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The illness and death of G.M. Zaslavsky caused significant shifts in the work carried out under this grant. Nonetheless it was possible for the remaining research scientists originally intended to be supported by the grant, Dr. Mark Edelman and Dr. Vasily Tarasov, to work in the broad directions laid out in the proposal. In three papers, which are attached, they were able to set up fractional differential equation to represent the evolution of the particle distribution function for systems of physical interest and relevance. The first paper dealt with a variation of the standard map, the second of a system with memory, and the third with a dissipative system. In each case it was possible to identify solutions of the continuous fractional kinetic equation with the properties of a discrete map. This identification provided, on the one hand, a reasonable solution algorithm for the fractional kinetic equation in question, and on the other hand a connection to discrete dynamical systems with potentially chaotic dynamics. In every case stochastic webs were identified and attractors and attractor basins were found. The detailed results are in the papers, the first published in Physics Letters A, the second in Journal of Physics A: Math & Theoretical, while the third is under review. The work has been presented at

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# Fractional Kinetics of Chaotic Dynamics of Particles in Complex Systems

**Final Report for Award Period 11/1/07 – 12/31/09**

**Award Number N00014-08-1-0121**

**Award Agency: Office of Naval Research**


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The illness and death of G.M. Zaslavsky caused significant shifts in the work carried out under this grant. Nonetheless it was possible for the remaining research scientists originally intended to be supported by the grant, Dr. Mark Edelman and Dr. Vasily Tarasov, to work in the broad directions laid out in the proposal. In three papers, which are attached, they were able to set up fractional differential equation to represent the evolution of the particle distribution function for systems of physical interest and relevance. The first paper dealt with a variation of the standard map, the second of a system with memory, and the third with a dissipative system. In each case it was possible to identify solutions of the continuous fractional kinetic equation with the properties of a discrete map. This identification provided, on the one hand, a reasonable solution algorithm for the fractional kinetic equation in question, and on the other hand a connection to discrete dynamical systems with potentially chaotic dynamics. In every case stochastic webs were identified and attractors and attractor basins were found. The detailed results are in the papers, the first published in Physics Letters A, the second in Journal of Physics, A, Math & Theoretical, while the third is under review. The work has been presented at a colloquium at Yeshiva University, and will be presented in the summer of 2010 at the Fourth International Conference 'Frontiers of Nonlinear Physics', Nizhny, Novgorod, Russia (http://www.fnp.sci-nnov.ru/ ). This work lends itself to substantial future developments.

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# Fractional standard map

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ABSTRACT

Properties of the phase space of the standard map with memory are investigated. This map was obtained from a kicked fractional differential equation. Depending on the value of the map parameter and the fractional order of the derivative in the original differential equation, this nonlinear dynamical system demonstrates attractors (fixed points, stable periodic trajectories, slow converging and slow diverging trajectories, ballistic trajectories, and fractal-like structures) and/or chaotic trajectories. At least one type of fractal-like sticky attractors in the chaotic sea was observed.



## 1. Introduction

The standard map (SM) can be derived from the differential equation describing kicked rotator. The description of many physical systems and effects (Fermi acceleration, comet dynamics, etc.) can be reduced to the studying of the SM [1]. The SM provides the simplest model of the universal generic area preserving map and it is one of the most widely studied maps. The topics examined include fixed points, elementary structures of islands and a chaotic sea, and fractional kinetics [1–3].

It was recently realized that many physical systems, including systems of oscillators with long range interaction [4,5], non-Markovian systems with memory ([6, Chapter 10], [7–11]), fractal media [12], etc., can be described by the fractional differential equations (FDE) [6,13,14]. As with the usual differential equations, the reduction of FDEs to the corresponding maps can provide a valuable tool for the analysis of the properties of the original systems. As in the case of the SM, the fractional standard map (FSM), derived in [15] from the fractional differential equation describing a kicked system, is perhaps the best candidate to start a general investigation of the properties of maps which can be obtained from FDEs.

As it was shown in [15], maps that can be derived from FDEs are of the type of discrete maps with memory. One-dimensional maps with memory, in which the present state of evolution depends on all past states, were studied previously in [16–21]. They were not derived from differential equations. Most results were obtained for the generalizations of the logistic map.

In the physical systems the transition from integer order time derivatives to fractional (of a lesser order) introduces additional damping and is similar in appearance to additional friction [6,22]. Accordingly, in the case of the FSM we may expect transformation of the islands of stability and the accelerator mode islands into attractors (points, attracting trajectories, strange attractors). Because the damping in systems with fractional derivatives is based on the internal causes different from the external forces of friction [22, 23], the corresponding attractors are also different from the attractors of the regular systems with friction and are called fractional attractors [22]. Even in one-dimensional cases [16–21] most of the results were obtained numerically. An additional dimension makes the problem even more complex and most of the results in the present Letter were obtained numerically.

## 2. FSM, initial conditions

The standard map in the form

$$p_{n+1} = p_n - K \sin x_n,$$
$$x_{n+1} = x_n + p_{n+1} \pmod{2\pi} \quad (1)$$

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can be derived from the differential equation

$$\ddot{x} + K\sin(x)\sum_{n=0}^{\infty}\delta\left(\frac{t}{T} - n\right) = 0. \tag{2}$$

By replacing the second-order time derivative in Eq. (2) with the Riemann–Liouville derivative ${}_0D_t^\alpha$ one obtains a fractional equation of motion in the form

$${}_0D_t^\alpha x + K\sin(x)\sum_{n=0}^{\infty}\delta\left(\frac{t}{T} - n\right) = 0 \quad (1 < \alpha \leqslant 2), \tag{3}$$

where

$$\begin{aligned}{}_0D_t^\alpha x(t) &= D_t^m\, {}_0I_t^{m-\alpha}x(t)\\ &= \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t \frac{x(\tau)\,d\tau}{(t-\tau)^{\alpha-m+1}} \quad (m-1 < \alpha \leqslant m),\end{aligned} \tag{4}$$

$D_t^m = d^m/dt^m$, and ${}_0I_t^\alpha$ is a fractional integral. The initial conditions for (3) are

$$\begin{aligned}&\left({}_0D_t^{\alpha-1}x\right)(0+) = p_1,\\ &\left({}_0D_t^{\alpha-2}x\right)(0+) = b.\end{aligned} \tag{5}$$

The Cauchy type problem (3) and (5) is equivalent to the Volterra integral equation of the second kind [24–26]

$$\begin{aligned}x(t) &= \frac{p_1}{\Gamma(\alpha)}t^{\alpha-1} + \frac{b}{\Gamma(\alpha-1)}t^{\alpha-2}\\ &\quad - \frac{K}{\Gamma(\alpha)}\int_0^t \frac{\sin[x(\tau)]\sum_{n=0}^{\infty}\delta(\frac{\tau}{T} - n)\,d\tau}{(t-\tau)^{1-\alpha}}.\end{aligned} \tag{6}$$

Defining the momentum as

$$p(t) = {}_0D_t^{\alpha-1}x(t), \tag{7}$$

and performing integration in (6) one can derive the equation for the FSM in the form (for the thorough derivation see [26])

$$p_{n+1} = p_n - K\sin x_n, \tag{8}$$

$$\begin{aligned}x_{n+1} &= \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n} p_{i+1}V_\alpha(n-i+1)\\ &\quad + \frac{b}{\Gamma(\alpha-1)}(n+1)^{\alpha-2} \pmod{2\pi},\end{aligned} \tag{9}$$

where

$$V_\alpha(m) = m^{\alpha-1} - (m-1)^{\alpha-1}. \tag{10}$$

Here it is assumed that $T = 1$ and $1 < \alpha \leqslant 2$. The form of Eq. (9) which provides a more clear correspondence with the SM ($\alpha = 2$) in the case $b = 0$ is presented in Section 4 (Eq. (31)).

The second initial condition in (5) can be written as

$$\begin{aligned}\left({}_0D_t^{\alpha-2}x\right)(0+) &= \lim_{t\to 0+} {}_0I_t^{2-\alpha}x(t)\\ &= \lim_{t\to 0+}\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{x(\tau)\,d\tau}{(t-\tau)^{\alpha-1}}\\ &= b \quad (1 < \alpha \leqslant 2),\end{aligned} \tag{11}$$

which requires $b = 0$ in order to have a solution bounded at $t = 0$ for $\alpha < 2$. The assumption $b = 0$ leads to the FSM equations which in the limiting case $\alpha = 2$ coincide with the equations for the standard map under the condition $x_0 = 0$.

In this Letter the FSM is taken in the form derived in [15] which coincides with (8) and (9) if $b = 0$. It is also assumed that $x_0 = 0$ and the results can be compared to those obtained for the SM with $x_0 = 0$ and arbitrary $p_0$. As a test, for the SM and for the FSM with $\alpha = 2$ and the same initial conditions numerical calculations show that phase portraits look identical.

System of Eqs. (8) and (9) can be considered either in a cylindrical phase space ($x$ mod $2\pi$) or in unbounded phase space. The second case is convenient to study transport. The trajectories in the second case are easily related to the first case. The FSM has no periodicity in $p$ (the SM does) and cannot be considered on a torus.

## 3. Stable fixed point

The SM has stable fixed points at $(0, 2\pi n)$ for $K < K_c = 4$. It is easy to see that point $(0, 0)$ is also a fixed point for the FSM. Direct computations using (8) and (9) demonstrate that for the small initial values of $p_0$ there is a clear transition from the convergence to the fixed point to divergence when the value of the parameter $K$ crosses the curve $K = K_c(\alpha)$ on Fig. 1(a) from smaller to larger values.

The following system describes the evolution of trajectories near fixed point $(0, 0)$

$$\delta p_{n+1} = \delta p_n - K\delta x_n, \tag{12}$$

$$\delta x_{n+1} = \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n}\delta p_{i+1}V_\alpha(n-i+1). \tag{13}$$

The solution can be found in the form

$$\delta p_n = p_0\sum_{i=0}^{n-1}p_{n,i}\left(\frac{2}{V_{\alpha l}}\right)^i\left(\frac{V_{\alpha l}K}{2\Gamma(\alpha)}\right)^i \quad (n > 0), \tag{14}$$

$$\delta x_n = \frac{p_0}{\Gamma(\alpha)}\sum_{i=0}^{n-1}x_{n,i}\left(\frac{2}{V_{\alpha l}}\right)^i\left(\frac{V_{\alpha l}K}{2\Gamma(\alpha)}\right)^i \quad (n > 0). \tag{15}$$

The origin of the terms in parentheses, as well as the definition

$$V_{\alpha l} = \sum_{k=1}^{\infty}(-1)^{k+1}V_\alpha(k) \tag{16}$$

will become clear in Section 5. Eqs. (12)–(16) lead to the following iterative relationships

$$x_{n+1,i} = -\sum_{m=i}^{n}(n-m+1)^{\alpha-1}x_{m,i-1} \quad (0 < i \leqslant n), \tag{17}$$

$$p_{n+1,i} = -\sum_{m=i}^{n}x_{m,i-1} \quad (0 < i < n) \tag{18}$$

with the initial and boundary conditions

$$\begin{aligned}&p_{n+1,n} = x_{n+1,n} = (-1)^n, \qquad p_{n+1,0} = 1,\\ &x_{n+1,0} = (n+1)^{\alpha-1}.\end{aligned} \tag{19}$$

From (17) and (18) it is clear that the series (14) and (15) are alternating and it is natural to apply the Dirichlet's test to verify their convergence. This can be done by considering the totals

$$S_n = \sum_{i=0}^{n-1}x_{n,i}\left(\frac{2}{V_{\alpha l}}\right)^i, \tag{20}$$

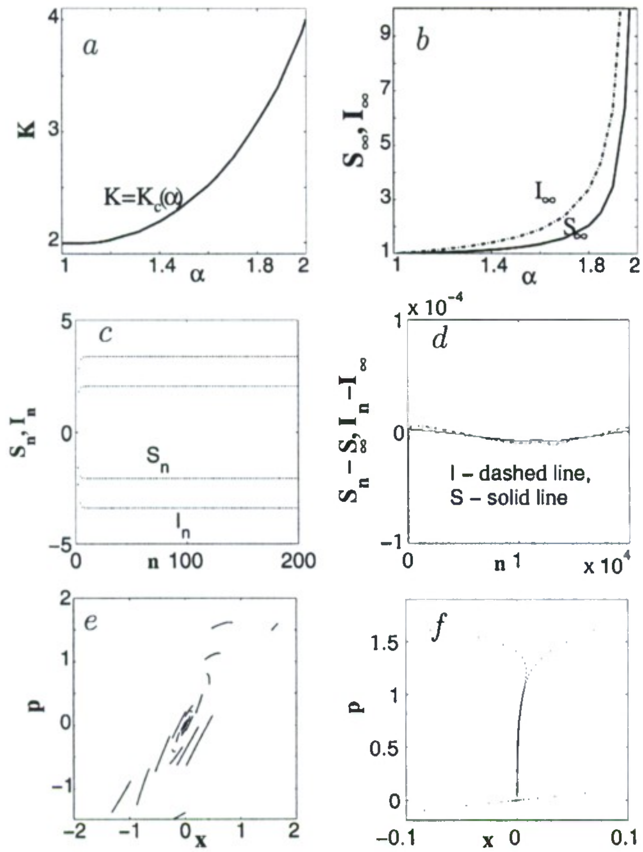


**Fig. 1.** Stability of the fixed point (0, 0): (a) The fixed point is stable below the curve $K = K_c(\alpha)$; (b) Values of $S_\infty$ and $I_\infty$ obtained after 20000 iterations of Eqs. (22) and (23). As $\alpha \to 2$ the values $S_\infty$ and $I_\infty$ increase rapidly. For $\alpha = 1.999$, $S_\infty \approx 276$ and $I_\infty \approx 552$ after 20000 iterations; (c) An example of the typical evolution of $S_\infty$ and $I_\infty$ over the first 200 iterations for $1 < \alpha < 2$. This particular figure corresponds to $\alpha = 1.8$; (d) Deviation of the values $S_n$ and $I_n$ from the values $S_\infty \approx 2.04337$ and $I_\infty \approx 3.37416$ for $\alpha = 1.8$ during the first 20000 iterations (this type of behavior remains for $1 < \alpha < 2$); (e) Evolution of trajectories with $p_0 = 1.5 + 0.0005i$, $0 \leqslant i < 200$ for the case $K = 3$, $\alpha = 1.9$. The line segments correspond to the nth iteration on the set of trajectories with close initial conditions. The evolution of the trajectories with smaller $p_0$ is similar; (f) $10^5$ iterations on both of two trajectories for $K = 2$, $\alpha = 1.4$. The one at the bottom with $p_0 = 0.3$ is a fast converging trajectory. The upper trajectory with $p_0 = 5.3$ is an example of the ASCT in which $p_{100000} \approx 0.042$.

$$I_n = \sum_{i=0}^{n-1} p_{n,i}\left(\frac{2}{V_{\alpha l}}\right)^i. \tag{21}$$

They obey the following iterative rules

$$S_n = n^{\alpha-1} - \frac{2}{V_{\alpha l}}\sum_{i=1}^{n-1}(n-i)^{\alpha-1}S_i, \quad S_1 = 1, \tag{22}$$

$$I_n = 1 - \frac{2}{V_{\alpha l}}\sum_{i=1}^{n-1}S_i. \tag{23}$$

Computer simulations show that values of $S_n$ and $I_n$ converge to the values $(-1)^{n+1}S_\infty$ and $(-1)^{n+1}I_\infty$ depicted on Fig. 1(b). Figs. 1(c), (d) show an example of the typical evolution of $S_n$ and $I_n$ over the first 20000 iterations. It means that the condition of convergence of $\delta p_n$ and $\delta x_n$ is

$$\frac{V_{\alpha l}K}{2\Gamma(\alpha)} < 1. \tag{24}$$

Numerical evaluation of the equality $K = 2\Gamma(\alpha)/V_{\alpha l}$ perfectly reproduces the curve on Fig. 1(a) obtained by the direct computations of (8) and (9).

Because not only the stability problem (12) and (13), but also the original map (8) and (9), contains convolutions, the use of generating functions [27], which allows transformations of sums of products into products of sums, could be utilized in the investigation of the FSM and some other maps with memory. As an example, in the particular case of the stability problem (12) and (13), the introduction of the generating functions

$$\tilde{W}_\alpha(t) = \frac{K}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\left[(i+1)^{\alpha-1} - i^{\alpha-1}\right]t^i, \tag{25}$$

$$\tilde{X}(t) = \sum_{i=0}^{\infty}\delta x_i t^i, \tag{26}$$

$$\tilde{P}(t) = \sum_{i=0}^{\infty}\delta p_i t^i, \tag{27}$$

leads to

$$\tilde{X}(t) = \frac{p_0\tilde{W}_\alpha(t)}{K}\frac{t}{1 - t(1 - \tilde{W}_\alpha(t))}, \tag{28}$$

$$\tilde{P}(t) = p_0\frac{1 + \tilde{W}_\alpha(t)}{1 - t(1 - \tilde{W}_\alpha(t))}. \tag{29}$$

Now the original problem is reduced to the problem of the asymptotic behavior at $t = 0$ of the derivatives of the analytic functions $\tilde{X}(t)$ and $\tilde{P}(t)$, which is still quite complex and is not considered in this Letter.

In the region of the parameter space where the fixed point is stable, the fixed point is surrounded by a finite basin of attraction, whose width $W$ depends on the values of $K$ and $\alpha$. For example, for $K = 3$ and $\alpha = 1.9$ the width of the basin of attraction is $1.6 < W < 1.7$. Simulations of thousands of trajectories with $p_0 < 1.6$ performed by the authors, of which only 200 (with $1.5 < p_0 < 1.6$) are presented in Fig. 1(e), show only converging trajectories, whereas among 200 trajectories with $1.6 < p_0 < 1.7$ in Fig. 2(a) there are trajectories converging to the fixed point as well as some trajectories converging to attracting slow diverging trajectories (ASDT), whose properties will be discussed in the following section. Trajectories in Fig. 1(e) converge very rapidly. In the case $K = 2$ and $\alpha = 1.4$ in addition to the trajectories which converge rapidly and ASDTs there exist attracting slow converging trajectories (ASCT) (Fig. 1(f)).

## 4. Attracting slow diverging trajectories (ASDT)

As it can be seen from Fig. 2(a), the phase portrait on a cylinder of the FSM with $K = 3$ and $\alpha = 1.9$ contains only one fixed point and ASDTs approximately equally spaced along the $p$-axis. This result corresponds to the fact that the standard map with $K = 3$ has only one central island. More complex structure of the standard map's phase space for smaller values of $K$ (for example for $K = 2$ and $K = 0.6$) can explain more complex structure of the FSM's phase space, where periodic attracting trajectories with periods $T = 4$ (Fig. 2(b)), $T = 2$, and $T = 3$ (Fig. 2(c)) are present.

Each ASDT has its own basin of attraction (see Fig. 2(d)). Between those basins two initially close trajectories at first diverge, but then converge to the same or different fixed point or ASDT.

Numerical evaluation shows that for ASDTs which converge to trajectories along the $p$-axis ($x \to x_{\lim} = 0$) in the area of stability (which is the same as for the stability of the fixed point) the following holds (for large n see Fig. 3(a))

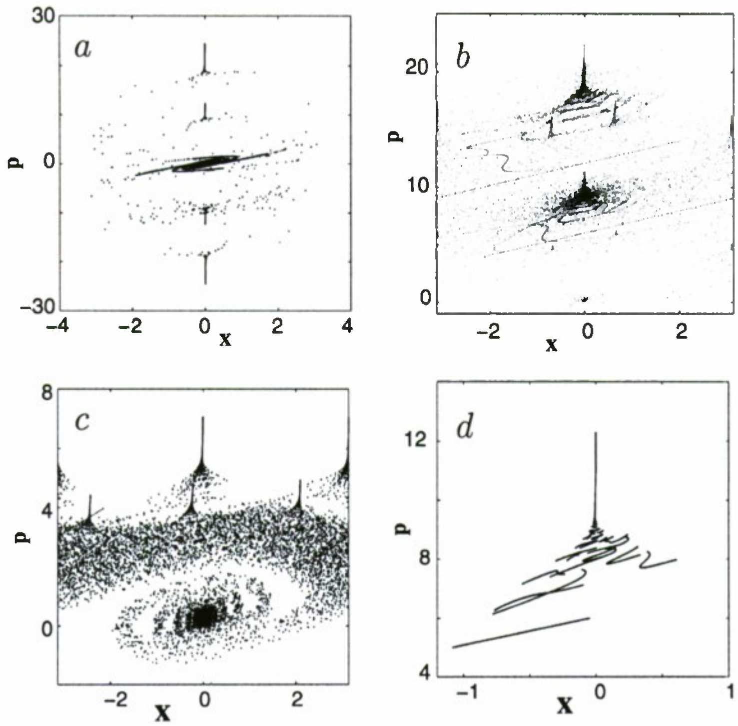


**Fig. 2.** Phase space with ASDTs: (a) The same values of parameters as in Fig. 1(e) but $p_0 = 1.6 + 0.0005i$; (b) 200 iterations on trajectories with $p_0 = 4 + 0.02i$, $0 \leqslant i < 500$ for the case $K = 2$, $\alpha = 1.9$. Trajectories converging to the fixed point, ASDTs with $x = 0$, and period 4 attracting trajectories are present; (c) 2000 iterations on trajectories with $p_0 = 2 + 0.04i$, $0 \leqslant i < 50$ for the case $K = 0.6$, $\alpha = 1.9$. Trajectories converging to the fixed point, period 2 and 3 attracting trajectories are present; (d) The same values of parameters as in Fig. 1(e) but $p_0 = 5 + 0.005i$.

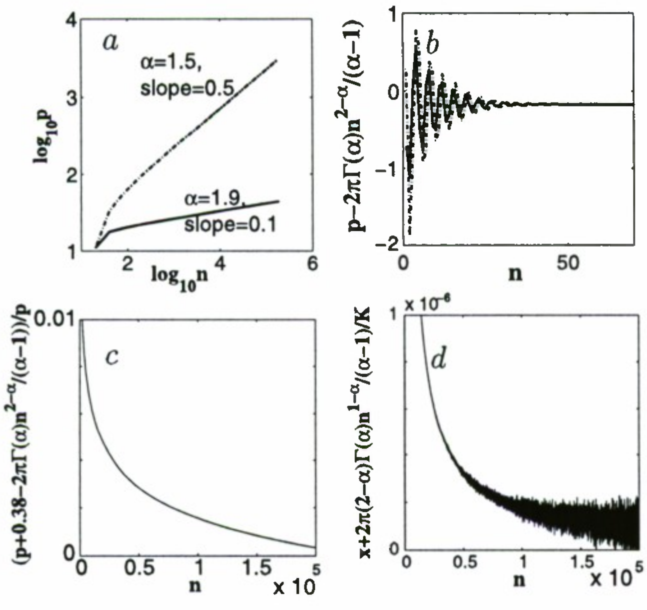


**Fig. 3.** Evaluation of the behavior of the ASDTs: (a) Momenta for two ASDTs with $x_n \approx 2\pi n$ in the unbounded space (in this example $K = 2$). The solid line is related to a trajectory with $\alpha = 1.9$ and its slope is 0.1. The dashed line corresponds to a trajectory with $\alpha = 1.5$ and its slope is 0.5; (b) Deviation of momenta from the asymptotic formula for two ASDTs with $x_n \approx 2\pi n$ in the unbounded space, $\alpha = 1.9$, and $K = 2$. The dashed line has $p_0 = 7$ and the solid one $p_0 = 6$; (c) Relative deviation of the momenta for the trajectories in (b) from the asymptotic formula; (d) Deviation of the $x$-coordinates for the trajectories in (b) from the asymptotic formula.

$$p_n = Cn^{2-\alpha}. \tag{30}$$

The constant C can be easily evaluated for $1.8 < \alpha < 2$. Consider an ASDT with $x_{lim} = 0$, $T = 1$, and $2\pi M$, where $M$ is an integer, constant step in $x$ in the unbounded space. Then Eq. (9) with $b = 0$ gives

$$x_{n+1} - x_n = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n} (p_{k+1} - p_k) V_\alpha(n-k+1) + \frac{p_1}{\Gamma(\alpha)} V_\alpha(n+1). \tag{31}$$

For large $n$ the last term is small ($\sim n^{\alpha-2}$) and the following holds

$$\sum_{k=1}^{n} (p_{k+1} - p_k) V_\alpha(n-k+1) = 2\pi M \Gamma(\alpha). \tag{32}$$

With the assumption $p_n \sim n^{2-\alpha}$ it can be shown that for values of $\alpha > 1.8$ considered the terms in the last sum with large $k$ are small and in the series representation of $V_\alpha(n-k+1)$ it is possible to keep only terms of the highest order in $k/n$. Thus, (32) leads to the approximations

$$p_n \approx p_0 + \frac{2\pi M \Gamma(\alpha) n^{2-\alpha}}{\alpha - 1}, \tag{33}$$

$$x_n \approx -\frac{2\pi M (2-\alpha) \Gamma(\alpha)}{K(\alpha-1) n^{\alpha-1}}. \tag{34}$$

In the case $K = 2$, $\alpha = 1.9$ Figs. 3(b)–(d) show two trajectories with $M = 1$ (initial momenta $p_0 = 6$ and $p_0 = 7$) approaching an ASDT: the deviation from the asymptotic (33) and (34) and the relative difference with respect to (33).

## 5. Period 2 stable trajectory

The SM has two stable points of the period $T = 2$ trajectory for $4 < K < 2\pi$ with the property

$$p_{n+1} = -p_n, \qquad x_{n+1} = -x_n. \tag{35}$$

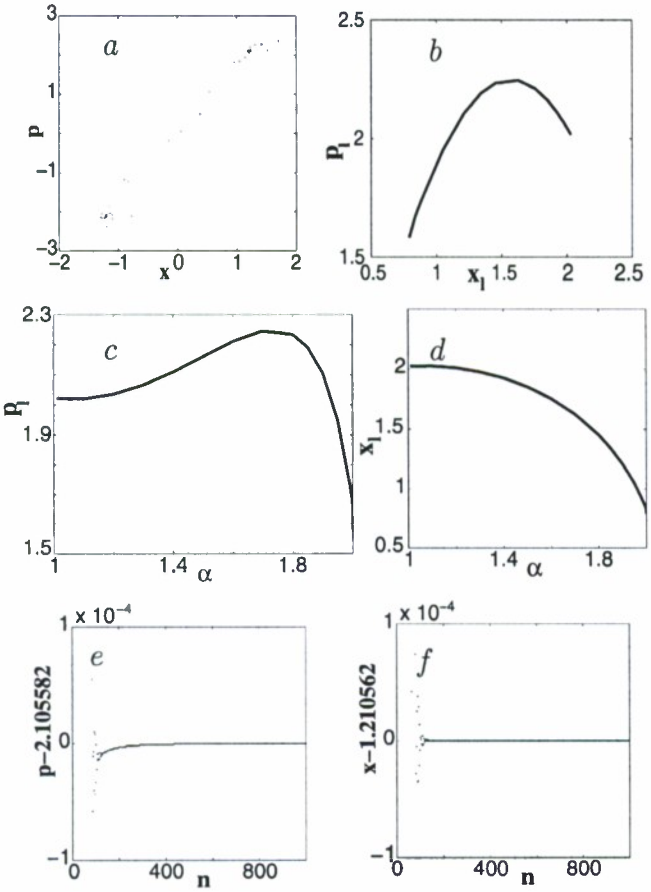


**Fig. 4.** Period 2 stable trajectory: (a) An example of $T = 2$ attractor for $K = 4.5$, $\alpha = 1.9$. One trajectory with $x_0 = 0$, $p_0 = 0.513$; (b) $p_l$ of $x_l$ for the case of $K = 4.5$; (c) $p_l$ of $\alpha$ for the case of $K = 4.5$; (d) $x_l$ of $\alpha$ for the case of $K = 4.5$; (e) $p_n - p_l$ for the trajectory in (a). After 1000 iterations $|p_n - p_l| < 10^{-7}$; (f) $x_n - x_l$ for the trajectory in (a). After 1000 iterations $|x_n - x_l| < 10^{-7}$.

The same points persist in the numerical experiments for the FSM (Fig. 4(a)). These points are attracting most of the trajectories with small $p_0$. Assuming the existence of a $T = 2$ attracting trajectory, it is possible to calculate the coordinates of its attracting points $(x_l, p_l)$ and $(-x_l, -p_l)$. In this case from (8) and (9)

$$p_l = \frac{K}{2}\sin(x_l), \tag{36}$$

$$x_l = \frac{K}{2\Gamma(\alpha)}\sin(x_l)\sum_{k=1}^{\infty}(-1)^{k+1}V_\alpha(k). \tag{37}$$

Finally, the equation for $x_l$ takes the form

$$x_l = \frac{K}{2\Gamma(\alpha)}V_{\alpha l}\sin(x_l), \tag{38}$$

where

$$V_{\alpha l} = \sum_{k=1}^{\infty}(-1)^{k+1}V_\alpha(k) \tag{39}$$

and can be easily calculated numerically. From (38) the condition of the existence of $T = 2$ trajectory

$$K > K_c(\alpha) = \frac{2\Gamma(\alpha)}{V_{\alpha l}}, \tag{40}$$

is exactly opposite to (24). It is satisfied above the curve $K = K_c(\alpha)$ on Fig. 1(a). For $\alpha = 2$ (40) produces the well-known condition

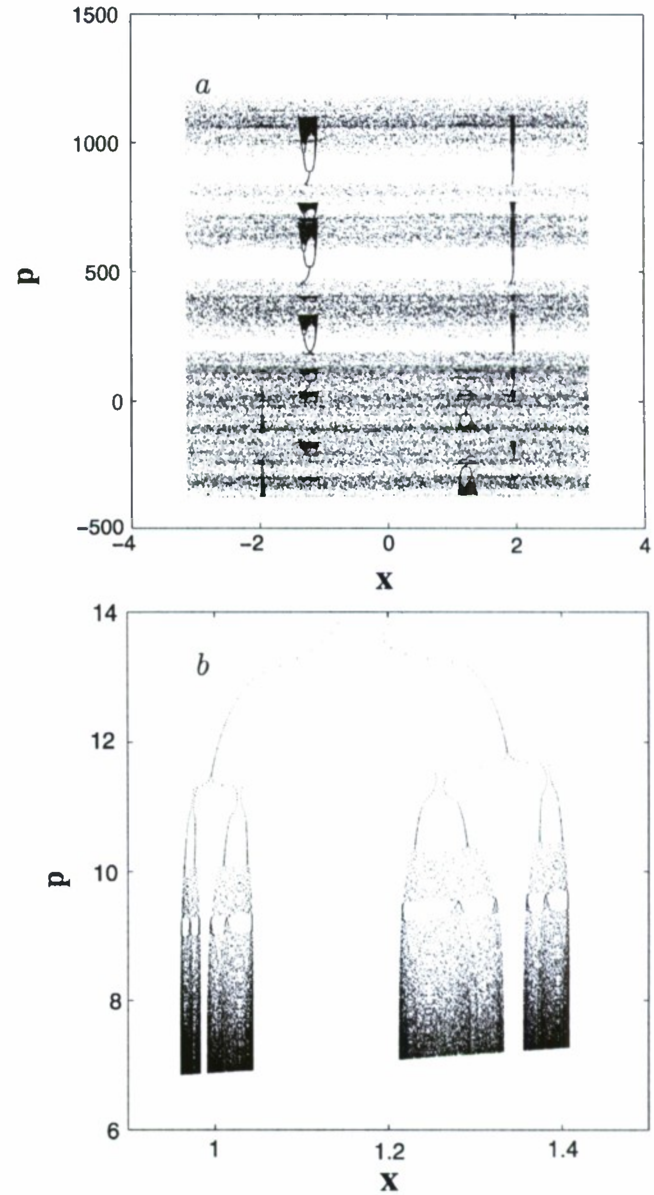


**Fig. 5.** Cascade of bifurcation type trajectories: (a) 120000 iterations on a single trajectory with $K = 4.5$, $\alpha = 1.65$, $p_0 = 0.3$. The trajectory occasionally sticks to a CBTT but then always recovers into the chaotic sea; (b) 100000 iterations on a single trajectory with $K = 3.5$, $\alpha = 1.1$, $p_0 = 20$. The trajectory very fast turns into a CBTT which slowly converges to a fractal type area.

$K > 4$ for the SM. The results of calculations of the $x_l$ and $p_l$ for the cases $K = 4.5$, $1 < \alpha < 2$ presented in Figs. 4(b)–(d) perfectly coincide with the results of the direct computations of (8) and (9) with $b = 0$. After 1000 iterations presented in Figs. 4(e), (f) the values of deviations $|p_n - p_l|$ and $|x_n - x_l|$ are less than $10^{-7}$.

## 6. Cascade of bifurcations type trajectories (CBTT)

Period 2 stable trajectories have limited basins of attraction. Trajectories that don't fall into those areas reveal a diverse variety of properties, from period two slow attracting trajectories to fractal type attractors and cascade of bifurcations type trajectories (CBTT). Fig. 5(a) presents a single chaotic trajectory which sticks to the areas similar to the cascade of bifurcations which are well known for the logistic map. In Fig. 5(b) a single trajectory falls very rapidly into one of the attracting CBTTs. Because the bifurcation di-

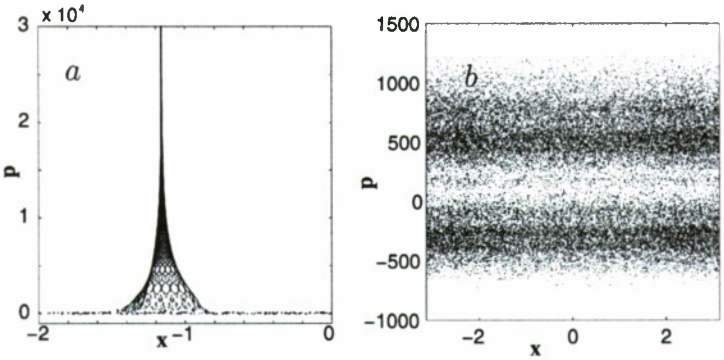


**Fig. 6.** Examples of phase space for $K > 2\pi$: (a) An attracting ballistic trajectory with $K = 6.908745$, $\alpha = 1.999$, $p_0 = 0.7$; (b) A chaotic trajectory for $K = 6.908745$, $\alpha = 1.9$.

agram of the logistic map has fractal properties (see for example Chapter 2 in [28]), it is expected that the structure to which this trajectory slowly converges also possesses fractal features.

The properties of this type of attractors, as well as the properties different types of observed during computer simulations chaotic, attracting, and ballistic trajectories for $K > 2\pi$ (see Fig. 6) will be considered in the subsequent article.

## 7. Fractional attractors and their stability

The problems of existence and stability of the fractional attractors for the systems described by the FDEs were addressed in a few recent papers. It was noticed in [22] that the properties of the fractional chaotic attractors are different from the properties of the "regular" chaotic attractors and may have some pseudochaotic features. The problem of existence of multi-scroll fractional chaotic attractors was considered in [29]. The problem of stability of the stationary solutions (fixed points for ODEs) of systems described by the fractional ODEs and PDEs was considered in [30–32]. In the above mentioned articles the equations contained the Caputo fractional derivatives, whereas in the present Letter the Riemann–Liouville fractional derivative is used. This fact does not allow a direct comparison of the results. The results [22,29–32] were supported by a relatively small number of computations and this is understandable, taking into account all the difficulties of performing numerical simulations for the equations with fractional derivatives.

The use of the FSM, which is equivalent to the original FDE, allows performing thousands of runs of simulations of the kicked fractional system with two parameters: $K$ and $\alpha$. The FSM also allows making some analytic deductions and revealing some properties of the fractional attractors which were not reported before:

(a) The stability of the fixed point (0, 0) of the FSM is different not only from the stability of the fixed point in the domain of the regular motion (zero Lyapunov exponent) of the SM, but also from the stability of fixed attracting points of the regular (not fractional) dissipative systems like, for example, the dissipative standard map (Zaslavsky map) [33]. The difference is in the way in which trajectories approach the attracting point. In the FSM this way depends on the initial conditions. For example, in Fig. 1(f) there are two trajectories approaching the same fixed point: one is fast spiraling into the attractor and the other is slowly converging.

(b) Stable period 2 attracting trajectories exist only in the asymptotic sense—they do not represent any real periodic solutions. If the initial condition is chosen in a period two stable attracting point, this trajectory will immediately jump out of this point and where it will end depends on the values of $K$ and $\alpha$.

(c) All the FSM attractors exist in the sense that there are trajectories which converge into those attractors. But if an initial condition is taken on any of the attracting trajectories (except for the fixed point), they will most likely not evolve along the same trajectory.

## 8. Conclusion

In this Letter properties of the phase space of the FSM were investigated. It was shown that islands of regular motion of the SM in the FSM turn into attractors (points, attracting trajectories, and fractal-like structures). Properties of the attracting fixed points, period two trajectories, ASCTs, and ASDTs were considered. This consideration allows the description of the evolution of the dynamical variable $x$ of the original fractional dynamical system, a system described by the FDE reducible to the FSM.

The explanation of the CBTTs, which are interesting phenomena, requires further detailed investigation. Chaotic trajectories that spend some time near CBTTs, which can be called "sticky attractors" in analogy to "sticky islands" of the SM, are good candidates for the investigation of anomalous diffusion. Phase space transport was not considered in this Letter. How general the properties of the phase space of the FSM are will become clear after further investigations of different fractional maps, maps with memory which can be derived from the FDEs, and particular those suggested in [15], will be conducted. The fact that so many physical systems can be reduced to studying of the SM gives a hope that those physical systems which can be reduced to studying the FSM will be found.

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We express our gratitude to H. Weitzner for many comments and helpful discussions. The authors thank A. Kheyfits for suggesting the use of generating functions to solve the FSM fixed point stability problem. This work was supported by the Office of Naval Research, Grant No. N00014-08-1-0121.

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Differential equations with fractional derivative and universal map with memory



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# Differential equations with fractional derivative and universal map with memory

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**Abstract**
Discrete maps with long-term memory are obtained from nonlinear differential equations with Riemann–Liouville and Caputo fractional derivatives. These maps are generalizations of the well-known universal map. The memory means that their present state is determined by all past states with special forms of weights. To obtain discrete maps from fractional differential equations, we use the equivalence of the Cauchy-type problems and to the nonlinear Volterra integral equations of the second kind. General forms of the universal maps with memory, which take into account general initial conditions for the cases of the Riemann–Liouville and Caputo fractional derivative, are suggested.

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Mathematics Subject Classification: 26A33, 37E05

## 1. Introduction

A dynamical system consists of a set of possible states, together with a rule that determines the present state in terms of past states. If we require that the rule be deterministic, then we can define the present state uniquely from the past states. A discrete-time system without memory takes the current state as input and updates the situation by producing a new state as output. All physical classical models are described by differential or integro-differential equations, and the discrete-time systems can be considered as a simplified version of these equations. A discrete form of the time evolution equation is called the map. Maps are important because they encode the behavior of deterministic systems. The assumption of determinism is that the output of the map can be uniquely determined from the input. In general, the present state is uniquely determined by all past states, and we have a discrete map with memory. Discrete maps are used for the study of evolution problems, possibly as a substitute of differential equations [1–3]. They lead to a much simpler formalism, which is particularly useful in simulations. The universal discrete map is one of the most widely studied maps. In this paper, we consider discrete maps with memory that can be used to study solutions of fractional differential equations [4–7].

The nonlinear dynamics can be considered in terms of discrete maps. It is a very important step in understanding the qualitative behavior of systems described by differential equations. The derivatives of non-integer orders are a generalization of the ordinary differentiation of integer order. Fractional differentiation with respect to time is characterized by long-term memory effects. The discrete maps with memory are considered in [8–14]. The interesting question is a connection of fractional differential equations and discrete maps with memory. It is important to derive discrete maps with memory from the equation of motion.

In [14], we prove that the discrete maps with memory can be obtained from differential equations with fractional derivatives. The fractional generalization of the universal map was obtained [14] from a differential equation with Riemann–Liouville fractional derivatives. The Riemann–Liouville derivative has some notable disadvantages such as the hyper-singular improper integral, where the order of singularity is higher than the dimension, and nonzero of the fractional derivative of constants, which would entail that dissipation does not vanish for a system in equilibrium. The desire to formulate initial value problems for mechanical systems leads to the use of Caputo fractional derivatives rather than the Riemann–Liouville fractional derivative.

It is possible to state that the Caputo fractional derivatives allow us to give more clear mechanical interpretation. At the same time, we cannot state that the Riemann–Liouville fractional derivative does not have a physical interpretation and that it shows unphysical behavior. Physical interpretations of the Riemann–Liouville fractional derivatives are more complicated than Caputo fractional derivatives. But the Riemann–Liouville fractional derivatives naturally appear for real physical systems in electrodynamics. We note that the dielectric susceptibility of a wide class of dielectric materials follows, over extended frequency ranges, a fractional power-law frequency dependence that is called the 'universal' response [15, 16]. As was proved in [17, 18], the electromagnetic fields in such dielectric media are described by differential equations with Riemann–Liouville fractional time derivatives. These fractional equations for 'universal' electromagnetic waves in dielectric media are common to a wide class of materials, regardless of the type of physical structure, chemical composition, or of the nature of the polarizing species. Therefore, we cannot state that Riemann–Liouville fractional time derivatives do not have a physical interpretation. The physical interpretation of these derivatives in electrodynamics is connected with the frequency dependence of the dielectric susceptibility. As a result, the discrete maps with memory that are connected with differential equations with Riemann–Liouville fractional derivatives are very important to physical applications, and these derivatives naturally appear for real physical systems.

For computer simulation and physical application, it is very important to take into account the initial conditions for discrete maps with memory that are obtained from differential equations with Riemann–Liouville fractional time derivatives. In [14], these conditions are not considered. In this paper to take into account the initial condition, we use the equivalence of the differential equation with Riemann–Liouville and Caputo fractional derivatives and the Volterra integral equation. This approach is more general than the auxiliary variable method that is used in [14]. The proof of the result for Riemann–Liouville fractional derivatives is more complicated in comparison with the results for the Caputo fractional derivative. In this paper, we prove that the discrete maps with memory can be obtained from differential equations with the Caputo fractional derivative. The fractional generalization of the universal map is obtained from a fractional differential equation with Caputo derivatives.

The universal maps with memory are obtained by using the equivalence of the fractional differential equation and the Volterra integral equation. We reduce the Cauchy-type problem for the differential equations with the Caputo and Riemann–Liouville fractional derivatives to nonlinear Volterra integral equations of second kind. The equivalence of this Cauchy-type problem for the fractional equations with the Caputo derivative and the correspondent Volterra integral equation was proved by Kilbas and Marzan in [19, 20]. We also use that the Cauchy-type problem for the differential equations with the Riemann–Liouville fractional derivative can be reduced to a Volterra integral equation. The equivalence of this Cauchy-type problem and the correspondent Volterra equation was proved by Kilbas, Bonilla and Trujillo in [21, 22].

In section 2, differential equations with integer derivative and universal maps without memory are considered to fix notations and provide convenient references. In section 3, fractional differential equations with the Riemann–Liouville derivative and universal maps with memory are discussed. In section 4, the difference between the Caputo and Riemann–Liouville fractional derivatives is discussed. In section 5, fractional differential equations with the Caputo derivative and correspondent discrete maps with memory are considered. A fractional generalization of the universal map is obtained from kicked differential equations with the Caputo fractional derivative of order $1 < \alpha \leqslant 2$. The usual universal map is a special case of the universal map with memory. Finally, a short conclusion is given in section 6.

## 2. Integer derivative and universal map without memory

In this section, differential equations with derivative of integer order and the universal map without memory are considered to fix notations and provide convenient references.

Let us consider the equation of motion

$$D_t^2 x(t) + K G[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right) = 0 \tag{1}$$

in which perturbation is a periodic sequence of delta-function-type pulses (kicks) following with period $T = 2\pi/\nu$, $K$ is an amplitude of the pulses, $D_t^2 = d^2/dt^2$, and $G[x]$ is some real-valued function. It is well known that this differential equation can be represented in the form of the discrete map

$$x_{n+1} - x_n = p_{n+1} T, \qquad p_{n+1} - p_n = -KT\, G[x_n]. \tag{2}$$

Equations (2) are called the universal map. For details, see for example [1–3].

The traditional method of derivation of the universal map equations from the differential equations is considered in section 5.1 of [2]. We use another method of derivation of these equations to fix notations and provide convenient references. We obtain the universal map by using the equivalence of the differential equation and the Volterra integral equation.

**Proposition 1.** *The Cauchy-type problem for the differential equations*

$$D_t^1 x(t) = p(t), \tag{3}$$

$$D_t^1 p(t) = -K G[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right) \tag{4}$$

*with the initial conditions*

$$x(0) = x_0, \qquad p(0) = p_0 \tag{5}$$

*is equivalent to the universal map equations of the form*

$$x_{n+1} = x_0 + p_0(n+1)T - KT^2 \sum_{k=1}^{n} G[x_k](n+1-k), \tag{6}$$

$$p_{n+1} = p_0 - KT \sum_{k=1}^{n} G[x_k]. \tag{7}$$

**Proof.** Consider the nonlinear differential equation of second order

$$D_t^2 x(t) = G[t, x(t)], \qquad (0 \leqslant t \leqslant t_f) \tag{8}$$

on a finite interval $[0, t_f]$ of the real axis, with the initial conditions

$$x(0) = x_0, \qquad \left(D_t^1 x\right)(0) = p_0. \tag{9}$$

The Cauchy-type problem of the form (8), (9) is equivalent to the Volterra integral equation of second kind

$$x(t) = x_0 + p_0 t + \int_0^t \mathrm{d}\tau\, G[\tau, x(\tau)](t-\tau). \tag{10}$$

Using the function

$$G[t, x(t)] = -KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right),$$

for $nT < t < (n+1)T$, we obtain

$$x(t) = x_0 + p_0 t - KT \sum_{k=1}^{n} G[x(kT)](t - kT). \tag{11}$$

For the momentum $p(t) = D_t^1 x(t)$, equation (11) gives

$$p(t) = p_0 - KT \sum_{k=1}^{n} G[x(kT)]. \tag{12}$$

The solution of the left side of the $(n+1)$th kick

$$x_{n+1} = x(t_{n+1} - 0) = \lim_{\varepsilon \to 0+} x(T(n+1) - \varepsilon), \tag{13}$$

$$p_{n+1} = p(t_{n+1} - 0) = \lim_{\varepsilon \to 0+} p(T(n+1) - \varepsilon), \tag{14}$$

where $t_{n+1} = (n+1)T$, has the form (6) and (7).

This ends the proof. □

**Remark 1.** We note that equations (6) and (7) can be rewritten in the form (2). Using equations (6) and (7), the differences $x_{n+1} - x_n$ and $p_{n+1} - p_n$ give equations (2) of the universal map.

**Remark 2.** If $G[x] = -x$, then equations (2) give the Anosov-type system

$$x_{n+1} - x_n = p_{n+1} T, \qquad p_{n+1} - p_n = KT x_n. \tag{15}$$

For $G[x] = \sin(x)$, equations (2) are

$$x_{n+1} - x_n = p_{n+1} T, \qquad p_{n+1} - p_n = -KT \sin(x_n). \tag{16}$$

This map is known as the standard or Chirikov–Taylor map [1].

## 3. Riemann–Liouville fractional derivative and universal map with memory

In this section, we discuss nonlinear differential equations with the left-sided Riemann–Liouville fractional derivative ${}_0D^{\alpha}_t$ defined for $\alpha > 0$ by

$$ {}_0D^{\alpha}_t x(t) = D^n_t \, {}_0I^{n-\alpha}_t x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{x(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\alpha-n+1}}, \qquad (n-1 < \alpha \leqslant n), \tag{17} $$

where $D^n_t = \mathrm{d}^n/\mathrm{d}t^n$, and ${}_0I^{\alpha}_t$ is a fractional integration [4, 6, 7].

We consider the fractional differential equation

$$ {}_0D^{\alpha}_t x(t) = G[t, x(t)], \tag{18} $$

where $G[t, x(t)]$ is a real-valued function, $0 \leqslant n-1 < \alpha \leqslant n$, and $t > 0$, with the initial conditions

$$ \left({}_0D^{\alpha-k}_t x\right)(0+) = c_k, \qquad k = 1, \ldots, n. \tag{19} $$

The notation $\left({}_0D^{\alpha-k}_t x\right)(0+)$ means that the limit is taken at almost all points of the right-sided neighborhood $(0, 0+\varepsilon)$, $\varepsilon > 0$, of zero as follows:

$$ \left({}_0D^{\alpha-k}_t x\right)(0+) = \lim_{t\to 0+} {}_0D^{\alpha-k}_t x(t), \qquad (k = 1, \ldots, n-1), $$

$$ \left({}_0D^{\alpha-n}_t x\right)(0+) = \lim_{t\to 0+} {}_0I^{n-\alpha}_t x(t). $$

The Cauchy-type problem (18) and (19) can be reduced to the nonlinear Volterra integral equation of second kind

$$ x(t) = \sum_{k=1}^{n} \frac{c_k}{\Gamma(\alpha-k+1)} t^{\alpha-k} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{G[\tau, x(\tau)]\,\mathrm{d}\tau}{(t-\tau)^{1-\alpha}}, \tag{20} $$

where $t > 0$. The result was obtained by Kilbas, Bonilla and Trujillo in [21, 22]. For $\alpha = n = 2$, equation (20) gives (10).

The Cauchy-type problem (18) and (19) and the nonlinear Volterra integral equation (20) are equivalent in the sense that, if $x(t) \in L(0, t_f)$ satisfies one of these relations, then it also satisfies the other. In [21, 22] (see also theorem 3.1. in section 3.2.1 of [7]), this result is proved by assuming that the function $G[t, x]$ belongs to $L(0, t_f)$ for any $x \in W \subset \mathbb{R}$.

Let us give the basic theorem regarding the nonlinear differential equation involving the Riemann–Liouville fractional derivative.

**Kilbas–Bonilla–Trujillo theorem.** *Let $W$ be an open set in $\mathbb{R}$ and let $G[t, x]$, where $t \in (0, t_f]$ and $x \in W$, be a real-valued function such that $G[t, x] \in L(0, t_f)$ for any $x \in W$. Let $x(t)$ be a Lebesgue measurable function on $(0, t_f)$. If $x(t) \in L(0, t_f)$, then $x(t)$ satisfies almost everywhere equation (18) and conditions (19) if, and only if, $x(t)$ satisfies almost everywhere the integral equation (20).*

**Proof.** This theorem is proved in [21, 22] (see also theorem 3.1. in section 3.2.1 of [7]). □

In [14] we consider a fractional generalization of equation (1) of the form

$$ {}_0D^{\alpha}_t x(t) + K\, G[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right) = 0, \qquad (1 < \alpha \leqslant 2), \tag{21} $$

where $t > 0$, and ${}_0D^{\alpha}_t$ is the Riemann–Liouville fractional derivative defined by (17). Let us give the following theorem for equation (21).

**Proposition 2.** *The Cauchy-type problem for the fractional differential equation of the form (21) with the initial conditions*

$$\left({}_0D_t^{\alpha-1}x\right)(0+) = c_1, \qquad \left({}_0D_t^{\alpha-2}x\right)(0+) = \left({}_0I_t^{2-\alpha}x\right)(0+) = c_2 \tag{22}$$

*is equivalent to the equation*

$$x(t) = \frac{c_1}{\Gamma(\alpha)}t^{\alpha-1} + \frac{c_2}{\Gamma(\alpha-1)}t^{\alpha-2} - \frac{KT}{\Gamma(\alpha)}\sum_{k=1}^{n} G[x(kT)]\,(t-kT)^{\alpha-1}, \tag{23}$$

*where* $nT < t < (n+1)T$.

**Proof.** Using the function

$$G[t, x(t)] = -KG[x]\sum_{k=1}^{\infty}\delta\left(\frac{t}{T}-k\right), \tag{24}$$

equation (21) has the form of (18) with the Riemann–Liouville fractional derivative of order $\alpha$, where $1 < \alpha \leqslant 2$. It allows us to use the Kilbas–Bonilla–Trujillo theorem. As a result, equation (21) with initial conditions (19) of the form (22) is equivalent to the nonlinear Volterra integral equation

$$x(t) = \frac{c_1}{\Gamma(\alpha)}t^{\alpha-1} + \frac{c_2}{\Gamma(\alpha-1)}t^{\alpha-2} - \frac{K}{\Gamma(\alpha)}\sum_{k=1}^{\infty}\int_0^t \mathrm{d}\tau\, G[x(\tau)](t-\tau)^{\alpha-1}\,\delta\left(\frac{\tau}{T}-k\right), \tag{25}$$

where $t > 0$. If $nT < t < (n+1)T$, then the integration in (25) with respect to $\tau$ gives (23).
This ends the proof. □

To obtain equations of discrete map a momentum must be defined. There are two possibilities of defining the momentum:

$$p(t) = {}_0D_t^{\alpha-1}x(t), \qquad p(t) = D_t^1 x(t). \tag{26}$$

Let us use the first definition. Then the momentum is defined by the fractional derivative of order $\alpha - 1$. Using the definition of the Riemann–Liouville fractional derivative (17) in the form

$${}_0D_t^{\alpha}x(t) = D_t^2\, {}_0I_t^{2-\alpha}x(t), \qquad (1 < \alpha \leqslant 2), \tag{27}$$

we define the momentum

$$p(t) = {}_0D_t^{\alpha-1}x(t) = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{x(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\alpha-1}}, \qquad (1 < \alpha \leqslant 2), \tag{28}$$

where $x(\tau)$ is defined for $\tau \in (0, t)$. Then

$${}_0D_t^{\alpha}x(t) = D_t^1 p(t), \qquad (1 < \alpha \leqslant 2). \tag{29}$$

Using momentum $p(t)$ and coordinate $x(t)$, equation (21) can be represented in the Hamiltonian form

$${}_0D_t^{\alpha-1}x(t) = p(t), \tag{30}$$

$$D_t^1 p(t) = -KG[x(t)]\sum_{k=1}^{\infty}\delta\left(\frac{t}{T}-k\right). \tag{31}$$

**Proposition 3.** *The Cauchy-type problem for the fractional differential equations of the form (30) and (31) with the initial conditions*

$$\left({}_0D_t^{\alpha-1}x\right)(0+) = c_1, \qquad \left({}_0D_t^{\alpha-2}x\right)(0+) = \left({}_0I_t^{2-\alpha}x\right)(0+) = c_2 \tag{32}$$

*is equivalent to the discrete map equations*

$$x_{n+1} = \frac{c_1 T^{\alpha-1}}{\Gamma(\alpha)}(n+1)^{\alpha-1} + \frac{c_2 T^{\alpha-2}}{\Gamma(\alpha-1)}(n+1)^{\alpha-2} - \frac{KT^\alpha}{\Gamma(\alpha)} \sum_{k=1}^{n} G[x_k]\,(n+1-k)^{\alpha-1}, \tag{33}$$

$$p_{n+1} = c_1 - KT \sum_{k=1}^{n} G[x_k]. \tag{34}$$

**Proof.** We use proposition 2 to prove this statement. To obtain an equation for the momentum (28), we use the following fractional derivatives of power functions (see section 2.1 in [7]):

$${}_aD_t^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-1-\alpha}, \qquad \alpha \geqslant 0, \quad \beta > 0, \quad t > a, \tag{35}$$

$${}_0D_t^\alpha t^{\alpha-k} = 0, \qquad k = 1, \ldots, n, \quad n-1 < \alpha \leqslant n. \tag{36}$$

These equations give

$${}_0D_t^\alpha t^{\alpha-1} = \Gamma(\alpha), \qquad {}_0D_t^\alpha t^{\alpha-2} = 0$$

and

$${}_aD_t^\alpha (t-a)^{\alpha-1} = \Gamma(\alpha).$$

We note that equation (23) for $x(\tau)$ can be used only if $\tau \in (nT, t)$, where $nT < t < (n+1)T$. The function $x(\tau)$ in the fractional derivative ${}_0D_t^\alpha$ of the form (28) must be defined for all $\tau \in (0, t)$. We cannot take the derivative ${}_0D_t^\alpha$ of the functions $(\tau - kT)^{\alpha-1}$ that are defined for $\tau \in (kT, t)$. In order to use equation (23) on the interval $(0, t)$, we must modify the sum in equation (23) by using the Heaviside step function. Then equation (23) has the form

$$x(\tau) = \frac{c_1}{\Gamma(\alpha)}\tau^{\alpha-1} + \frac{c_2}{\Gamma(\alpha-1)}\tau^{\alpha-2} - \frac{KT}{\Gamma(\alpha)} \sum_{k=1}^{n} G[x(kT)](\tau-kT)^{\alpha-1}\theta(\tau-kT), \tag{37}$$

where $\tau \in (0, t)$. Using the relation

$${}_0D_t^\alpha((t-a)^{\alpha-1}\theta(t-a)) = {}_aD_t^\alpha (t-a)^{\alpha-1} = \Gamma(\alpha), \tag{38}$$

equations (28) and (37) give

$$p(t) = c_1 - KT \sum_{k=1}^{n} G[x(kT)], \tag{39}$$

where $nT < t < (n+1)T$. Then the solution of the left side of the $(n+1)$th kick

$$p_{n+1} = c_1 - KT \sum_{k=1}^{n} G[x_k]. \tag{40}$$

As a result, we obtain a universal map with memory in the form of equations (33) and (34).

This ends the proof. □

**Remark 3.** For $\alpha = n = 2$ equations (33) and (34) give the usual universal map (6) and (7).

**Remark 4.** We note that the map (33) and (34) with

$$c_1 = p_1, \qquad c_2 = 0$$

was obtained in [14] in the form

$$x_{n+1} = \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{n} p_{k+1} V_\alpha(n-k+1), \tag{41}$$

$$p_{n+1} = p_n - KTG(x_n), \qquad (1 < \alpha \leqslant 2), \tag{42}$$

where $p_1 = c_1$, and the function $V_\alpha(z)$ is defined by

$$V_\alpha(z) = z^{\alpha-1} - (z-1)^{\alpha-1}, \qquad (z \geqslant 1). \tag{43}$$

In [14], we obtain these map equations by using an auxiliary variable $\xi(t)$ such that

$${}_0^C D_t^{2-\alpha} \xi(t) = x(t).$$

The nonlinear Volterra integral equations and the general initial conditions (32) are not used in [14]. In the general case, the fractional differential equation of the kicked system (21) is equivalent to the discrete map equations

$$x_{n+1} = \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{n} p_{k+1} V_\alpha(n-k+1) + \frac{c_2 T^{\alpha-2}}{\Gamma(\alpha-1)}(n+1)^{\alpha-2}, \tag{44}$$

$$p_{n+1} = p_n - KTG(x_n), \qquad (1 < \alpha \leqslant 2), \tag{45}$$

where $p_1 = c_1$. Here we take into account the initial conditions (32). The second term of the right-hand side of equation (44) is not considered in [14]. Using $-1 < \alpha - 2 < 0$, we have

$$\lim_{n\to\infty} (n+1)^{\alpha-2} = 0.$$

Therefore, the case of large values of $n$ is equivalent to $c_2 = 0$.

Let us give the proposition regarding the second definition of the momentum $p(t) = D_t^1 x(t)$.

**Proposition 4.** *The Cauchy-type problem for the fractional differential equations*

$$D_t^1 x(t) = p(t), \tag{46}$$

$${}_0D_t^\alpha x(t) = -KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right), \qquad (1 < \alpha < 2) \tag{47}$$

*with the initial conditions*

$$\left({}_0D_t^{\alpha-1} x\right)(0+) = c_1, \qquad \left({}_0D_t^{\alpha-2} x\right)(0+) = \left({}_0I_t^{2-\alpha} x\right)(0+) = c_2 \tag{48}$$

*is equivalent to the discrete map equations*

$$x_{n+1} = \frac{c_1 T^{\alpha-1}}{\Gamma(\alpha)}(n+1)^{\alpha-1} + \frac{c_2 T^{\alpha-2}}{\Gamma(\alpha-1)}(n+1)^{\alpha-2} - \frac{KT^\alpha}{\Gamma(\alpha)} \sum_{k=1}^{n} G[x_k](n+1-k)^{\alpha-1}, \tag{49}$$

$$p_{n+1} = \frac{c_1 T^{\alpha-2}}{\Gamma(\alpha-1)}(n+1)^{\alpha-2} + \frac{c_2(\alpha-2)T^{\alpha-3}}{\Gamma(\alpha-1)}(n+1)^{\alpha-3} - \frac{KT^{\alpha-1}}{\Gamma(\alpha-1)} \sum_{k=1}^{n} G[x_k](n+1-k)^{\alpha-2}. \tag{50}$$

**Proof.** We define the momentum

$$p(t) = D_t^1 x(t).$$

If $nT < t < (n+1)T$, then the differentiation of (23) with respect to $t$ gives

$$p(t) = \frac{c_1}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{c_2(\alpha-2)}{\Gamma(\alpha-1)} t^{\alpha-3} - \frac{KT}{\Gamma(\alpha-1)} \sum_{k=1}^{n} G[x(kT)]\,(t-kT)^{\alpha-2}. \tag{51}$$

Here we use the relation

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1), \qquad (1 < \alpha \leqslant 2).$$

Using equations (23) and (51), we can obtain the solution of the left side of the $(n+1)$th kick (13) and (14). As a result, we have equations (49) and (50).

This ends the proof. □

**Remark 5.** Equations (49) and (50) describe a generalization of equations (6) and (7). If $\alpha = n = 2$ and $c_2 = x_0$, $c_1 = p_0$, then equation (49) gives (6) and (7).

**Remark 6.** In equations (50) and (51), we can use

$$\frac{c_2(\alpha-2)}{\Gamma(\alpha-1)} = \frac{c_2}{\Gamma(\alpha-2)}$$

for $1 < \alpha < 2$.

**Remark 7.** If we use the definition $p(t) = D_t^1 x(t)$, then the Hamiltonian form of the equations of motion will be more complicated than (30) and (31) since

$$D_t^2\, {}_0I_t^{2-\alpha} x(t) \neq {}_0I_t^{2-\alpha} D_t^2 x(t).$$

**Remark 8.** Note that we use the usual momentum $p(t) = D_t^1 x(t)$. In this case, the values $c_1$ and $c_2$ are not connected with $p(0)$ and $x(0)$. If we use the momentum $p(t) = {}_0D_t^{\alpha-1} x(t)$, then $c_1 = p(0)$.

## 4. Riemann–Liouville and Caputo fractional derivatives

In [14] we consider nonlinear differential equations with Riemann–Liouville fractional derivatives. The discrete maps with memory are obtained from these equations. The problems with initial conditions for the Riemann–Liouville fractional derivative are not discussed.

The Riemann–Liouville fractional derivative has some notable disadvantages in applications in mechanics such as the hyper-singular improper integral, where the order of singularity is higher than the dimension, and nonzero of the fractional derivative of constants, which would entail that dissipation does not vanish for a system in equilibrium. The desire to use the usual initial value problems for mechanical systems leads to the use of Caputo fractional derivatives [7, 6] rather than the Riemann–Liouville fractional derivative.

The left-sided Caputo fractional derivative [7, 23–25] of order $\alpha > 0$ is defined by

$$ {}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\mathrm{d}\tau \, D_\tau^n f(\tau)}{(t-\tau)^{\alpha-n+1}} = {}_0 I_t^{n-\alpha} D_t^n f(t), \tag{52} $$

where $n-1 < \alpha < n$, and ${}_0 I_t^{\alpha}$ is the left-sided Riemann–Liouville fractional integral of order $\alpha > 0$ that is defined by

$$ {}_0 I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)\,\mathrm{d}\tau}{(t-\tau)^{1-\alpha}}, \qquad (t > 0). \tag{53} $$

This definition is, of course, more restrictive than the Riemann–Liouville fractional derivative [4, 7] in that it requires the absolute integrability of the derivative of order $n$. The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desire order of fractional derivative. The Riemann–Liouville fractional derivative is computed in the reverse order. Integration by part of (52) will lead to

$$ {}_0^C D_t^{\alpha} x(t) = {}_0 D_t^{\alpha} x(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} x^{(k)}(0). \tag{54} $$

It is observed that the second term in equation (54) regularizes the Caputo fractional derivative to avoid the potentially divergence from singular integration at $t = 0$. In addition, the Caputo fractional differentiation of a constant results in zero

$$ {}_0^C D_t^{\alpha} C = 0. $$

Note that the Riemann–Liouville fractional derivative of a constant need not be zero, and we have

$$ {}_0 D_t^{\alpha} C = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} C. $$

If the Caputo fractional derivative is used instead of the Riemann–Liouville fractional derivative, then the initial conditions for fractional dynamical systems are the same as those for the usual dynamical systems. The Caputo formulation of fractional calculus can be more applicable in mechanics than the Riemann–Liouville formulation.

## 5. Caputo fractional derivative and universal map with memory

In this section, we study a generalization of differential equation (1) by the Caputo fractional derivative. The universal map with memory is derived from this fractional equation.

We consider the nonlinear differential equation of order $\alpha$, where $0 \leqslant n-1 < \alpha \leqslant n$,

$$ {}_0^C D^{\alpha} x(t) = G[t, x(t)], \qquad (0 \leqslant t \leqslant t_f), \tag{55} $$

involving the Caputo fractional derivative ${}_0^C D_t^{\alpha}$ on a finite interval $[0, t_f]$ of the real axis, with the initial conditions

$$ \left(D_t^k x\right)(0) = c_k, \qquad k = 0, \ldots, n-1. \tag{56} $$

Kilbas and Marzan [19, 20] proved the equivalence of the Cauchy-type problem of the form (55), (56) and the Volterra integral equation of second kind

$$ x(t) = \sum_{k=0}^{n-1} \frac{c_k}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t \mathrm{d}\tau \, G[\tau, x(\tau)] \, (t-\tau)^{\alpha-1} \tag{57} $$

in the space $C^{n-1}[0, t_f]$. For $\alpha = n = 2$ equation (57) gives (10).

Let us give the basic theorem regarding the nonlinear differential equation involving the Caputo fractional derivative.

**Kilbas–Marzan theorem.** *The Cauchy-type problem (55) and (56) and the nonlinear Volterra integral equation (57) are equivalent in the sense that, if $x(t) \in C[0, t_f]$ satisfies one of these relations, then it also satisfies the other.*

**Proof.** In [19, 20] (see also [7], theorem 3.24.) this theorem is proved by assuming that a function $G[t, x]$ for any $x \in W \subset \mathbb{R}$ belongs to $C_\gamma(0, t_f)$ with $0 \leqslant \gamma < 1$, $\gamma < \alpha$. Here $C_\gamma(0, t_f)$ is the weighted space of functions $f[t]$ given on $(0, t_f]$, such that $t^\gamma f[t] \in C(0, t_f)$. This ends the proof. □

We consider the fractional differential equation of the form

$$ {}_0^C D_t^\alpha x(t) + KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right) = 0, \qquad (1 < \alpha < 2), \tag{58} $$

where ${}_0^C D_t^\alpha$ is the Caputo fractional derivative, with the initial conditions

$$ x(0) = x_0, \qquad (D^1 x)(0) = p_0. \tag{59} $$

Using $p(t) = D_t^1 x(t)$, equation (58) can be rewritten in the Hamilton form.

**Proposition 5.** *The Cauchy-type problem for the fractional differential equations*

$$ D_t^1 x(t) = p(t), \tag{60} $$

$$ {}_0^C D_t^{\alpha-1} p(t) = -KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right), \qquad (1 < \alpha < 2) \tag{61} $$

*with the initial conditions*

$$ x(0) = x_0, \qquad p(0) = p_0 \tag{62} $$

*is equivalent to the discrete map equations*

$$ x_{n+1} = x_0 + p_0(n+1)T - \frac{KT^\alpha}{\Gamma(\alpha)} \sum_{k=1}^{n} (n+1-k)^{\alpha-1} G[x_k], \tag{63} $$

$$ p_{n+1} = p_0 - \frac{KT^{\alpha-1}}{\Gamma(\alpha-1)} \sum_{k=1}^{n} (n+1-k)^{\alpha-2} G[x_k]. \tag{64} $$

**Proof.** We use the Kilbas–Marzan theorem with the function

$$ G[t, x(t)] = -KG[x(t)] \sum_{k=1}^{\infty} \delta\left(\frac{t}{T} - k\right). $$

The Cauchy-type problem (58) and (59) is equivalent to the Volterra integral equation of second kind

$$ x(t) = x_0 + p_0 t - \frac{K}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \int_0^t \mathrm{d}\tau (t-\tau)^{\alpha-1} G[x(\tau)] \delta\left(\frac{t}{T} - k\right), \tag{65} $$

in the space of continuously differentiable functions $x(t) \in C^1[0, t_f]$.

If $nT < t < (n+1)T$, then equation (65) gives

$$x(t) = x_0 + p_0 t - \frac{KT}{\Gamma(\alpha)} \sum_{k=1}^{n} (t - kT)^{\alpha-1} G[x(kT)]. \tag{66}$$

We define the momenta

$$p(t) = D_t^1 x(t). \tag{67}$$

Then equations (66) and (67) give

$$p(t) = p_0 - \frac{KT}{\Gamma(\alpha-1)} \sum_{k=1}^{n} (t - kT)^{\alpha-2} G[x(kT)], \qquad (nT < t < (n+1)T), \tag{68}$$

where we use $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$.

The solution of the left side of the $(n+1)$th kick (13) and (14) can be represented by equations (63) and (64), where we use the condition of continuity $x(t_n + 0) = x(t_n - 0)$.

This ends the proof. □

**Remark 9.** Equations (63) and (64) define a generalization of the universal map. This map is derived from a fractional differential equation with Caputo derivatives without any approximations. The main property of the suggested map is a long-term memory that means that their present state depends on all past states with a power-law form of weights.

**Remark 10.** If $\alpha = 2$, then equations (63) and (64) give the universal map of the form (6) and (7) that is equivalent to equations (2). As a result, the usual universal map is a special case of this universal map with memory.

**Remark 11.** By analogy with proposition 5, it is easy to obtain the universal map with memory from fractional equation (58) with $\alpha > 2$.

## 6. Conclusion

The suggested discrete maps with memory are generalizations of the universal map. These maps describe fractional dynamics of complex physical systems. The suggested universal maps with memory are equivalent to the correspondent fractional kicked differential equations. We obtain a discrete map from a fractional differential equation by using the equivalence of the Cauchy-type problem and the nonlinear Volterra integral equation of second kind. An approximation for fractional derivatives of these equations is not used.

It is important to obtain and to study discrete maps which correspond to the real physical systems described by the fractional differential equations. In mechanics and electrodynamics, we can consider viscoelastic and dielectric materials as media with memory. We note that the dielectric susceptibility of a wide class of dielectric materials follows, over extended frequency ranges, a fractional power-law frequency dependence that is called the 'universal' response [15, 16]. As was proved in [17, 18], the electromagnetic fields in such dielectric media are described by differential equations with fractional time derivatives. These fractional equations for electromagnetic waves in dielectric media are common to a wide class of materials, regardless of the type of physical structure, chemical composition, or of the nature of the polarizing species, whether dipoles, electrons or ions. We hope that it is possible to obtain the discrete maps with memory which correspond to the real dielectric media described by the fractional differential equations.

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# Fractional Dissipative Standard Map

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## Abstract

Using kicked differential equations of motion with derivatives of non-integer orders, we obtain generalizations of the dissipative standard map. The main property of these generalized maps, which are called fractional maps, is long-term memory. The memory effect in the fractional maps means that their present state of evolution depends on all past states with special forms of weights. The fractional dissipative standard maps are used to study attractors of the systems described by kicked fractional differential equations.

**Discrete maps are widely used to study general properties of dynamical systems. In those cases when they can be derived from differential equations, their analysis gives the exact properties of the corresponding systems. In this article we derive discrete maps (fractional maps) from the fractional differential equations, which correspond to the fractional generalizations of the dissipative standard map [1]. We demonstrate how the attractors of the fractional maps are different from the attractors of the dissipative standard map.**

## I. INTRODUCTION

There is a number of distinct areas of physics where basic problems can be reduced to the study of simple discrete maps. Discrete maps as substitutes of differential equations have been used to study evolution problems in [2-6]. They lead to a simpler formalism, which is particularly useful in simulations. The dissipative standard map is one of the most widely studied maps. In this paper we consider fractional generalizations of the dissipative standard map which are described by fractional differential equations [7-9].

The treatment of nonlinear dynamics in terms of discrete maps is a very important step in understanding the qualitative behavior of systems described by differential equations. The derivatives of non-integer orders are a natural generalization of the ordinary differentiation of the integer order. The fractional differentiation with respect to time is characterized by long-term memory effects which correspond to intrinsic dissipative processes in physical systems. The application of memory effects to discrete maps means that their present state of evolution depends on all past states [10-15, 17].

Discrete maps with memory can be derived (see [17]) from equations of motion with fractional derivatives. In Ref. [17] a fractional generalization of the standard map has been derived from a fractional differential equation. A fractional generalization of the dissipative standard map was also suggested in [17]. Unfortunately, in that generalization a dissipation was introduced by the change of the variable $p_n \to -bp_n$. The map equations were not directly connected with a fractional equation of motion. In this paper we propose two generalizations of the dissipative standard map. The first one is derived from a differential equation with fractional damped kicks. The second generalization of the dissipative standard map is derived from a fractional differential equation (kickes are not fractional) . A nonlinear system with fractional derivatives perturbed by a periodic force exhibits a new type of chaotic motion which can be called the fractional chaotic attractor. Fractional discrete maps [17] are used to study new types of attractors of fractional dynamics described by kicked fractional equations. In this paper some fractional differential equations of motion of kicked systems with friction are considered. Corresponding discrete maps with memory are derived from these equations. The fractional generalizations of the dissipative standard map are suggested and these maps are used in computer simulations.

## II. DISCRETE MAPS WITHOUT MEMORY

In this section, a brief review of discrete maps is considered to fix notations and provide convenient references. For details, see [2-6].

### A. Standard map

Let us consider the equation of motion

$$\ddot{x} + K \sin(x) \sum_{n=0}^{\infty} \delta\Big(t - n\Big) = 0, \tag{1}$$

in which perturbation is a periodic sequence of delta-function type pulses (kicks) following with period $T = 1$, $K$ is the amplitude of the pulses. This equation can be presented in the Hamiltonian form

$$\dot{x} = p, \quad \dot{p} + K \sin(x) \sum_{n=0}^{\infty} \delta\Big(t - n\Big) = 0. \tag{2}$$

It is well-known that these equations can be represented (see for example Chapter 5 in [3]) in the form of discrete map

$$x_{n+1} = x_n + p_{n+1}, \tag{3}$$

$$p_{n+1} = p_n - K \sin(x_n). \tag{4}$$

Equations (3) and (4) are called the standard map. This map is also called the Chirikov map [4].

### B. Dissipative standard map

The dissipative standard map [1, 18, 19] is

$$X_{n+1} = X_n + \mu Y_{n+1} + \Omega, \tag{5}$$

$$Y_{n+1} = e^{-q}\left[Y_n + \varepsilon\ \sin(X_n)\right], \tag{6}$$

where $\mu = (e^q - 1)/q$. The dissipative standard map is also called the Zaslavsky map. Note that a shift $\Omega$ does not play an important role and it can be put to zero ($\Omega = 0$). The dissipative standard map with $\Omega = 0$ can be represented by the equations

$$X_{n+1} = X_n + P_{n+1}, \tag{7}$$

$$P_{n+1} = -bP_n - Z\ \sin(X_n). \tag{8}$$

For the parameters

$$Z = -\varepsilon\mu e^{-q}, \quad P_n = \mu Y_n, \quad b = -e^{-q} \tag{9}$$

equations (7) and (8) give Eqs. (5) and (6) with $\Omega = 0$.

For $b = -1$ and $Z = K$, we get the standard map which is described by Eqs. (3) and (4) with $T = 1$.

Note that for large $q \to \infty$ (for small $b \to 0$) Eqs. (7) and (8) with $Z = -K$ shrink to the proposed by Arnold [20] one-dimensional sine-map

$$X_{n+1} = X_n + K \sin(X_n). \tag{10}$$

### C. Kicked damped rotator map

The equation of motion for kicked damped rotator is

$$\ddot{x} + q\dot{x} = KG(x) \sum_{n=0}^{\infty} \delta\Big(t - nT\Big). \tag{11}$$

It is well-known [5] that Eq. (11) gives the two-dimensional map

$$x_{n+1} = x_n + \frac{1 - e^{-qT}}{q}\left[p_n + KG(x_n)\right], \tag{12}$$

$$p_{n+1} = e^{-qT}\left[p_n + KG(x_n)\right]. \tag{13}$$

This map is known as the kicked damped rotator map. The phase volume shrinks each time step by a factor $\exp(-q)$. The map is defined by two important parameters, dissipation constant $q$ and force amplitude $K$. These equations can be rewritten in the form

$$x_{n+1} = x_n + \frac{e^{qT} - 1}{q} p_{n+1},$$

$$p_{n+1} = e^{-qT}[p_n + KG(x_n)].$$

It is easy to see that these equations give the dissipative standard map (7) and (8) with $\Omega = 0$ if we use

$$X_n = x_n, \quad Y_n = p_n, \quad \varepsilon = K, \quad T = 1, \quad G(x) = \sin(x).$$

This allows us to derive dissipative standard map (5) and (6) from the differential equation.

$$\ddot{X} + q\dot{X} = \varepsilon \sin(X) \sum_{n=0}^{\infty} \delta\Big(t - n\Big). \tag{14}$$

These equations give the discrete map defined by Eqs. (5) and (6) with $\Omega = 0$.

## III. FRACTIONAL STANDARD MAP AND DISSIPATION

A fractional generalization of the differential equation (1) has been suggested in [17]. The discrete map which corresponds to the fractional equation of order $1 < \alpha \leq 2$ was derived. This map can be considered as a generalization of the standard map for the case $1 < \alpha \leq 2$.

We consider a fractional generalization of (1) in the form

$$_0D_t^\alpha x + K \sin(x) \sum_{n=0}^{\infty} \delta\Big(t - n\Big) = 0, \quad (1 < \alpha \leq 2), \tag{15}$$

where $_0D_t^\alpha$ is the Riemann-Liouville fractional derivative [7–9], which is defined by

$$_0D_t^\alpha x = D_t^2 \, {_0I_t^{2-\alpha}} x = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t \frac{x(\tau)d\tau}{(t-\tau)^{\alpha-1}}, \quad (1 < \alpha \leq 2). \tag{16}$$

Here we use the notation $D_t^2 = d^2/dt^2$, and $_0I_t^\alpha$ is a fractional integration [7–9]. Defining the momentum as

$$p(t) = {_0D_t^{\alpha-1}} x(t),$$

and using the initial conditions

$$({_0D_t^{\alpha-1}} x)(0+) = p_1, \quad ({_0D_t^{\alpha-2}} x)(0+) = b, \tag{17}$$

it is possible to derive the equation for the fractional standard map.

**Proposition 1.** *The fractional differential equation of the kicked system (15) is equivalent to the discrete map*

$$x_{n+1} = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} p_{k+1} V_\alpha(n-k+1) + \frac{b}{\Gamma(\alpha-1)}(n+1)^{\alpha-2}, \tag{18}$$

$$p_{n+1} = p_n - K \sin(x_n), \quad (1 < \alpha \leq 2), \tag{19}$$

*where the function $V_\alpha(z)$ is defined by*

$$V_\alpha(z) = z^{\alpha-1} - (z-1)^{\alpha-1}. \tag{20}$$

Proof of this Proposition is given in [21].

A fractional generalization of the dissipative standard map suggested in [1, 18] can be defined by

$$x_{n+1} = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} p_{k+1} V_\alpha(n-k+1), \quad (1 < \alpha \leq 2), \tag{21}$$

$$p_{n+1} = -bp_n - Z \sin(x_n), \tag{22}$$

where the parameters are defined by conditions (9). For $b = -1$ and $Z = K$ Eqs. (21) and (22) give the fractional standard map with $T = 1$. Note that this fractional dissipative standard map is not derived from a fractional differential equation. This map is derived by $p_n \to -bp_n$ in the fractional standard map. Fractional dissipative standard map can be derived from fractional differential equations. In this paper, we derive two fractional generalizations of the dissipative standard map which are obtained from fractional differential equations.

## IV. FRACTIONAL DERIVATIVE IN THE KICKED TERM AND THE FIRST FRACTIONAL DISSIPATIVE STANDARD MAP

In this section we suggest the first fractional generalization of differential equation (11) for a kicked damped rotator. In this generalization we introduce a fractional derivative in the kicked damped term, i.e. the term of a periodic sequence of delta-function type pulses (kicks), and derive the corresponding discrete map.

Consider the fractional generalization of equation (11) in the form

$$D_t^2 X(t) - q D_t^1 X(t) = \varepsilon \sin\left({}_0^C D_t^\beta X\right) \sum_{n=0}^{\infty} \delta(t-n), \quad (0 \le \beta < 1), \tag{23}$$

where $q \in \mathbb{R}$, and ${}_0^C D_t^\beta$ is the Caputo fractional derivative [9] of the order $0 \le \beta < 1$ defined by

$${}_0^C D_t^\beta X = {}_0 I_t^{1-\beta} D_t^1 X = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{d\tau}{(t-\tau)^\beta} \frac{dX(\tau)}{d\tau}, \quad (0 \le \beta < 1). \tag{24}$$

Here we use the notation $D_t^1 = d/dt$, and ${}_0 I_t^\alpha$ is a fractional integration [7–9]. For $\beta = 0$ fractional equation (23) gives equation (11). Note that we use the minus on the left-hand side of Eq. (23), where $q$ can be a positive or negative value. Fractional derivative ${}_0^C D_t^\beta X$ is presented in the kicked damped term.

**Proposition 2.** *The fractional differential equation of the kicked system (23) is equivalent to the discrete map*

$$X_{n+1} = X_n + \frac{1-e^{-q}}{q} Y_{n+1}, \tag{25}$$

$$Y_{n+1} = e^q \left[ Y_n + \varepsilon \sin\left( \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{n-1} Y_{k+1} W_{2-\beta}(q, k-n) \right) \right], \tag{26}$$

*where the functions $W_{2-\beta}(a,b)$ are defined by*

$$W_{2-\beta}(a,b) = a^{\beta-1} e^{a(b+1)} \Big[ \Gamma(1-\beta, ab) - \Gamma(1-\beta, a(b+1)) \Big],$$

*and $\Gamma(a,b)$ is the incomplete Gamma function*

$$\Gamma(a,b) = \int_b^\infty y^{a-1} e^{-y} dy. \tag{27}$$

**Proof.** Fractional equation (23) can be presented in the Hamiltonian form

$$\dot{X} = Y,$$

$$\dot{Y} - qY = \varepsilon \sin({}_0^C D_t^\beta X) \sum_{n=0}^{\infty} \delta(t-n), \tag{28}$$

where $0 < \beta < 0$, and $q \in \mathbb{R}$.

Between any two kicks

$$\dot{Y} - qY = 0. \tag{29}$$

For $t \in (t_n + 0, t_{n+1} - 0)$, the solution of Eq. (29) is

$$Y(t_{n+1} - 0) = Y(t_n + 0)e^q. \tag{30}$$

Let us use the notations $t_n = nT$, with $T = 1$ and

$$X_n = X(t_n - 0) = \lim_{\epsilon \to 0} X(n - \epsilon),$$

$$Y_n = Y(t_n - 0) = \lim_{\epsilon \to 0} Y(n - \epsilon). \tag{31}$$

For $t \in (t_n - \epsilon, t_{n+1} - \epsilon)$, the general solution of (28) is

$$Y(t) = Y_n e^{q(t-t_n)} + \varepsilon \sum_{m=0}^{\infty} \sin({}_0^C D^\beta_{t_m} X) \int_{t_n - \epsilon}^{t} d\tau e^{q(t-\tau)} \delta(\tau - m). \tag{32}$$

Then

$$Y_{n+1} = e^q \Big[ Y_n + \varepsilon \sin({}_0^C D^\beta_{t_n} X) \Big]. \tag{33}$$

Using (33), the integration of the first equation of (28) gives

$$X_{n+1} = X_n - \frac{1 - e^q}{q} \Big[ Y_n + \varepsilon \sin({}_0^C D^\beta_{t_n} X) \Big]. \tag{34}$$

Let us consider the Caputo fractional derivative from Eqs. (33) and (34). It is defined by the equation

$${}_0^C D^\beta_{t_n} X = {}_0 I^{1-\beta}_t D^1_t X = \frac{1}{\Gamma(1-\beta)} \int_0^{t_n} \frac{d\tau}{(t_n - \tau)^\beta} \frac{dX(\tau)}{d\tau}, \quad (0 \leq \beta < 1).$$

Using $Y(\tau) = dX(\tau)/d\tau$, this relation can be rewritten as

$${}_0^C D^\beta_{t_n} X = \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{Y(\tau) d\tau}{(t_n - \tau)^\beta}, \tag{35}$$

where $t_{k+1} = t_k + 1 = (k+1)$, and $t_k = k$, such that $t_0 = 0$. For $\tau \in (t_k, t_{k+1})$, equations (30) and (31) give

$$Y(\tau) = Y(t_k + 0) e^{q(\tau - t_k)} = Y(t_{k+1} - 0) e^{-q} e^{q(\tau - t_k)} =$$

$$= Y_{k+1} e^{q(\tau - t_k - 1)} = Y_{k+1} e^{q(\tau - t_{k+1})}.$$

Then

$$\int_{t_k}^{t_{k+1}} \frac{Y(\tau) d\tau}{(t_n - \tau)^{-\beta}} = Y_{k+1} \int_{t_k}^{t_{k+1}} e^{q(\tau - t_{k+1})} (t_n - \tau)^{-\beta} d\tau =$$

$$= Y_{k+1} \int_{t_n-t_{k+1}}^{t_n-t_k} e^{q(t_n-t_{k+1}-z)} z^{-\beta} dz = Y_{k+1} e^{q(t_n-t_{k+1})} \int_{t_n-t_{k+1}}^{t_n-t_k} z^{-\beta} e^{-qz} dz =$$

$$= Y_{k+1} q^{\beta-1} e^{q(n-k-1)} \int_{q(t_n-t_{k+1})}^{q(t_n-t_k)} y^{-\beta} e^{-y} dy. \tag{36}$$

As a result, equation (36) gives

$$\int_{t_k}^{t_{k+1}} \frac{Y(\tau)d\tau}{(t_n-\tau)^{-\beta}} =$$

$$= Y_{k+1} q^{\beta-1} e^{q(n-k-1)} \Big[\Gamma(1-\beta, q(t_n-t_{k+1})) - \Gamma(1-\beta, q(t_n-t_k))\Big]. \tag{37}$$

Here $\Gamma(a,b)$ is the incomplete Gamma function (27), where $a$ and $b$ are complex numbers. Using (35) and (37), we obtain

$$ {}_0^C D_{t_n}^{\beta} X = \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{n-1} Y_{k+1} W_{2-\beta}(q, k-n), \quad (0 \le \beta < 1), \tag{38}$$

where

$$W_{2-\beta}(a,b) = a^{\beta-1} e^{a(b+1)} \Big[\Gamma(1-\beta, ab) - \Gamma(1-\beta, a(b+1))\Big]. \tag{39}$$

Substitution of (38) into (33) and (34) gives

$$Y_{n+1} = e^q \Big[Y_n + \varepsilon \sin\Big(\frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{n-1} Y_{k+1} W_{2-\beta}(q, k-n)\Big)\Big], \tag{40}$$

$$X_{n+1} = X_n - \frac{1-e^q}{q} \Big[Y_n + \varepsilon \sin\Big(\frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{n-1} Y_{k+1} W_{2-\beta}(q, k-n)\Big)\Big]. \tag{41}$$

Equations (40) and (41) can be presented in the form of Eqs. (25) and (26).

This ends the proof. □

The iteration equations (25) and (26) define a fractional generalization of the dissipative standard map. For $\beta = 0$ this map gives the Zaslavsky map (5) and (6) with

$$\mu = (1-e^{-q})/q \tag{42}$$

and $\Omega = 0$.

## V. FRACTIONAL DERIVATIVE IN THE UNKICKED TERMS AND THE SECOND FRACTIONAL DISSIPATIVE STANDARD MAP

In this section we suggest a fractional generalization of the differential equation for a kicked damped rotator with fractional derivatives in the unkicked terms and derive the corresponding discrete map.

We consider the fractional generalization of equation (11) in the form

$$ {}_0D^{\alpha}_t X(t) - q\, {}_0D^{\beta}_t X(t) = \varepsilon \sin(X) \sum_{n=0}^{\infty} \delta(t-n), \tag{43} $$

where

$$ q \in \mathbb{R}, \quad 1 < \alpha \leq 2, \quad \beta = \alpha - 1, $$

and ${}_0D^{\alpha}_t$ is the Riemann-Liouville fractional derivative [7-9], which is defined by Eq. (16). This equation has fractional derivatives in the unkicked terms, i.e. on the left-hand side of Eq. (43). We use the minus in the left-hand side of Eq. (43), where $q$ can have a positive or negative value.

**Proposition 3.** *The fractional differential equation of the kicked system (43) is equivalent to the discrete map*

$$ X_{n+1} = \frac{1}{\Gamma(\alpha-1)} \sum_{k=0}^{n} Y_{k+1}\, W_{\alpha}(q, k-n-1), \tag{44} $$

$$ Y_{n+1} = e^{q} \Big[ Y_n + \varepsilon \sin(X_n) \Big], \tag{45} $$

*where the functions $W_{\alpha}(a,b)$ are defined by*

$$ W_{\alpha}(a,b) = a^{1-\alpha} e^{a(b+1)} \Big[ \Gamma(\alpha-1, ab) - \Gamma(\alpha-1, a(b+1)) \Big], \tag{46} $$

*and $\Gamma(a,b)$ is the incomplete Gamma function (27).*

**Proof.** Let us define an auxiliary variable $\xi(t)$ such that

$$ {}^{C}_{0}D^{2-\alpha}_t \xi = X(t), \tag{47} $$

where ${}^{C}_{0}D^{2-\alpha}_t$ is the Caputo fractional derivative (24). Using

$$ {}_0I^{2-\alpha}_t\, {}^{C}_{0}D^{2-\alpha}_t \xi = \xi(t) - \xi(0), \quad (0 \leq 2-\alpha < 1), \tag{48} $$

we obtain

$$ {}_0D_t^{\alpha} X = D_t^2 \, {}_0I_t^{2-\alpha} X = D_t^2 \, {}_0I_t^{2-\alpha} \, {}_0^C D_t^{2-\alpha} \xi = D_t^2(\xi(t) - \xi(0)) = D_t^2 \xi, \qquad (49) $$

and

$$ {}_0D_t^{\beta} X = D_t^1 \, {}_0I_t^{1-\beta} X = D_t^1 \, {}_0I_t^{2-\alpha} X = $$

$$ = D_t^1 \, {}_0I_t^{2-\alpha} \, {}_0^C D_t^{2-\alpha} \xi = D_t^1(\xi(t) - \xi(0)) = D_t^1 \xi. \qquad (50) $$

Substitution of (49), (50) and (47) into Eq. (43) gives

$$ D_t^2 \xi - q D_t^1 \xi = \varepsilon \sin({}_0^C D_t^{2-\alpha} \xi) \sum_{n=0}^{\infty} \delta(t-n), \quad (1 < \alpha \leq 2). \qquad (51) $$

The fractional equation (51) can be presented in the Hamiltonian form

$$ \dot{\xi} = Y, $$

$$ \dot{Y} - qY = \varepsilon \sin({}_0^C D_t^{2-\alpha} \xi) \sum_{n=0}^{\infty} \delta(t-n), \quad (1 < \alpha < 2, \quad q \in \mathbb{R}). \qquad (52) $$

Using Eq. (26) of Proposition 2, we obtain

$$ Y_{n+1} = e^q \Big[ Y_n + \varepsilon \sin\Big( \frac{1}{\Gamma(\alpha-1)} \sum_{k=0}^{n-1} Y_{k+1} W_{\alpha}(q, k-n) \Big) \Big]. $$

For $(X_n, Y_n)$, we use equation (38) in the form

$$ X_n = {}_0^C D_{t_n}^{2-\alpha} \xi = \frac{1}{\Gamma(\alpha-1)} \sum_{k=0}^{n-1} Y_{k+1} W_{\alpha}(q, k-n). $$

As a result, we have

$$ X_{n+1} = \frac{1}{\Gamma(\alpha-1)} \sum_{k=0}^{n} Y_{k+1} W_{\alpha}(q, k-n-1), \qquad (53) $$

$$ Y_{n+1} = e^q \Big[ Y_n + \varepsilon \sin(X_n) \Big], \qquad (54) $$

where $W_{\alpha}(a, b)$ is defined in (46). This ends the proof. □

If we use the variables

$$ P_n = \mu Y_n, \quad b = -e^q, \quad Z = -\mu \varepsilon e^q, $$

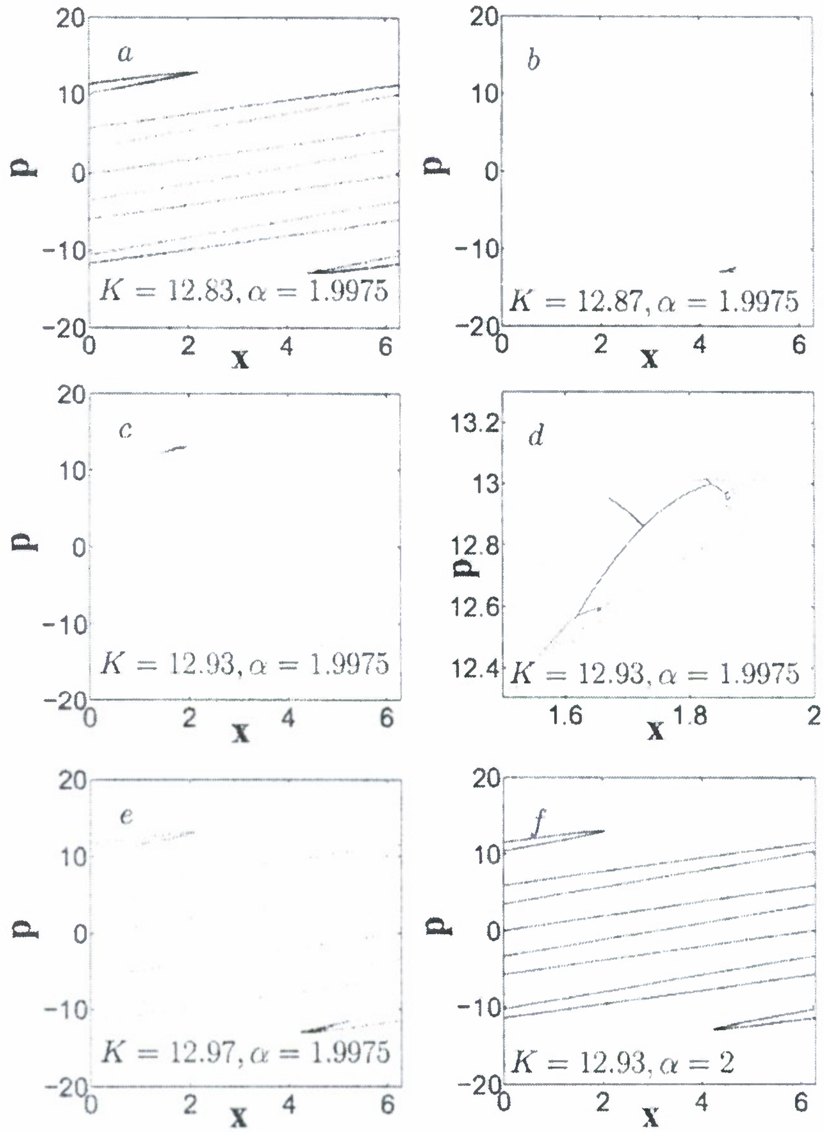


FIG. 1: Structures of the chaotic attractors for different values of $K$ obtained after $10^5$ iterations ($\Gamma = 5$, $\Omega = 0$); $\alpha = 1.9975$ in Figs. 1a-e. a) $K = 12.83$; b) $K = 12.87$; c) $K = 12.93$; d) zoom of Fig. 1c; e) $K = 12.97$; f) $\alpha = 2$, $K = 12.93$.

then equations (44) and (45) give

$$X_{n+1} = \frac{\mu^{-1}}{\Gamma(\alpha - 1)} \sum_{k=0}^{n} P_{k+1} W_\alpha(q, k - n - 1). \tag{55}$$

$$P_{n+1} = -bP_n - Z \sin(X_n), \tag{56}$$

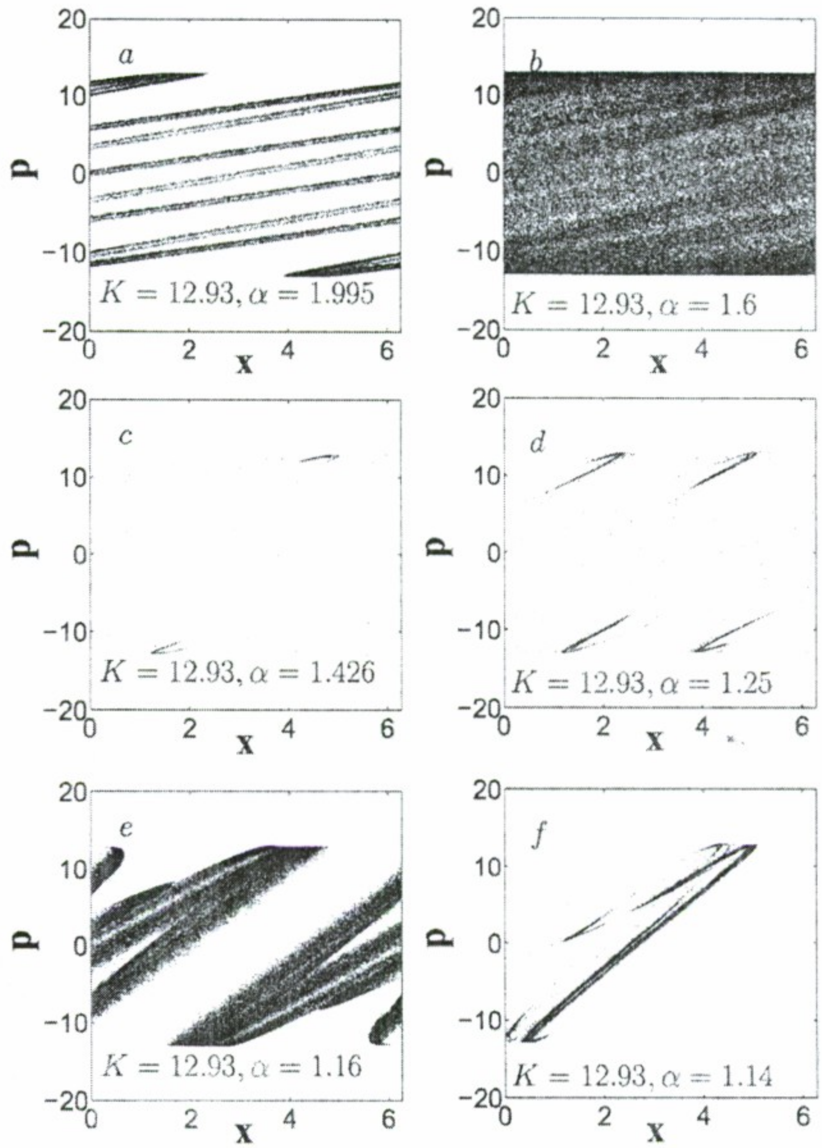


FIG. 2: Structures of the fractional chaotic attractors for $K = 12.93$ and different values of $\alpha$ obtained after $10^5$ iterations ($\Gamma = 5$, $\Omega = 0$).

These equations can be considered as a fractional generalization of the dissipative standard map equations (7) and (8) with $\Omega = 0$. For $\alpha = 2$, this fractional dissipative standard map gives the dissipative standard map that is described by Eqs. (7) and (8).

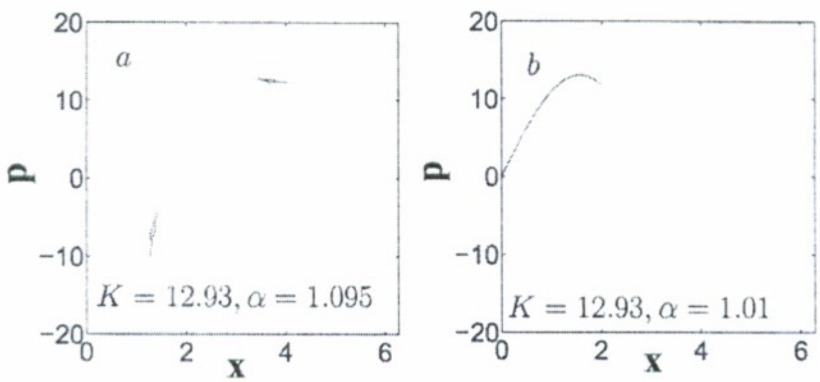


FIG. 3: Attracting trajectories for $K = 12.93$ and small values of $\alpha$ obtained after $10^5$ iterations ($\Gamma = 5$, $\Omega = 0$).

## VI. NUMERICAL SIMULATIONS

Numerical simulations were performed for the second fractional dissipative standard map (Eqs. (55) and (56)). First we used our code to reproduce the results presented in Fig. 1 from [19] for the structures of the chaotic attractors of the dissipative standard map at the window of the ballistic motion near $K \approx 4\pi$ ($q = -5$, $K = \varepsilon \exp(q)$, and used in [19] $\Gamma$ is equal to $-q$) for the fractional standard map with $\alpha = 2$ and obtained a perfect agreement (an example is given in Fig. 1f). As $\alpha$ decreases slightly from $\alpha = 2$ to $\alpha = 1.9975$, the window of the ballistic motion shrinks and moves to the higher values of $K$. Already for $\alpha = 1.9975$ in Figs. 1a-e the ballistic motion appears for $K > 12.86$ and disappears at $K = 12.97$. The window is completely closed at $\alpha \approx 1.9969$. The structures of two symmetric attractors with disjoint basins which appear within the window (Figs. 1b,c) is also very different from the structures of the dying attractors of the dissipative standard map [1, 19]. The attractor in Fig. 1d evolves from period 8 trajectory to period 4, period 2, and, finally, period 1 trajectory slowly moving in the direction of the upper left corner with the step of the order of $10^{-7}$.

When $\alpha$ decreases further, the structures of the fractional chaotic attractors evolve in the manner presented in Fig. 2, where one can find one-scroll, two-scroll, and four-scroll fractional chaotic attractors, strongly deviating from the chaotic attractor of the dissipative standard map Fig. 1f (see also [19]). The problem of existence of multi-scroll fractional chaotic attractors was considered in [22] but for the fractional differential equations with the Caputo derivatives. For values of $\alpha$ near 1 fractional chaotic attractor turns into period two and for smaller values period one attracting trajectories Fig. 3.

## VII. CONCLUSION

The suggested discrete maps with memory are generalizations of the dissipative standard map. These maps describe fractional dynamics of complex physical systems. The suggested fractional dissipative standard maps demonstrate a chaotic behavior with a new type of attractors. The interesting property of these fractional maps is long-term memory. As a result, a present state of evolution depends on all past states with the weight functions. The fractional dissipative standard maps are equivalent to the correspondent fractional kicked differential equations. Note that to derive discrete maps an approximation for fractional derivatives of these equations is not used.

Computer simulations of the suggested discrete maps with memory prove that the nonlinear dynamical systems, which are described by the equations with fractional derivatives, exhibit a new type of chaotic motion. This type of motion demonstrates a fractional generalization of attractors.

The special cases of discrete maps have been studied to describe the properties of the fractional chaotic attractors of these differential equations. Under a wide range of circumstances such maps give rise to chaotic behavior.

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