(8\cdot\rho\cdot(V_k-u_k)\cdot(V_x-u_x)\cdot(V_z-u_z)\cdot f\right)\cdot\frac{\partial u_y}{\partial x_k}+\left(8\cdot\rho\cdot u_k\cdot(V_x-u_x)\cdot(V_z-u_z)\cdot f\right)\cdot\frac{\partial u_y}{\partial x_k}+$$
$$+\left(8\cdot\rho\cdot(V_k-u_k)\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot f\right)\cdot\frac{\partial u_z}{\partial x_k}+\left(8\cdot\rho\cdot u_k\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot f\right)\cdot\frac{\partial u_z}{\partial x_k}=$$
$$=8\cdot\rho\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot(V_z-u_z)\cdot St(f)\ . \tag{96}$$

Substituting Eqs. (24) and (84) into Eq. (96) and integrating the obtained equation over the entire velocity domain, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(8\cdot\rho\cdot V_T^3\cdot\int_{\vec{\chi}}\chi_x\cdot\chi_y\cdot\chi_z\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)+\left(8\cdot\rho\cdot V_T^2\cdot\int_{\vec{\chi}}\chi_y\cdot\chi_z\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)\cdot\frac{\partial u_x}{\partial t}+\\
&+\left(8\cdot\rho\cdot V_T^2\cdot\int_{\vec{\chi}}\chi_x\cdot\chi_z\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)\cdot\frac{\partial u_y}{\partial t}+\left(8\cdot\rho\cdot V_T^2\cdot\int_{\vec{\chi}}\chi_x\cdot\chi_y\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)\cdot\frac{\partial u_z}{\partial t}+\\
&+\frac{\partial}{\partial x_k}\left(8\cdot\rho\cdot V_T^4\cdot\int_{\vec{\chi}}\chi_k\cdot\chi_x\cdot\chi_y\cdot\chi_z\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)+\frac{\partial}{\partial x_k}\left(u_k\cdot 8\cdot\rho\cdot V_T^3\cdot\int_{\vec{\chi}}\chi_x\cdot\chi_y\cdot\chi_z\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)+\\
&+\left(8\cdot\rho\cdot V_T^3\cdot\int_{\vec{\chi}}\chi_k\cdot\chi_y\cdot\chi_z\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)\cdot\frac{\partial u_x}{\partial x_k}+\left[u_k\cdot 8\cdot\rho\cdot V_T^2\cdot\int_{\vec{\chi}}\chi_y\cdot\chi_z\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right]\cdot\frac{\partial u_x}{\partial x_k}+\\
&+\left(8\cdot\rho\cdot V_T^3\cdot\int_{\vec{\chi}}\chi_k\cdot\chi_x\cdot\chi_z\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)\cdot\frac{\partial u_y}{\partial x_k}+\left(8\cdot\rho\cdot u_k\cdot V_T^2\cdot\int_{\vec{\chi}}\chi_x\cdot\chi_z\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)\cdot\frac{\partial u_y}{\partial x_k}+\\
&+\left(8\cdot\rho\cdot V_T^3\cdot\int_{\vec{\chi}}\chi_k\cdot\chi_x\cdot\chi_y\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)\cdot\frac{\partial u_z}{\partial x_k}+\left(8\cdot\rho\cdot u_k\cdot V_T^2\cdot\int_{\vec{\chi}}\chi_x\cdot\chi_y\cdot\left(f_{Grad}+\tau\cdot f_M\cdot f_1\right)\cdot d^3\vec{\chi}\right)\cdot\frac{\partial u_z}{\partial x_k}=\\
&\qquad=-8\cdot\rho\cdot V_T^3\cdot\int_{\vec{\chi}}\chi_x\cdot\chi_y\cdot\chi_z\cdot f_M\cdot f_1\cdot d^3\vec{\chi}\ .
\end{aligned}
\tag{97}
$$

Following the Chapman-Enskog recipe [5, 6], let us put $\tau\to 0$ in Eq. (97); then substituting Eq. (58) for f_{Grad} in the left-hand side of the obtained equation we obtain the following equation for M_{1x1y1z} (by integrating via Mathematica):

$$
\begin{aligned}
&-\frac{M_{1x1y1z}}{\tau}=-8\cdot\rho\cdot V_T^3\cdot\int_{\vec{\chi}}\chi_x\cdot\chi_y\cdot\chi_z\cdot f_M\cdot f_1\cdot d^3\vec{\chi}\ =\left(8\cdot\sigma_{yz}\right)\cdot\frac{\partial u_x}{\partial t}+\left(8\cdot\sigma_{xz}\right)\frac{\partial u_y}{\partial t}+\left(8\cdot\sigma_{xy}\right)\cdot\frac{\partial u_z}{\partial t}+\\
&\quad+\left(8\cdot\sigma_{yz}\cdot u_k\right)\cdot\frac{\partial u_x}{\partial x_k}+\left(\sigma_{xz}\cdot u_k\right)\cdot\frac{\partial u_y}{\partial x_k}+\left(8\cdot\sigma_{xy}\cdot u_k\right)\cdot\frac{\partial u_z}{\partial x_k}+\left(q_z\cdot\frac{16}{5}\right)\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\ +\left(q_y\cdot\frac{16}{5}\right)\cdot\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)+\\
&\quad+\left(q_x\cdot\frac{16}{5}\right)\cdot\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)+\frac{\partial}{\partial x}\left(4\cdot V_T^2\cdot\sigma_{yz}\right)+\frac{\partial}{\partial y}\left(4\cdot V_T^2\cdot\sigma_{xz}\right)+\frac{\partial}{\partial z}\left(4\cdot V_T^2\cdot\sigma_{yx}\right)\ .
\end{aligned}
\tag{98}
$$

Substituting the expression for $\partial u_i/\partial t$, Eq. (15), into Eq. (98) we obtain the final expression for the M_{1x1y1z}

$$
\begin{aligned}
&-\frac{M_{1x1y1z}}{\tau}=4\cdot\rho\cdot V_T^2\cdot\left[\frac{\partial}{\partial x}\left(\frac{\sigma_{yz}}{\rho}\right)+\frac{\partial}{\partial y}\left(\frac{\sigma_{xz}}{\rho}\right)+\frac{\partial}{\partial z}\left(\frac{\sigma_{xy}}{\rho}\right)\right]-\left(\frac{8\cdot\sigma_{yz}}{\rho}\right)\cdot\frac{\partial\sigma_{kx}}{\partial x_k}-\left(\frac{8\cdot\sigma_{xz}}{\rho}\right)\cdot\frac{\partial\sigma_{ky}}{\partial x_k}\ -\left(\frac{8\cdot\sigma_{xy}}{\rho}\right)\cdot\frac{\partial\sigma_{kz}}{\partial x_k}+\\
&\qquad+\left(q_z\cdot\frac{16}{5}\right)\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+\left(q_y\cdot\frac{16}{5}\right)\cdot\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)+\left(q_x\cdot\frac{16}{5}\right)\cdot\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)\ .
\end{aligned}
\tag{99}
$$

Taking into account that

$$
\sigma_{ij}\propto\tau\cdot\frac{\rho\cdot V_T^2\cdot u}{L}\qquad\text{and}\qquad q_i\propto\tau\cdot\frac{\tau\cdot\rho\cdot V_T^4}{L}\ , \tag{100}
$$

we can drop terms that are of the order of τ^2; in the right-hand side Eq. (99). This yields

$$-\frac{M_{1x1y1z}}{\tau}=4\cdot\rho\cdot V_T^2\cdot\left[\frac{\partial}{\partial x}\left(\frac{\sigma_{yz}}{\rho}\right)+\frac{\partial}{\partial y}\left(\frac{\sigma_{xz}}{\rho}\right)+\frac{\partial}{\partial z}\left(\frac{\sigma_{xy}}{\rho}\right)\right]+\left(q_z\cdot\frac{16}{5}\right)\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+ \\ +\left(q_y\cdot\frac{16}{5}\right)\cdot\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)+\left(q_x\cdot\frac{16}{5}\right)\cdot\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right). \tag{101}$$

All other M-moments, Eqs. (86) – (89), are derived in Appendix B. After collecting all M-moments in Eqs. (91) and (93) together,

$$\frac{\partial}{\partial x}\left(\frac{1}{32}\cdot M_{4x}+\frac{1}{8}\cdot M_{2x2y}+\frac{1}{8}\cdot M_{2x2z}\right)=\frac{\partial}{\partial x}\left(\tau\cdot\left\{\frac{V_T^2}{\rho}\cdot\left(\frac{14}{5}\cdot q_x\cdot\frac{\partial\rho}{\partial x}+\frac{4}{5}\cdot q_y\cdot\frac{\partial\rho}{\partial y}+\frac{4}{5}\cdot q_z\cdot\frac{\partial\rho}{\partial z}\right)-\right.\right. \\ -\left(5\cdot q_x\cdot\frac{\partial V_T^2}{\partial x}+3\cdot q_y\cdot\frac{\partial V_T^2}{\partial y}+3\cdot q_z\cdot\frac{\partial V_T^2}{\partial z}\right)-V_T^2\cdot\left(\frac{14}{5}\cdot\frac{\partial q_x}{\partial x}+\frac{4}{5}\cdot\frac{\partial q_y}{\partial y}+\frac{4}{5}\cdot\frac{\partial q_z}{\partial z}\right)- \\ -V_T^2\cdot\left[\left(\frac{2}{3}\cdot\sigma_{xx}+\frac{4}{3}\cdot\sigma_{yy}+\frac{4}{3}\cdot\sigma_{zz}\right)\cdot\frac{\partial u_x}{\partial x}-\left(\frac{1}{3}\cdot\sigma_{xx}+\frac{2}{3}\cdot\sigma_{yy}+\frac{2}{3}\cdot\sigma_{zz}\right)\cdot\frac{\partial u_y}{\partial y}-\right. \\ \left.\left.\left.-\left(\frac{1}{3}\cdot\sigma_{xx}+\frac{2}{3}\cdot\sigma_{yy}+\frac{2}{3}\cdot\sigma_{zz}\right)\cdot\frac{\partial u_z}{\partial z}+4\cdot\sigma_{xy}\cdot\frac{\partial u_x}{\partial y}+4\cdot\sigma_{xy}\cdot\frac{\partial u_y}{\partial x}+4\cdot\sigma_{xz}\cdot\frac{\partial u_x}{\partial z}+4\cdot\sigma_{xz}\cdot\frac{\partial u_z}{\partial x}\right]\right\}\right) \tag{102}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{32}\cdot M_{3x1y}+\frac{1}{32}\cdot M_{1x3y}+\frac{1}{32}\cdot M_{1x1y2z}\right)=-\frac{\partial}{\partial y}\left(\tau\cdot\left\{\left(\frac{7}{10}\cdot q_y\right)\cdot\frac{\partial V_T^2}{\partial x}+\left(\frac{7}{10}\cdot q_x\right)\cdot\frac{\partial V_T^2}{\partial y}+\left(\frac{7}{10}\cdot V_T^2\right)\cdot\frac{\partial q_y}{\partial x}+\right.\right. \\ +\left(\frac{7}{10}\cdot V_T^2\right)\cdot\frac{\partial q_x}{\partial y}-\left(\frac{7}{10}\cdot\frac{q_y\cdot V_T^2}{\rho}\right)\cdot\frac{\partial\rho}{\partial x}-\left(\frac{7}{10}\cdot\frac{q_x\cdot V_T^2}{\rho}\right)\cdot\frac{\partial\rho}{\partial y}+V_T^2\cdot\left[\frac{1}{3}\cdot\sigma_{xy}\cdot\left(\frac{\partial u_y}{\partial y}+\frac{\partial u_x}{\partial x}-2\cdot\frac{\partial u_z}{\partial z}\right)+\right. \\ \left.\left.\left.+\left(\frac{3}{4}\cdot\sigma_{xx}+\frac{3}{4}\cdot\sigma_{yy}+\frac{1}{4}\cdot\sigma_{zz}\right)\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+\frac{1}{2}\cdot\sigma_{yz}\cdot\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)+\frac{1}{2}\cdot\sigma_{xz}\cdot\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)\right]\right\}\right) \tag{103}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{32}\cdot M_{3x1z}+\frac{1}{32}\cdot M_{1x3z}+\frac{1}{32}\cdot M_{1x1z2y}\right)=-\frac{\partial}{\partial y}\left(\tau\cdot\left\{\left(\frac{7}{10}\cdot q_z\right)\cdot\frac{\partial V_T^2}{\partial x}+\left(\frac{7}{10}\cdot q_x\right)\cdot\frac{\partial V_T^2}{\partial z}+\left(\frac{7}{10}\cdot V_T^2\right)\cdot\frac{\partial q_z}{\partial x}+\right.\right. \\ +\left(\frac{7}{10}\cdot V_T^2\right)\cdot\frac{\partial q_x}{\partial z}-\left(\frac{7}{10}\cdot\frac{q_z\cdot V_T^2}{\rho}\right)\cdot\frac{\partial\rho}{\partial x}-\left(\frac{7}{10}\cdot\frac{q_x\cdot V_T^2}{\rho}\right)\cdot\frac{\partial\rho}{\partial z}+V_T^2\cdot\left[\frac{1}{3}\cdot\sigma_{xz}\cdot\left(\frac{\partial u_z}{\partial z}+\frac{\partial u_x}{\partial x}-2\cdot\frac{\partial u_y}{\partial y}\right)+\right. \\ \left.\left.\left.+\left(\frac{3}{4}\cdot\sigma_{xx}+\frac{3}{4}\cdot\sigma_{zz}+\frac{1}{4}\cdot\sigma_{yy}\right)\cdot\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)+\frac{1}{2}\cdot\sigma_{yz}\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+\frac{1}{2}\cdot\sigma_{xy}\cdot\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right)\right]\right\}\right) \tag{104}$$

$$\frac{1}{8}\cdot\frac{\partial}{\partial z}\left(M_{1x1y1z}\right)=-\frac{\partial}{\partial z}\left\{\tau\cdot\left[\frac{\rho\cdot V_T^2}{2}\cdot\left(\frac{\partial}{\partial x}\left(\frac{\sigma_{yz}}{\rho}\right)+\frac{\partial}{\partial y}\left(\frac{\sigma_{xz}}{\rho}\right)+\frac{\partial}{\partial z}\left(\frac{\sigma_{xy}}{\rho}\right)\right)+\right.\right.$$
$$\left.\left.+\left(q_z\cdot\frac{2}{5}\right)\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+\left(q_y\cdot\frac{2}{5}\right)\cdot\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)+\left(q_x\cdot\frac{2}{5}\right)\cdot\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)\right]\right\}, \quad (105)$$

and substituting them into Eqs. (91) and (93) we obtain the final equations for q_x and σ_{xy}. By changing the order of indexes (x,y,z) to (y,x,z) in Eqs. (91), (102) – (104) we obtain the equation for q_y, and, similarly, by using (z,y,x) the equation for q_y; by using (x,z,y) and (y,z,x) in Eqs. (92) and (105) we obtain the equations for σ_{xz} and σ_{yz}, respectively; and by using (y,x,z) in Eq. (92) fwe obtain the equation for σ_{yy}. Thus, the resulting set of 13 equations, Eqs. (14) – (16), (91) – (93), (102) – (105) with rotations of indexes as described above, represents a complete set of the 13 regularized Grad's moment equations for ρ, (u_x,u_y,u_z), V_T^2, (q_x,q_y,q_z), $(\sigma_{xx},\sigma_{xy},\sigma_{xz},\sigma_{yy},\sigma_{yz})$.

Finally, the coefficients Λ_{14} - Λ_{29} in the velocity distribution function f_H, Eq. (31), can be obtained from Eq. (86) – (90) by substituting f_H for f; which yields:

$$\Lambda_{14}=\frac{M_{4x}}{384\cdot\rho\cdot V_T^4},\quad \Lambda_{15}=\frac{M_{4y}}{384\cdot\rho\cdot V_T^4},\quad \Lambda_{16}=\frac{M_{4z}}{384\cdot\rho\cdot V_T^4}, \quad (106)$$

$$\Lambda_{17}=\frac{M_{2x2y}}{64\cdot\rho\cdot V_T^4},\quad \Lambda_{18}=\frac{M_{2x2z}}{64\cdot\rho\cdot V_T^4},\quad \Lambda_{19}=\frac{M_{2y2z}}{64\cdot\rho\cdot V_T^4}, \quad (107)$$

$$\Lambda_{20}=\frac{M_{3x1y}}{96\cdot\rho\cdot V_T^4},\ \Lambda_{21}=\frac{M_{3x1z}}{96\cdot\rho\cdot V_T^4},\ \Lambda_{22}=\frac{M_{3y1x}}{96\cdot\rho\cdot V_T^4},\ \Lambda_{23}=\frac{M_{3y1z}}{96\cdot\rho\cdot V_T^4},\ \Lambda_{24}=\frac{M_{3z1x}}{96\cdot\rho\cdot V_T^4},\ \Lambda_{25}=\frac{M_{3z1y}}{96\cdot\rho\cdot V_T^4}, \quad (108)$$

$$\Lambda_{26}=\frac{M_{1x1y2z}}{32\cdot\rho\cdot V_T^4},\quad \Lambda_{27}=\frac{M_{1y1z2x}}{32\cdot\rho\cdot V_T^4},\quad \Lambda_{28}=\frac{M_{1z1y2x}}{32\cdot\rho\cdot V_T^4}, \quad (109)$$

$$\Lambda_{29}=\frac{M_{1x1y1z}}{8\cdot\rho\cdot V_T^3}. \quad (110)$$

V. CONCLUSIONS

We have presented a new set of moment equations for rarefied gas dynamics. Our equations are a closure for Grad's 13 moment equations extended to the third order of the Knudsen number. We have assume a Hermite polynomial approximation for the monatomic gas velocity distribution function, the BGK approximation of the collision term in the Boltzmann kinetic equation and used the the well-known Chapman-Enskog regularization method that has been previously used to derive a closure of the Euler's gas dynamic equations. We have shown that the selected 29-term Hermite polynomial representation of the velocity distribution function makes good physical sense and obtained the coefficients for this polynomial. Our regularized Grad's 13 moment equations differ from a similar set of equations obtained by Struchtrup and Horrilhon [4], who have used a very different and complicated method and have not assumed a Hermite polynomial representation of the velocity distribution function. On the contrary, the closure method presented in this paper turns out to be quite straightforward and comprehensible.

APPENDIX A

Let us obtain the general moment equation for q_i. Multiplying the Boltzmann Equation, Eq. (1), by

$$\frac{m}{2}\cdot(V_i-u_i)\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right] \quad , \tag{A1}$$

we obtain

$$\begin{aligned}&\frac{m}{2}\cdot(V_i-u_i)\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot\frac{\partial(n\cdot f)}{\partial t}+\\&\quad+\frac{m}{2}\cdot(V_i-u_i)\cdot V_k\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot\frac{\partial(f\cdot n)}{\partial x_k}=\\&\quad=\frac{m}{2}\cdot(V_i-u_i)\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot St(n\cdot f)\ .\end{aligned} \tag{A2}$$

Transferring the terms in front of the derivatives inside the derivative brackets in Eq. (A2), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\rho}{2}\cdot(V_i-u_i)\cdot[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2]\cdot f\right)+\left(\frac{\rho}{2}\cdot[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2]\cdot f\right)\cdot\frac{\partial u_i}{\partial t}+\\
&+\left(\rho\cdot(V_i-u_i)\cdot(V_k-u_k)\cdot f\right)\cdot\frac{\partial u_k}{\partial t}+\frac{\partial}{\partial x_k}\left(\frac{\rho}{2}\cdot(V_i-u_i)\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot(V_k-u_k)\cdot f\right)+\\
&+\frac{\partial}{\partial x_k}\left(u_k\cdot\frac{\rho}{2}\cdot(V_i-u_i)\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot f\right)+\\
&+\left(f\cdot(V_k-u_k)\cdot\frac{\rho}{2}\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\right)\cdot\frac{\partial u_i}{\partial x_k}+\\
&+u_k\cdot\left(f\cdot\frac{\rho}{2}\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\right)\cdot\frac{\partial u_i}{\partial x_k}+\left(\rho\cdot(V_i-u_i)\cdot(V_k-u_k)\cdot f\cdot(V_j-u_j)\right)\cdot\frac{\partial u_j}{\partial x_k}+\\
&+u_k\cdot\left(\rho\cdot(V_i-u_i)\cdot f\cdot(V_j-u_j)\right)\cdot\frac{\partial u_j}{\partial x_k}=\frac{\rho}{2}\cdot(V_i-u_i)\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot St(n\cdot f)\ .
\end{aligned}
\tag{A3}
$$

Integrating Eq. (A3) over the entire velocity domain, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\rho}{2}\cdot\int_{\vec V}(V_i-u_i)\cdot[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2]\cdot f\cdot d^3\vec V\right)+\\
&+\left(\frac{\rho}{2}\cdot\int_{\vec V}[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2]\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_i}{\partial t}+\left(\rho\cdot\int_{\vec V}(V_i-u_i)\cdot(V_k-u_k)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_k}{\partial t}+\\
&+\frac{\partial}{\partial x_k}\left(\frac{\rho}{2}\cdot\int_{\vec V}(V_i-u_i)\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot(V_k-u_k)\cdot f\cdot d^3\vec V\right)+\\
&+\frac{\partial}{\partial x_k}\left(u_k\cdot\frac{\rho}{2}\cdot\int_{\vec V}(V_i-u_i)\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot f\cdot d^3\vec V\right)+\\
&+\left(\frac{\rho}{2}\cdot\int_{\vec V}(V_k-u_k)\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_i}{\partial x_k}+\\
&+\left(u_k\cdot\frac{\rho}{2}\cdot\int_{\vec V}\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_i}{\partial x_k}+\\
&+\left(\rho\cdot\int_{\vec V}(V_i-u_i)\cdot(V_k-u_k)\cdot(V_j-u_j)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_j}{\partial x_k}+\left(u_k\cdot\rho\cdot\int_{\vec V}(V_i-u_i)\cdot(V_j-u_j)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_j}{\partial x_k}=\\
&=\frac{\rho}{2}\cdot\int_{\vec V}(V_i-u_i)\cdot\left[(V_x-u_x)^2+(V_y-u_y)^2+(V_z-u_z)^2\right]\cdot St(f)\cdot d^3\vec V\ \cdot
\end{aligned}
\tag{A4}
$$

Expressing the third and ninth terms in the left-hand side of Eq. (A4) as

$$
\left(\int_{\vec V}\rho\cdot(V_i-u_i)\cdot(V_k-u_k)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_k}{\partial t}=\left(\int_{\vec V}\rho\cdot\left((V_i-u_i)\cdot(V_k-u_k)-\delta_{ik}\cdot\frac{V_T^2}{2}\right)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_k}{\partial t}+\left(\rho\cdot\frac{V_T^2}{2}\right)\cdot\frac{\partial u_i}{\partial t}\ ,
$$

$$\left(u_k \cdot \rho \cdot \int_{\vec{V}} (V_i - u_i)\cdot(V_j - u_j)\cdot f \cdot d^3\vec{V}\right)\cdot\frac{\partial u_j}{\partial x_k} = \left(u_k \cdot \rho \cdot \int_{\vec{V}} \left((V_i - u_i)\cdot(V_j - u_j) - \delta_{ij}\cdot\frac{V_T^2}{2}\right)\cdot f \cdot d^3\vec{V}\right)\cdot\frac{\partial u_j}{\partial x_k} + \left(u_k \cdot \rho \cdot \frac{V_T^2}{2}\right)\cdot\frac{\partial u_i}{\partial x_k},$$

and taking into account Eqs. (6) and (7) and the fact that $\sigma_{ij} = \sigma_{ji}$, we obtain

$$\frac{\partial q_i}{\partial t} + \left(\frac{5}{2}\cdot\frac{\rho\cdot V_T^2}{2}\right)\cdot\frac{\partial u_i}{\partial t} + \sigma_{ik}\cdot\frac{\partial u_k}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k\cdot q_i\right) + q_k\cdot\frac{\partial u_i}{\partial x_k} + \left(u_k\cdot\frac{5}{2}\cdot\frac{\rho\cdot V_T^2}{2}\right)\cdot\frac{\partial u_i}{\partial x_k} + \left(u_k\cdot\sigma_{ij}\right)\cdot\frac{\partial u_j}{\partial x_k} +$$
$$+\frac{\partial}{\partial x_k}\left(\frac{\rho}{2}\cdot\int_{\vec{V}}(V_i - u_i)\cdot(\vec{V}-\vec{u})^2\cdot(V_k - u_k)\cdot f\cdot d^3\vec{V}\right) + \left(\rho\cdot\int_{\vec{V}}(V_i - u_i)\cdot(V_k - u_k)\cdot(V_j - u_j)\cdot f\cdot d^3\vec{V}\right)\cdot\frac{\partial u_j}{\partial x_k} = \quad \text{(A5)}$$
$$= \frac{\rho}{2}\cdot\int_{\vec{V}}(V_i - u_i)\cdot(\vec{V}-\vec{u})^2\cdot St(f)\cdot d^3\vec{V},$$

where

$$\sigma_{zz} = \rho\cdot\int_{\vec{V}} f\cdot(V_z - u_z)^2\cdot d^3\vec{V} - \frac{\rho\cdot V_T^2}{2} = -\sigma_{yy} - \sigma_{zz}. \quad \text{(A6)}$$

Here we have used the general expression for stress tensor σ_{ij} given by Eq. (7). Substituting $\partial u_k / \partial t$ from Eq. (15) into Eq. (A6) and after some simple algebra, the equation for the heat flux q_i turns into

$$\frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k\cdot q_i\right) + q_k\cdot\frac{\partial u_i}{\partial x_k} - \left(\frac{5}{2}\cdot\frac{V_T^2}{2}\right)\cdot\left(\frac{\partial}{\partial x_i}\left(\frac{\rho\cdot V_T^2}{2}\right) + \frac{\partial\sigma_{ji}}{\partial x_j}\right) - \frac{\sigma_{ik}}{\rho}\cdot\left(\frac{\partial}{\partial x_k}\left(\frac{\rho\cdot V_T^2}{2}\right) + \frac{\partial\sigma_{jk}}{\partial x_j}\right) +$$
$$+\frac{\partial}{\partial x_k}\left(\frac{\rho}{2}\cdot\int_{\vec{V}}(V_i - u_i)\cdot(\vec{V}-\vec{u})^2\cdot(V_k - u_k)\cdot f\cdot d^3\vec{V}\right) + \left(\rho\cdot\int_{\vec{V}}(V_i - u_i)\cdot(V_k - u_k)\cdot(V_j - u_j)\cdot f\cdot d^3\vec{V}\right)\cdot\frac{\partial u_j}{\partial x_k} = \quad \text{(A7)}$$
$$= \frac{\rho}{2}\cdot\int_{\vec{V}}(V_i - u_i)\cdot(\vec{V}-\vec{u})^2\cdot St(f).$$

Next, let us derive a general equation for σ_{xy}. Multiplying the Boltzmann Equation, Eq. (1) by

$$m\cdot(V_x - u_x)\cdot(V_y - u_y) \quad , \quad \text{(A8)}$$

we obtain

$$m\cdot(V_x - u_y)\cdot(V_x - u_y)\cdot\frac{\partial(n\cdot f)}{\partial t} + m\cdot(V_x - u_y)\cdot(V_x - u_y)\cdot V_k\cdot\frac{\partial(f\cdot n)}{\partial x_k} = m\cdot(V_x - u_y)\cdot(V_x - u_y)\cdot St(n\cdot f). \quad \text{(A9)}$$

Transferring the terms in front of the derivatives inside the derivative brackets in Eq. (A9), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot f\right)+\left(\rho\cdot(V_y-u_y)\cdot f\right)\cdot\frac{\partial u_x}{\partial t}+\left(\rho\cdot(V_x-u_x)\cdot f\right)\cdot\frac{\partial u_y}{\partial t}+\\
&\quad+\frac{\partial}{\partial x_k}\left(\rho\cdot(V_k-u_k)\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot f\right)+\frac{\partial}{\partial x_k}\left(u_k\cdot\rho\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot f\right)+\\
&\quad+\left(\rho\cdot(V_k-u_k)\cdot(V_y-u_y)\cdot f\right)\cdot\frac{\partial u_x}{\partial x_k}+\left(\rho\cdot u_k\cdot(V_y-u_y)\cdot f\right)\cdot\frac{\partial u_x}{\partial x_k}+\\
&\quad+\left(\rho\cdot(V_k-u_k)\cdot(V_x-u_x)\cdot f\right)\cdot\frac{\partial u_y}{\partial x_k}+\left(u_k\cdot\rho\cdot(V_x-u_x)\cdot f\right)\cdot\frac{\partial u_y}{\partial x_k}=\rho\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot St(f)\ .
\end{aligned}\tag{A10}$$

Integrating Eq. (A10) over the entire velocity domain, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho\cdot\int_{\vec V}(V_x-u_x)\cdot(V_y-u_y)\cdot f\cdot d^3\vec V\right)+\left(\rho\cdot\int_{\vec V}(V_y-u_y)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_x}{\partial t}+\left(\rho\cdot\int_{\vec V}(V_x-u_x)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_y}{\partial t}+\\
&\quad+\frac{\partial}{\partial x_k}\left(\rho\cdot\int_{\vec V}(V_k-u_k)\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot f\cdot d^3\vec V\right)+\frac{\partial}{\partial x_k}\left(u_k\cdot\rho\cdot\int_{\vec V}(V_x-u_x)\cdot(V_y-u_y)\cdot f\cdot d^3\vec V\right)+\\
&\quad+\left(\rho\cdot\int_{\vec V}(V_k-u_k)\cdot(V_y-u_y)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_x}{\partial x_k}+\left(u_k\cdot\rho\cdot\int_{\vec V}(V_y-u_y)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_x}{\partial x_k}+\\
&\quad+\left(\rho\cdot\int_{\vec V}(V_k-u_k)\cdot(V_x-u_x)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_y}{\partial x_k}+\left(u_k\cdot\rho\cdot\int_{\vec V}(V_x-u_x)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_y}{\partial x_k}=\\
&\qquad=\rho\cdot\int_{\vec V}(V_x-u_x)\cdot(V_y-u_y)\cdot St(f)\cdot d^3\vec V\ .
\end{aligned}\tag{A11}$$

Expressing the sixth and eighth terms in the left-hand side of Eq. (A11) as

$$\left(\int_{\vec V}\rho\cdot(V_k-u_k)\cdot(V_i-u_i)f\cdot d^3\vec V\right)\cdot\frac{\partial u_j}{\partial x_k}=\left(\int_{\vec V}\rho\cdot\left((V_i-u_i)\cdot(V_k-u_k)-\delta_{ik}\cdot\frac{V_T^2}{2}\right)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_j}{\partial x_k}+\left(\rho\cdot\frac{V_T^2}{2}\right)\cdot\frac{\partial u_j}{\partial x_i}\ ,\tag{A12}$$

where indexes $ij=(xy,yx)$, and taking into account Eq. (4) and the general expression for the stress tensor, Eq. (7) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\sigma_{xy}\right)+\sigma_{ky}\cdot\frac{\partial u_x}{\partial x_k}+\sigma_{kx}\cdot\frac{\partial u_y}{\partial x_k}+\frac{\partial}{\partial x_k}\left(u_k\cdot\sigma_{xy}\right)+\frac{\rho\cdot V_T^2}{2}\cdot\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+\\
&+\frac{\partial}{\partial x_k}\left(\rho\cdot\int_{\vec V}(V_k-u_k)\cdot(V_x-u_x)\cdot(V_y-u_y)\cdot f\cdot d^3\vec V\right)=\rho\cdot\int_{\vec V}(V_x-u_x)\cdot(V_y-u_y)\cdot St(f)\cdot d^3\vec V\ .
\end{aligned}\tag{A13}$$

Now, let us derive a general equation for σ_{xx}. Multiplying the Boltzmann Equation, Eq. (1) by

$$m\cdot(V_x-u_x)^2-\frac{m}{2}\cdot V_T^2 \tag{A14}$$

we obtain

$$m\cdot\left((V_x-u_y)^2-\frac{V_T^2}{2}\right)\cdot\frac{\partial(n\cdot f)}{\partial t}+m\cdot\left((V_x-u_y)^2-\frac{V_T^2}{2}\right)\cdot V_k\cdot\frac{\partial(f\cdot n)}{\partial x_k}=m\cdot\left((V_x-u_y)^2-\frac{V_T^2}{2}\right)\cdot St(n\cdot f)\ . \tag{A15}$$

Transferring the terms in front of the derivatives inside the derivative brackets in Eq. (A15), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho\cdot\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot f\right)+\left(2\cdot\rho\cdot(V_x-u_x)\cdot f\right)\cdot\frac{\partial u_x}{\partial t}+\left(\frac{\rho\cdot f}{2}\right)\cdot\frac{\partial V_T^2}{\partial t}+\\
&\quad+\frac{\partial}{\partial x_k}\left(\rho\cdot(V_k-u_k)\cdot\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot f\right)+\frac{\partial}{\partial x_k}\left(u_k\cdot\rho\cdot\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot f\right)+\\
&\quad+\left(2\cdot\rho\cdot(V_k-u_k)\cdot(V_x-u_x)\cdot f\right)\cdot\frac{\partial u_x}{\partial x_k}+\left(\rho\cdot u_k\cdot(V_x-u_x)\cdot f\right)\cdot\frac{\partial u_x}{\partial x_k}+\\
&\quad+\left(\frac{\rho\cdot(V_k-u_k)}{2}\cdot f\right)\cdot\frac{\partial V_T^2}{\partial x_k}+\left(\frac{u_k\cdot\rho}{2}\cdot f\right)\cdot\frac{\partial u_y}{\partial x_k}=\rho\cdot\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot St(f)\ .
\end{aligned} \tag{A16}$$

Integrating Eq. (A16) over the entire velocity domain, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho\cdot\int_{\vec V}\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot f\cdot d^3\vec V\right)+\left(2\cdot\rho\cdot\int_{\vec V}(V_x-u_x)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_x}{\partial t}+\left(\frac{\rho}{2}\cdot\int_{\vec V}f\cdot d^3\vec V\right)\cdot\frac{\partial V_T^2}{\partial t}+\\
&\quad+\frac{\partial}{\partial x_k}\left(\rho\cdot\int_{\vec V}(V_k-u_k)\cdot\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot f\cdot d^3\vec V\right)+\frac{\partial}{\partial x_k}\left(u_k\cdot\rho\cdot\int_{\vec V}\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot f\cdot d^3\vec V\right)+\\
&\quad+\left(2\cdot\rho\cdot\int_{\vec V}(V_k-u_k)\cdot(V_x-u_x)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_x}{\partial x_k}+\left(\rho\cdot u_k\cdot\int_{\vec V}(V_x-u_x)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial u_x}{\partial x_k}+\\
&\quad+\left(\frac{\rho}{2}\cdot\int_{\vec V}(V_k-u_k)\cdot f\cdot d^3\vec V\right)\cdot\frac{\partial V_T^2}{\partial x_k}+\left(\frac{u_k\cdot\rho}{2}\cdot\int_{\vec V}f\cdot d^3\vec V\right)\cdot\frac{\partial u_y}{\partial x_k}=\rho\cdot\int_{\vec V}\left((V_x-u_x)^2-\frac{V_T^2}{2}\right)\cdot St(f)\cdot d^3\vec V\ .
\end{aligned} \tag{A17}$$

Using Eq. (A12) in the sixth term in the left-hand side of Eq. (A17), substituting $\partial V_T^2/\partial t$ from Eq. (16), and then taking into account Eqs. (3) and (4) and the general expression for the stress tensor, Eq. (7), we obtain

$$\frac{\partial \sigma_{yy}}{\partial t}-\frac{1}{3}\cdot\rho\cdot V_T^2\cdot\frac{\partial u_i}{\partial x_i}-\frac{2}{3}\cdot\frac{\partial q_i}{\partial x_i}-\frac{2}{3}\cdot\sigma_{ij}\cdot\frac{\partial u_i}{\partial x_i}+\frac{\partial}{\partial x_k}\left(u_k\cdot\sigma_{yy}\right)+2\cdot\sigma_{yk}\cdot\frac{\partial u_y}{\partial x_k}+\rho\cdot V_T^2\cdot\frac{\partial u_y}{\partial y}+$$
$$+\frac{\partial}{\partial x_k}\left(\rho\cdot\int_{\vec V}\left((V_y-u_y)^2-\frac{V_T^2}{2}\right)\cdot(V_k-u_k)\cdot f\cdot d^3\vec V\right)=\rho\cdot\int_{\vec V}\left((V_y-u_y)^2-\frac{V_T^2}{2}\right)\cdot St(f)\cdot d^3\vec V\,. \qquad \text{(A18)}$$

APPENDIX B

Let us obtain an equation for M_{2x2y}. Multiplying the Boltzmann Equation, Eq. (1), with the BGK approximation substituted for the collision term, Eq. (24), by $m\cdot\left(4\cdot(V_x-u_x)^2-2\cdot V_T^2\right)\cdot\left(4\cdot(V_y-u_y)^2-2\cdot V_T^2\right)$ we obtain

$$m\cdot\left(4\cdot(V_x-u_x)^2-2\cdot V_T^2\right)\cdot\left(4\cdot(V_y-u_y)^2-2\cdot V_T^2\right)\cdot\frac{\partial(n\cdot f)}{\partial t}+$$
$$+m\cdot V_k\cdot\left(4\cdot(V_x-u_x)^2-2\cdot V_T^2\right)\cdot\left(4\cdot(V_y-u_y)^2-2\cdot V_T^2\right)\cdot\frac{\partial(f\cdot n)}{\partial x_k}= \qquad \text{(B1)}$$
$$=m\cdot\left(4\cdot(V_x-u_x)^2-2\cdot V_T^2\right)\cdot\left(4\cdot(V_y-u_y)^2-2\cdot V_T^2\right)\cdot n\cdot\frac{(f_M-f)}{\tau}\,.$$

Transferring the terms in front of the derivatives inside the derivatives brackets in Eq. (B1), we obtain

$$\frac{\partial}{\partial t}\left(4\cdot\rho\cdot\left(2\cdot(V_x-u_x)^2-V_T^2\right)\cdot\left(2\cdot(V_y-u_y)^2-V_T^2\right)\cdot f\right)+\left(\rho\cdot 8\cdot\left((V_x-u_x)^2+(V_y-u_y)^2-V_T^2\right)\cdot f\right)\cdot\frac{\partial V_T^2}{\partial t}+$$
$$+\left(8\cdot\rho\cdot(V_x-u_x)\cdot\left(2\cdot(V_y-u_y)^2-V_T^2\right)\cdot f\right)\cdot\frac{\partial u_x}{\partial t}+\left(8\cdot\rho\cdot(V_y-u_y)\cdot\left(2\cdot(V_x-u_x)^2-V_T^2\right)\cdot f\right)\cdot\frac{\partial u_y}{\partial t}+$$
$$+\frac{\partial}{\partial x_k}\left(4\cdot\rho\cdot(V_k-u_k)\cdot\left(2\cdot(V_x-u_x)^2-V_T^2\right)\cdot\left(2\cdot(V_y-u_y)^2-V_T^2\right)\cdot f\right)+$$
$$+\frac{\partial}{\partial x_k}\left(4\cdot\rho\cdot u_k\cdot\left(2\cdot(V_x-u_x)^2-V_T^2\right)\cdot\left(2\cdot(V_y-u_y)^2-V_T^2\right)\cdot f\right)+$$
$$+8\cdot\rho\cdot(V_k-u_k)\cdot\left[\left((V_x-u_x)^2+(V_y-u_y)^2-V_T^2\right)\cdot f\right]\cdot\frac{\partial V_T^2}{\partial x_k}+8\cdot\rho\cdot u_k\cdot\left[\left((V_x-u_x)^2+(V_y-u_y)^2-V_T^2\right)\cdot f\right]\cdot\frac{\partial V_T^2}{\partial x_k}+ \qquad \text{(B2)}$$
$$+16\cdot\rho\cdot(V_k-u_k)\cdot\left[(V_x-u_x)\cdot\left(2\cdot(V_y-u_y)^2-V_T^2\right)f\right]\cdot\frac{\partial u_x}{\partial x_k}+16\cdot\rho\cdot u_k\cdot\left[(V_x-u_x)\cdot\left(2\cdot(V_y-u_y)^2-V_T^2\right)f\right]\cdot\frac{\partial u_x}{\partial x_k}+$$
$$+16\cdot\rho\cdot(V_k-u_k)\cdot\left[(V_y-u_y)\cdot\left(2\cdot(V_x-u_x)^2-V_T^2\right)f\right]\cdot\frac{\partial u_y}{\partial x_k}+16\cdot\rho\cdot u_k\cdot\left[(V_y-u_y)\cdot\left(2\cdot(V_x-u_x)^2-V_T^2\right)f\right]\cdot\frac{\partial u_y}{\partial x_k}+$$
$$=\frac{1}{\tau}\cdot\rho\cdot\left(4\cdot(V_x-u_x)^2-2\cdot V_T^2\right)\cdot\left(4\cdot(V_y-u_y)^2-2\cdot V_T^2\right)\cdot(f_M-f)\,.$$

Following the closure method described in Section IV for calculating moment M_{1x1y1z}, we substitute f_{Grad} for f into the left-hand side of Eq. (B3) and $f_{Grad} + \tau \cdot f_M \cdot f_1$ into the right-hand side and then integrate the resulting equation over the entire velocity domain using Mathematica. This yields

$$\begin{aligned}
&\left(\frac{64}{5} \cdot q_x\right) \cdot \frac{\partial u_x}{\partial t} + \left(\frac{64}{5} \cdot q_y\right) \cdot \frac{\partial u_y}{\partial t} + \left(8 \cdot \sigma_{xx} + 8 \cdot \sigma_{yy}\right) \cdot \frac{\partial V_T^2}{\partial t} + \frac{\partial}{\partial x}\left(\frac{32}{5} \cdot V_T^2 \cdot q_x\right) + \frac{\partial}{\partial y}\left(\frac{32}{5} \cdot V_T^2 \cdot q_y\right) + \left(16 \cdot V_T^2 \cdot \sigma_{yy}\right) \cdot \frac{\partial u_x}{\partial x} + \\
&+ \left(16 \cdot V_T^2 \cdot \sigma_{xx}\right) \cdot \frac{\partial u_y}{\partial y} + \left(32 \cdot V_T^2 \cdot \sigma_{xy}\right) \cdot \frac{\partial u_x}{\partial y} + \left(32 \cdot V_T^2 \cdot \sigma_{xy}\right) \cdot \frac{\partial u_y}{\partial x} + \left(\frac{64}{5} \cdot \frac{u_k \cdot q_x}{V_T^3}\right) \cdot \frac{\partial u_x}{\partial x_k} + \left(\frac{64}{5} \cdot \frac{u_k \cdot q_y}{V_T^3}\right) \cdot \frac{\partial u_y}{\partial x_k} + \\
&+ \left(\frac{64}{5} \cdot q_x\right) \cdot \frac{\partial V_T^2}{\partial x} + \left(\frac{64}{5} \cdot q_y\right) \cdot \frac{\partial V_T^2}{\partial y} + \left(\frac{32}{5} \cdot q_z\right) \cdot \frac{\partial V_T^2}{\partial z} + \left(8 \cdot u_k \cdot \sigma_{xx} + 8 \cdot u_k \cdot \sigma_{yy}\right) \cdot \frac{\partial V_T^2}{\partial x_k} = -\frac{M_{2x2y}}{\tau} \ .
\end{aligned} \tag{B4}$$

Substituting $\partial u_i / \partial t$ and $\partial V_T^2 / \partial t$ from the momentum and energy conservation law equations, Eqs. (15) and (16), into Eq. (B4) we obtain

$$\begin{aligned}
&-\left(\left(\frac{32}{5} \cdot \frac{q_x \cdot V_T^2}{\rho}\right) \cdot \frac{\partial \rho}{\partial x} + \left(\frac{32}{5} \cdot \frac{q_y \cdot V_T^2}{\rho}\right) \cdot \frac{\partial \rho}{\partial y}\right) + \left(\frac{32}{5} \cdot V_T^2\right) \cdot \frac{\partial q_x}{\partial x} + \left(\frac{32}{5} \cdot V_T^2\right) \cdot \frac{\partial q_y}{\partial y} + \\
&\quad + \left(\frac{64}{5} \cdot q_x\right) \cdot \frac{\partial V_T^2}{\partial x} + \left(\frac{64}{5} \cdot q_y\right) \cdot \frac{\partial V_T^2}{\partial y} + \left(\frac{32}{5} \cdot q_z\right) \cdot \frac{\partial V_T^2}{\partial z} + \frac{16}{3} \cdot V_T^2 \cdot \left(2 \cdot \sigma_{yy} - \sigma_{xx}\right) \cdot \frac{\partial u_x}{\partial x} + \\
&\quad + \frac{16}{3} \cdot V_T^2 \cdot \left(2 \cdot \sigma_{xx} - \sigma_{yy}\right) \cdot \frac{\partial u_y}{\partial y} - \frac{16}{3} \cdot V_T^2 \cdot \left(\sigma_{xx} + \sigma_{yy}\right) \cdot \frac{\partial u_z}{\partial z} + \left(32 \cdot V_T^2 \cdot \sigma_{xy}\right) \cdot \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - \\
&\quad - \left(\frac{32}{3} \cdot \frac{\sigma_{xx}}{\rho} + \frac{32}{3} \cdot \frac{\sigma_{yy}}{\rho}\right) \cdot \frac{\partial q_k}{\partial x_k} - \left(\left(\frac{64}{5} \cdot \frac{q_x}{\rho}\right) \cdot \frac{\partial \sigma_{kx}}{\partial x_k} + \left(\frac{64}{5} \cdot \frac{q_y}{\rho}\right) \cdot \frac{\partial \sigma_{ky}}{\partial x_k}\right) - \\
&\quad - \left(\frac{32}{3} \cdot \frac{\sigma_{xx}}{\rho} + \frac{32}{3} \cdot \frac{\sigma_{yy}}{\rho}\right) \cdot \sigma_{ik} \cdot \frac{\partial u_i}{\partial x_k} = -\frac{M_{2x2y}}{\tau}
\end{aligned} \tag{B5}$$

Taking into account Eq. (100) and dropping all terms of order τ^2 in Eq. (B5) we obtain,

$$\begin{aligned}
&-\frac{M_{2x2y}}{\tau} = -\frac{32}{5} \cdot \frac{V_T^2}{\rho}\left(q_x \cdot \frac{\partial \rho}{\partial x} + q_y \cdot \frac{\partial \rho}{\partial y}\right) + \left(\frac{32}{5} \cdot V_T^2\right) \cdot \frac{\partial q_x}{\partial x} + \left(\frac{32}{5} \cdot V_T^2\right) \cdot \frac{\partial q_y}{\partial y} + \\
&\quad + \left(\frac{32}{5} \cdot q_z\right) \cdot \frac{\partial V_T^2}{\partial z} + \left(\frac{64}{5} \cdot q_x\right) \cdot \frac{\partial V_T^2}{\partial x} + \left(\frac{64}{5} \cdot q_y\right) \cdot \frac{\partial V_T^2}{\partial y} + \frac{16}{3} \cdot V_T^2 \cdot \left(2 \cdot \sigma_{yy} - \sigma_{xx}\right) \cdot \frac{\partial u_x}{\partial x} + \\
&\quad + \frac{16}{3} \cdot V_T^2 \cdot \left(2 \cdot \sigma_{xx} - \sigma_{yy}\right) \cdot \frac{\partial u_y}{\partial y} - \frac{16}{3} \cdot V_T^2 \cdot \left(\sigma_{xx} + \sigma_{yy}\right) \cdot \frac{\partial u_z}{\partial z} + \left(32 \cdot V_T^2 \cdot \sigma_{xy}\right) \cdot \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)
\end{aligned} \tag{B6}$$

Changing the order of indexes (x, y, z) to (x, z, y) in Eq. (B6), we obtain an equation the M_{2x2z} moment, and, similarly, for the M_{2y2z} moment.

Let us obtain an equation for M_{4x}. Multiplying the Boltzmann Equation, Eq. (1), with the BGK approximation of the collision term, Eq. (24) by $m\cdot\left(16\cdot(V_x-u_x)^4-48\cdot V_T^2\cdot(V_x-u_x)^2+12\cdot V_T^4\right)$ we obtain

$$\begin{aligned} & m\cdot\left(16\cdot(V_x-u_x)^4-48\cdot V_T^2\cdot(V_x-u_x)^2+12\cdot V_T^4\right)\cdot\frac{\partial(n\cdot f)}{\partial t}+ \\ & +m\cdot V_k\cdot\left(16\cdot(V_x-u_x)^4-48\cdot V_T^2\cdot(V_x-u_x)^2+12\cdot V_T^4\right)\cdot\frac{\partial(f\cdot n)}{\partial x_k}= \\ & =m\cdot\left(16\cdot(V_x-u_x)^4-48\cdot V_T^2\cdot(V_x-u_x)^2+12\cdot V_T^4\right)\cdot n\cdot\frac{f_M-f}{\tau}\ . \end{aligned} \tag{B7}$$

Transferring the terms in front of the derivatives inside the derivatives brackets in Eq. (B7), we obtain

$$\begin{aligned} & \frac{\partial}{\partial t}\left(\rho\cdot\left(16\cdot(V_x-u_x)^4-48\cdot V_T^2\cdot(V_x-u_x)^2+12\cdot V_T^4\right)\cdot f\right)+\left(\rho\cdot\left(48\cdot(V_x-u_x)^2-24\cdot V_T^2\right)\cdot f\right)\cdot\frac{\partial V_T^2}{\partial t}+ \\ & +\left(\rho\cdot\left(64\cdot(V_x-u_x)^3-96\cdot V_T^2\cdot(V_x-u_x)\right)\cdot f\right)\cdot\frac{\partial u_x}{\partial t}+\frac{\partial}{\partial x_k}\left(\rho\cdot(V_k-u_k)\cdot\left(16\cdot(V_x-u_x)^4-48\cdot V_T^2\cdot(V_x-u_x)^2+12\cdot V_T^4\right)\cdot f\right)+ \\ & +\frac{\partial}{\partial x_k}\left(\rho\cdot u_k\cdot\left(16\cdot(V_x-u_x)^4-48\cdot V_T^2\cdot(V_x-u_x)^2+12\cdot V_T^4\right)\cdot f\right)+\rho\cdot(V_k-u_k)\cdot\left[\left(48\cdot(V_x-u_x)^2-24\cdot V_T^2\right)\cdot f\right]\cdot\frac{\partial V_T^2}{\partial x_k}+ \\ & +\rho\cdot u_k\cdot\left[\left(48\cdot(V_x-u_x)^2-24\cdot V_T^2\right)\cdot f\right]\cdot\frac{\partial V_T^2}{\partial x_k}+\rho\cdot\left[(V_k-u_k)\cdot\left(64\cdot(V_x-u_x)^3-96\cdot V_T^2\cdot(V_x-u_x)\right)\cdot f\right]\cdot\frac{\partial u_x}{\partial x_k}+ \\ & +\rho\cdot u_k\cdot\left[\left(64\cdot(V_x-u_x)^3-96\cdot V_T^2\cdot(V_x-u_x)\right)\cdot f\right]\cdot\frac{\partial u_x}{\partial x_k}=\rho\cdot\left(16\cdot(V_x-u_x)^4-48\cdot V_T^2\cdot(V_x-u_x)^2+12\cdot V_T^4\right)\cdot\frac{f_M-f}{\tau}\ . \end{aligned} \tag{B8}$$

Following the same procedure as for the moment M_{2x2y} described in the paragraph following after Eq. (B2) we obtain

$$\begin{aligned} -\frac{M_{4x}}{\tau} & =\left(\frac{384}{5}\cdot q_x\right)\cdot\frac{\partial u_x}{\partial t}+\left(48\cdot\sigma_{xx}\right)\cdot\frac{\partial V_T^2}{\partial t}+\left(\frac{480}{5}\cdot q_x+48\cdot u_x\cdot\sigma_{xx}\right)\cdot\frac{\partial V_T^2}{\partial x}+\left(\frac{192}{5}\cdot V_T^2\right)\cdot\frac{\partial q_x}{\partial x}+ \\ & +\left(96\cdot\sigma_{xx}\cdot V_T^2+\frac{384}{5}\cdot u_x\cdot q_x\right)\cdot\frac{\partial u_x}{\partial x}+\left(\frac{384}{5}\cdot u_y\cdot q_x\right)\cdot\frac{\partial u_x}{\partial y}+\left(\frac{384}{5}\cdot u_z\cdot q_x\right)\cdot\frac{\partial u_x}{\partial z}+ \\ & +\left(\frac{96}{5}\cdot q_y+48\cdot u_y\cdot\sigma_{xx}\right)\cdot\frac{\partial V_T^2}{\partial y}+\left(\frac{96}{5}\cdot q_z+48\cdot u_z\cdot\sigma_{xx}\right)\cdot\frac{\partial V_T^2}{\partial z}\ . \end{aligned} \tag{B9}$$

Substituting $\partial u_i/\partial t$ and $\partial V_T^2/\partial t$ from the momentum and energy conservation law equations, Eqs. (15) and (16), into Eq. (B9) and then dropping all terms of the order τ^2 in the resulting equation, we obtain

$$-\frac{M_{4x}}{\tau}=\left(\frac{288}{5}\cdot q_x\right)\cdot\frac{\partial V_T^2}{\partial x}+\left(\frac{96}{5}\cdot q_y\right)\cdot\frac{\partial V_T^2}{\partial y}+\left(\frac{96}{5}\cdot q_z\right)\cdot\frac{\partial V_T^2}{\partial z}+\left(\frac{192}{5}\cdot V_T^2\right)\cdot\frac{\partial q_x}{\partial x}-$$
$$-\left(\frac{192}{5}\cdot\frac{q_x\cdot V_T^2}{\rho}\right)\cdot\frac{\partial\rho}{\partial x}+\left(32\cdot\sigma_{xx}\cdot V_T^2\right)\cdot\left(2\cdot\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}-\frac{\partial u_z}{\partial z}\right). \tag{B10}$$

Changing the order of indexes (x,y,z) to (y,x,z) in Eq. (B10), we obtain an equation the M_{4y} moment, similarly, for the M_{4z} moment.

Let us obtain an equation for M_{3x1y}. Multiplying the Boltzmann Equation, Eq. (1) with the BGK approximation of the collision term, Eq. (24) by $m\cdot 8\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot\left(V_y-u_y\right)$ we obtain

$$8\cdot m\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot\left(V_y-u_y\right)\cdot\frac{\partial(n\cdot f)}{\partial t}+$$
$$+8\cdot m\cdot V_k\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot\left(V_y-u_y\right)\cdot\frac{\partial(f\cdot n)}{\partial x_k}=$$
$$=8\cdot m\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot\left(V_y-u_y\right)\cdot n\cdot\frac{f_M-f}{\tau}. \tag{B11}$$

Transferring the terms in front of the derivatives inside the derivatives brackets in Eq. (B11), we obtain

$$\frac{\partial}{\partial t}\left(8\cdot\rho\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot\left(V_y-u_y\right)\cdot f\right)+\left(24\cdot\rho\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot f\right)\cdot\frac{\partial V_T^2}{\partial t}+$$
$$+\left(8\cdot\rho\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot f\right)\cdot\frac{\partial u_y}{\partial t}+\left(24\cdot\rho\cdot\left(2\cdot(V_x-u_x)^2-V_T^2\right)\cdot\left(V_y-u_y\right)\cdot f\right)\cdot\frac{\partial u_x}{\partial t}+$$
$$+\frac{\partial}{\partial x_k}\left(8\cdot\rho\cdot(V_k-u_k)\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot\left(V_y-u_y\right)\cdot f\right)+$$
$$+\frac{\partial}{\partial x_k}\left(8\cdot\rho\cdot u_k\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot\left(V_y-u_y\right)\cdot f\right)+$$
$$+24\cdot\rho\cdot\left[(V_k-u_k)\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot f\right]\cdot\frac{\partial V_T^2}{\partial x_k}+24\cdot\rho\cdot u_k\cdot\left[(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot f\right]\cdot\frac{\partial V_T^2}{\partial x_k}+$$
$$+\left(8\cdot\rho\cdot(V_k-u_k)\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot f\right)\cdot\frac{\partial u_y}{\partial x_k}+\left(8\cdot\rho\cdot u_k\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot f\right)\cdot\frac{\partial u_y}{\partial x_k}+$$
$$+\left(24\cdot\rho\cdot(V_k-u_k)\left(2\cdot(V_x-u_x)^2-V_T^2\right)\cdot\left(V_y-u_y\right)\cdot f\right)\cdot\frac{\partial u_x}{\partial x_k}+\left(24\cdot\rho\cdot u_k\cdot\left(2\cdot(V_x-u_x)^2-V_T^2\right)\cdot\left(V_y-u_y\right)\cdot f\right)\cdot\frac{\partial u_x}{\partial x_k}=$$
$$=8\cdot\rho\cdot\left(2\cdot(V_x-u_x)^3-3\cdot V_T^2\cdot(V_x-u_x)\right)\cdot\left(V_y-u_y\right)\cdot\frac{f_M-f}{\tau}. \tag{B12}$$

Following the same procedure as for the moment M_{2x2y} described in the paragraph after Eq. (B2) we obtain

$$\left(24\cdot\sigma_{xy}\right)\cdot\frac{\partial V_T^2}{\partial t}+\left(\frac{96}{5}\cdot q_y\right)\cdot\frac{\partial u_x}{\partial t}+\left(\frac{96}{5}\cdot q_x\right)\cdot\frac{\partial u_y}{\partial t}+\left(\frac{96}{5}\cdot q_y\right)\cdot\frac{\partial V_T^2}{\partial x}+\left(\frac{96}{5}\cdot q_x\right)\cdot\frac{\partial V_T^2}{\partial y}++\left(\frac{48}{5}\cdot V_T^2\right)\cdot\frac{\partial q_y}{\partial x}+\left(\frac{48}{5}\cdot V_T^2\right)\cdot\frac{\partial q_x}{\partial y}$$
$$+\left(\frac{48}{5}\cdot V_T^2\right)\cdot\frac{\partial q_y}{\partial x}+\left(\frac{48}{5}\cdot V_T^2\right)\cdot\frac{\partial q_x}{\partial y}+\left(24\cdot u_k\cdot\sigma_{xy}\right)\cdot\frac{\partial V_T^2}{\partial x_k}+\left(48\cdot\sigma_{xy}\cdot V_T^2\right)\cdot\frac{\partial u_x}{\partial x}+ \tag{B13}$$
$$+\left(24\cdot\sigma_{xx}\cdot V_T^2\right)\cdot\frac{\partial u_x}{\partial y}+\left(\frac{96}{5}\cdot q_y\cdot u_k\right)\cdot\frac{\partial u_x}{\partial x_k}+\left(24\cdot\sigma_{xx}\cdot V_T^2\right)\cdot\frac{\partial u_y}{\partial x}+\left(\frac{96}{5}\cdot q_x\cdot u_k\right)\cdot\frac{\partial u_y}{\partial x_k}=-\frac{M_{3x1y}}{\tau}\ .$$

Substituting $\partial u_i/\partial t$ and $\partial V_T^2/\partial t$ from the momentum and energy conservation law equations, Eqs. (15) and (16), into Eq. (B13) and then dropping all terms of order τ^2 in the resulting equation we obtain

$$-\frac{M_{3x1y}}{\tau}=-\left(16\cdot\sigma_{xy}\cdot V_T^2\right)\cdot\frac{\partial u_y}{\partial y}-\left(16\cdot\sigma_{xy}\cdot V_T^2\right)\cdot\frac{\partial u_z}{\partial z}+\left(32\cdot\sigma_{xy}\cdot V_T^2\right)\cdot\frac{\partial u_x}{\partial x}+\left(24\cdot\sigma_{xx}\cdot V_T^2\right)\cdot\frac{\partial u_x}{\partial y}+\left(24\cdot\sigma_{xx}\cdot V_T^2\right)\cdot\frac{\partial u_y}{\partial x}-$$
$$-\left(\frac{48}{5}\cdot\frac{q_y\cdot V_T^2}{\rho}\right)\cdot\frac{\partial\rho}{\partial x}-\left(\frac{48}{5}\cdot\frac{q_x\cdot V_T^2}{\rho}\right)\cdot\frac{\partial\rho}{\partial y}+\left(\frac{48}{5}\cdot V_T^2\right)\cdot\frac{\partial q_y}{\partial x}+\left(\frac{48}{5}\cdot q_y\right)\cdot\frac{\partial V_T^2}{\partial x}+\left(\frac{48}{5}\cdot V_T^2\right)\cdot\frac{\partial q_x}{\partial y}+\left(\frac{48}{5}\cdot q_x\right)\cdot\frac{\partial V_T^2}{\partial y}\ . \tag{B14}$$

Changing the order of indexes (x,y,z) to (x,z,y) in Eq. (B14), we obtain an equation the M_{3x1z} moment; for (y,x,z) we obtain an equation the M_{3y1x} moment, and similarly for the M_{3y1z}, M_{3z1y}, and M_{3z1x} moments.

Finally, let us obtain an equation for M_{1x1y2z}. Multiplying the Boltzmann Equation, Eq. (1) with the BGK approximation of the collision term, Eq. (24) by $m\cdot 8\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)$ we obtain

$$m\cdot 8\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot\frac{\partial(n\cdot f)}{\partial t}+$$
$$+8\cdot m\cdot V_k\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot\frac{\partial(f\cdot n)}{\partial x_k}= \tag{B15}$$
$$=8\cdot m\cdot m\cdot 8\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot n\cdot\frac{f_M-f}{\tau}\ .$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(8\cdot\rho\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot f\right)+\left(8\cdot\rho\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot f\right)\cdot\frac{\partial V_T^2}{\partial t}+\\
&+\left(8\cdot\rho\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot f\right)\cdot\frac{\partial u_x}{\partial t}+\left(8\cdot\rho\cdot(V_x-u_x)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot f\right)\cdot\frac{\partial u_y}{\partial t}+\\
&+\left(16\cdot\rho\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot(V_z-u_z)\cdot f\right)\cdot\frac{\partial u_z}{\partial t}+\\
&+\frac{\partial}{\partial x_k}\left(8\cdot\rho\cdot(V_k-u_k)\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot f\right)+\\
&+\frac{\partial}{\partial x_k}\left(8\cdot\rho\cdot u_k\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot f\right)+\\
&+\left(8\cdot\rho\cdot(V_k-u_k)\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot f\right)\cdot\frac{\partial V_T^2}{\partial x_k}+\left(8\cdot\rho\cdot u_k\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot f\right)\cdot\frac{\partial V_T^2}{\partial x_k}+\\
&+\left(8\cdot\rho\cdot(V_k-u_k)\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot f\right)\cdot\frac{\partial u_x}{\partial x_k}+\left(8\cdot\rho\cdot u_k\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot f\right)\cdot\frac{\partial u_x}{\partial x_k}+\\
&+\left(8\cdot\rho\cdot(V_k-u_k)\cdot(V_x-u_x)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot f\right)\cdot\frac{\partial u_y}{\partial x_k}+\left(8\cdot\rho\cdot u_k\cdot(V_x-u_x)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot f\right)\cdot\frac{\partial u_y}{\partial x_k}+\\
&+\left(16\cdot\rho\cdot(V_k-u_k)\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot(V_z-u_z)\cdot f\right)\cdot\frac{\partial u_z}{\partial x_k}+\left(16\cdot\rho\cdot u_k\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot(V_z-u_z)\cdot f\right)\cdot\frac{\partial u_z}{\partial x_k}=\\
&\qquad=8\cdot\rho\cdot(V_x-u_x)\cdot\left(V_y-u_y\right)\cdot\left(2\cdot(V_z-u_z)^2-V_T^2\right)\cdot\frac{f_M-f}{\tau}\ .
\end{aligned}
\tag{B16}
$$

Following the same procedure as for the moment M_{2x2y} described in the paragraph after Eq. (B2) we obtain

$$
\begin{aligned}
&\left(8\cdot\sigma_{xy}\right)\cdot\frac{\partial V_T^2}{\partial t}+\left(\frac{32}{5}\cdot q_y\right)\cdot\frac{\partial u_x}{\partial t}+\left(\frac{32}{5}\cdot q_x\right)\cdot\frac{\partial u_y}{\partial t}+\frac{\partial}{\partial x}\left(\frac{16}{5}\cdot q_y\cdot V_T^2\right)+\frac{\partial}{\partial y}\left(\frac{16}{5}\cdot q_x\cdot V_T^2\right)+\left(\frac{16}{5}\cdot q_y\right)\cdot\frac{\partial V_T^2}{\partial x}+\left(\frac{16}{5}\cdot q_x\right)\cdot\frac{\partial V_T^2}{\partial y}+\\
&+\left(8\cdot\sigma_{xy}\cdot u_k\right)\cdot\frac{\partial V_T^2}{\partial x_k}+\left(8\cdot\sigma_{zz}\cdot V_T^2\right)\cdot\frac{\partial u_x}{\partial y}+\left(16\cdot\sigma_{yz}\cdot V_T^2\right)\cdot\frac{\partial u_x}{\partial z}+\left(\frac{32}{5}\cdot q_y\cdot u_k\right)\cdot\frac{\partial u_x}{\partial x_k}+\left(8\cdot\sigma_{zz}\cdot V_T^2\right)\cdot\frac{\partial u_y}{\partial x}+\\
&+\left(16\cdot\sigma_{xz}\cdot V_T^2\right)\cdot\frac{\partial u_y}{\partial z}+\left(\frac{32}{5}\cdot q_x\cdot u_k\right)\cdot\frac{\partial u_y}{\partial x_k}+\left(16\cdot\sigma_{yz}\cdot V_T^2\right)\cdot\frac{\partial u_z}{\partial x}+\left(16\cdot\sigma_{xz}\cdot V_T^2\right)\cdot\frac{\partial u_z}{\partial y}+\left(16\cdot\sigma_{xy}\cdot V_T^2\right)\cdot\frac{\partial u_z}{\partial z}=-\frac{M_{1x1y2z}}{\tau}\ .
\end{aligned}
\tag{B17}
$$

Substituting $\partial u_i/\partial t$ and $\partial V_T^2/\partial t$ from the momentum and energy conservation law equations, Eqs. (15) and (16), into Eq. (B17) and then dropping all terms of the order τ^2 in the resulting equation we obtain

$$
\begin{aligned}
&-\frac{M_{1x1y2z}}{\tau}=-\left(\frac{16}{3}\cdot\sigma_{xy}\cdot V_T^2\right)\cdot\frac{\partial u_x}{\partial x}-\left(\frac{16}{3}\cdot\sigma_{xy}\cdot V_T^2\right)\cdot\frac{\partial u_y}{\partial y}+\left(8\cdot\sigma_{zz}\cdot V_T^2\right)\cdot\frac{\partial u_x}{\partial y}+\left(16\cdot\sigma_{yz}\cdot V_T^2\right)\cdot\frac{\partial u_x}{\partial z}+\left(8\cdot\sigma_{zz}\cdot V_T^2\right)\cdot\frac{\partial u_y}{\partial x}+\\
&\quad+\left(16\cdot\sigma_{xz}\cdot V_T^2\right)\cdot\frac{\partial u_y}{\partial z}+\left(16\cdot\sigma_{yz}\cdot V_T^2\right)\cdot\frac{\partial u_z}{\partial x}+\left(16\cdot\sigma_{xz}\cdot V_T^2\right)\cdot\frac{\partial u_z}{\partial y}+\left(\frac{32}{3}\cdot\sigma_{xy}\cdot V_T^2\right)\cdot\frac{\partial u_z}{\partial z}-\left(\frac{16}{5}\cdot\frac{q_y\cdot V_T^2}{\rho}\right)\cdot\frac{\partial\rho}{\partial x}-\\
&\quad-\left(\frac{16}{5}\cdot\frac{q_x\cdot V_T^2}{\rho}\right)\cdot\frac{\partial\rho}{\partial y}+\left(\frac{16}{5}\cdot V_T^2\right)\cdot\frac{\partial q_y}{\partial x}+\left(\frac{16}{5}\cdot V_T^2\right)\cdot\frac{\partial q_x}{\partial y}+\left(\frac{16}{5}\cdot q_y\right)\cdot\frac{\partial V_T^2}{\partial x}+\left(\frac{16}{5}\cdot q_x\right)\cdot\frac{\partial V_T^2}{\partial y}\ .
\end{aligned}
\tag{B18}
$$

Changing the order of indexes (x, y, z) to (x, z, y) in Eq. (B18) we obtain an equation for the M_{1x1z3y} moment, and using (z, y, x) we obtain an equation for the M_{1y1z3x} moment.

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