Characterizing the vibration of an elastically point supported rectangular plate using eigensensitivity analysis

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Abstract

Normalized frequencies are computed for a rectangular, isotropic plate resting on elastic supports. The normalized frequencies are determined using eigensensitivity analysis, which approximates the eigenparameters in a Maclaurin series, yielding an approximate closed-form expression. One benefit of the approximate closed-form expression is its computational efficiency and yet another is its application of re-analysis. Accuracy of the approximate expression is assessed by comparing results with the widely used Rayleigh–Ritz method using orthogonal polynomials and beam shape functions in both approaches. Consideration for a variety of edge conditions is given through a combination of simply supported, clamped and free boundary conditions. Results indicate that the accuracy of higher frequencies computed by the sensitivity approach is highly dependent upon choice of basis function.

1. Background

Point supported plates are plates that have prescribed displacements at a number of discrete locations within its domain. Rectangular, circular as well as elliptical plates represent geometries of interest. Rigid point supports have a prescribed displacement of zero while elastically point supported have displacements dependent on the stiffness at the support. Electronic circuit boards, solar panels, and concrete slabs represent applications that can be effectively modeled as point supported plates. A brief overview of recent contributions is presented. Altekin [1] consider both bucking and free vibration of elliptical plates which have point supports along the symmetric diagonals. A Ritz approach is used to solve for the fundamental frequency and critical buckling load. Huang et al. [2] studied the free vibration of tapered isotropic plates on rigid point supports. A Mindlin plate theory is used and a Green’s function approach is used to generate the governing equations. Yu [3] used the method of superposition established by Gorman [4] to analyze the free vibration of cantilever plates containing an attached mass. Zhou et al. [5] utilized a three-dimensional theory to study the frequency analysis of both isotropic and composites plates. A finite layer formulation is used to model the structure and a hybrid basis function is introduced to adequately satisfy the displacement constraints at the point supports.

Sensitivity analysis seeks to assess the effect of a parameter on the response of a system. Application of sensitivity analysis covers all fields of economics, business, science, mathematics, and engineering. Structural eigensensitivity analysis provides a direct method to assess the effect of system parameters on the eigenvalues, typically the frequencies and buckling loads. An early contributor in this area of research was Hearmon [6] who developed a one-term formula to approximate the fundamental frequency of orthotropic plates. Bert [7] studied the optimal design of composite plates for maximum fundamental frequency. Barton and Reiss [8] provided approximate closed-formed formula for uni-axial and bi-axial buckling of symmetric composite plates. Recently Barton [9][10] has applied this technique to investigate the bucking of isotropic plates subject to combined in-plane loading and the thermal buckling of composite plates with clamped-free boundary conditions.

In this paper, eigensensitivity analysis is used to determine an approximate closed-form expression which is used to compute frequencies of a square isotropic plate resting on four elastic point supports. A combination of boundary conditions is considered including simply supported, clamped and free. The article is organized into three sections beginning the problem formulation, an overview of the eigensensitivity approach, and the results and discussion section.

2. Problem formulation

There are many ways to formulate the equations that govern the free vibration of the elastically point supported plate including the Newtonian mechanics and the principles of work-
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energy. Here the principle of minimum potential energy is preferred.

The total potential energy $II$ for the system containing elastic supports and discrete masses is given as

$$II = U - T$$  \hspace{1cm} (1)$$

Here $U$ contains the strain energy of the plate and the supporting springs given by

$$U = \frac{D}{2} \iint_{R} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \cdot \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \, dx \, dy$$

$$+ \left\{ \sum_{p=1}^{N} \int_{0}^{1} \int_{0}^{1} k_{p} \delta(x-x_{p}) \delta(y-y_{p}) \cdot w^{2}(x, y) \, dx \, dy \right\}$$  \hspace{1cm} (2)$$

where $D$ is the flexural stiffness, $k_{p}$ is the stiffness of the $p$th spring, $v$ is Poisson’s ratio and $w(x,y)$ is the transverse displacement. The kinetic energy consists of energy from the plate and from added discrete masses. Here $T$ is given by

$$T_{plate} = \frac{\rho b c l^{2}}{2} \iint_{R} w^{2}(x,y) \, dx \, dy + \frac{\rho a b c l^{2}}{2} \iint_{R} \sum_{q=0}^{N} \sum_{r=0}^{N} m_{q} \delta(x-x_{q}) \delta(y-y_{r}) \cdot w^{2}(x,y) \, dx \, dy$$  \hspace{1cm} (3)$$

Above $m_{q}$ is the $q$th discrete mass in the paper, all stiffnesses are taken to be equal as $k_{p}$ and all masses are taken to be equal as $m$.

The displacement $w(x,y)$ may be expressed as

$$w(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \phi(x/a) \phi(y/b)$$  \hspace{1cm} (4)$$

where $\phi(x/a)$ and $\delta(y/b)$ are shape function in the x- and y-directions, respectively. The shape functions are selected to satisfy kinematic and static boundary conditions.

The governing differential equation comes by extremizing Eq. (1) with respect to the kinematic variable $w(x,y)$. Substituting Eq. (4) into Eq. (1) and minimizing the total potential energy with respect to the coefficients $w_{ij}$ results in the eigenvalue problem of

$$\sum_{m} \sum_{n} (K_{ijmn} - \alpha^2 M_{ijmn}) w_{mn} = 0$$  \hspace{1cm} (5)$$

where

$$K_{ijmn} = \frac{\partial^{2} D}{\partial x^{2}} A_{m} b_{n} + \frac{\partial^{2} C_{m}}{\partial y^{2}} C_{n} a_{p} + 2(1-v) \frac{\partial^{2} C_{m}}{\partial x \partial y} C_{n} a_{p} + k \Gamma_{ijmn}$$

$$M_{ijmn} = M B_{m} a_{n} + M \Gamma_{ijmn}$$

and $\Gamma_{ijmn} = \sum_{p=1}^{N} \int \phi(x_{p}) \delta(y_{p}) \phi(x_{p}) \delta(y_{p}) \phi(x_{p}) \delta(y_{p}) \, dx \, dy$ is the product of the basis functions and is evaluated where the springs and masses are located. Several parameters are used in the above definitions including $R$ which is the aspect ratio given as $a/b$ and a set of matrices called boundary condition matrices. These boundary condition matrices are defined by

$$A_{m} = \begin{cases} a_{m} & \text{if } x_{m} = x_{p} \text{ and } y_{m} = y_{p} \\ b_{m} & \text{if } x_{p} = x_{m} \text{ and } y_{m} = y_{p} \\ c_{m} & \text{if } x_{p} = x_{m} \text{ and } y_{p} = y_{m} \\ e_{m} & \text{if } x_{p} = x_{m} \text{ and } y_{p} = y_{m} \end{cases}$$\hspace{1cm} (7)$$

where the prime represents the spatial derivative and $(\cdots)$ is used to represent the $L_{2}$ inner product on $[0,1]$. The above equations for $K_{ijmn}$ and $M_{ijmn}$ are the quite general. They are independent of any particular set of basis functions. Therefore they can be used if $\phi_{a}$ and $\delta_{b}$ are kinematically admissible polynomials, beam shape functions, or any other set of kinematically admissible functions.

3. Sensitivity analysis

A complete presentation of the sensitivity approach used in this paper can be referenced in Barton and Reiss [8]. Eq. (5), written as

$$[K]_{[z]_{1}} = \lambda_{i} [M]_{[z]_{i}}$$  \hspace{1cm} (8)$$

provides the basis to apply the sensitivity approach. Here we define $[z]_{i}$ as the $(i,j)$th component of the eigenvector and $\lambda_{i}$ is its corresponding eigenvalue. An approximate expression for the eigenvalue $\lambda_{i}$ can be determined by introducing parameters $S_{1}$ and $S_{2}$ into Eq. (8) and considering

$$[K](S_{1})[z]_{i}(S_{1}, S_{2}) = \lambda_{i}(S_{1}, S_{2})[M](S_{2})[z]_{i}(S_{1}, S_{2})$$  \hspace{1cm} (9)$$

where

$$\left[ \begin{array}{c} [K](S_{1}) = [K]_{0} + S_{1}[\Delta K] \\ [M](S_{2}) = [M]_{0} + S_{2}[\Delta M] \end{array} \right]$$

Here $[K]_{0}$ and $[M]_{0}$ are a diagonal matrices obtained from $[K]$ and $[M]$ by deleting all off-diagonal elements; $[\Delta K]$ and $[\Delta M]$ are matrices which have zeros on the diagonal and contain only the off-diagonal elements of $[K]$ and $[M]$, respectively. The parameters $S_{1}$ and $S_{2}$ take on values of either 0 or 1. If both $S_{1}$ and $S_{2}=0$, the solution to Eq. (9) becomes the ratio of the diagonal elements of the stiffness matrix $[K]_{0}$ and matrix $[M]_{0}$. If both $S_{1}$ and $S_{2}=1$, the original eigenvalue problem, Eq. (9), is recovered. The desired eigenvalue $\lambda_{i}$ is obtained by expanding $\lambda_{i}$ in a Maclaurin series about $(S_{1},S_{2})=(0,0)$ and evaluating at $(S_{1},S_{2})=(1,1)$.

$$\lambda_{i} = \lambda_{i}(1,1) \geq \lambda_{i}(0,0) + \frac{1}{2} \delta^{2} \lambda_{i}(0,0)$$  \hspace{1cm} (10)$$

The expressions appearing on the right-hand side of Eq. (10) were presented in [8] and are

$$\frac{\partial \lambda_{i}(0,0)}{\partial S_{1}} = \frac{K_{ij}}{M_{ij}}$$

$$\frac{\partial \lambda_{i}(0,0)}{\partial S_{2}} = 0$$

$$\delta^{2} \lambda_{i}(0,0) = -\frac{2}{M_{ij}^{2}} \sum_{k \neq i, j} \left( \frac{[K_{ij} \Delta M_{ij} - M_{ij} \Delta K_{ij}]^{2}}{K_{kij} M_{ij} - K_{ij} M_{kij}} \right)$$

Substituting Eq. (11) into Eq. (10) provides the desired quadratic approximate closed-form expression of

$$\lambda_{i} = \frac{K_{ij}}{M_{ij}} - \frac{1}{M_{ij}} \sum_{k \neq i, j} \left( \frac{[K_{ij} \Delta M_{ij} - M_{ij} \Delta K_{ij}]^{2}}{K_{kij} M_{ij} - K_{ij} M_{kij}} \right)$$  \hspace{1cm} (12)$$

4. Discussion

In this section, the general form of the approximate closed-form expression, Eq. (11), is presented for combinations of simply supported, clamped or free boundary conditions and for two types of basis function. Boundary conditions specific results are then evaluated based upon chosen basis functions.

Eq. (12) is specialized for a plate consisting of four springs of equal stiffness and up to four equal masses can be determined by first evaluating the stiffness and mass matrix elements given through Eq. (6) and then substituting these results into Eq. (12). Evaluating Eq. (6) requires determining the diagonal matrices for
both \([K]\) and \([M]\) which are

\[
K_{ij} = (A_{ij} b_{ij} + 2v^2 R^2 C_{ij} c_{ij} + 2(1-v)R^2 E_{ij} e_{ij} + R^4 B_{ij} a_{ij} + k_p a^2 R^2 \Gamma_{ij})
\]

\[
M_{ij} = B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij}
\]

and computing the off-diagonal matrices for both \([K]\) and \([M]\) are

\[
\Delta K_{ij} = v R^2 (C_{ik} c_{ij} + C_{jk} c_{ij}) + 2(1-v)R^2 E_{ik} e_{ij} + k_p a^2 R^2 \Gamma_{ij}
\]

\[
\Delta M_{ij} = \frac{m}{M} \Gamma_{ij}
\]

Substituting both Eqs. (13) and (14) into the general expression given in Eq. (12) yields

\[
\frac{\omega^2 \alpha^2 \rho h}{D} = \left[ A_{ij} b_{ij} + 2v^2 R^2 C_{ij} c_{ij} + 2(1-v)R^2 E_{ij} e_{ij} + R^4 B_{ij} a_{ij} + k_p a^2 R^2 \Gamma_{ij} \right] 
- \frac{1}{(B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij})^2} \times \left\{ \frac{\sum_{k \neq i} \sum_{l \neq j} \left\{ \left[ A_{ik} b_{ij} + 2v^2 R^2 C_{ik} c_{ij} + 2(1-v)R^2 E_{ik} e_{ij} + R^4 B_{ik} a_{ij} + k_p a^2 R^2 \Gamma_{ij} \right] 
- \frac{m}{M} \Gamma_{lk} \times \left[ B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij} \right] - \left[ \left[ A_{ik} b_{ij} + 2v^2 R^2 C_{ik} c_{ij} + 2(1-v)R^2 E_{ik} e_{ij} + R^4 B_{ik} a_{ij} + k_p a^2 R^2 \Gamma_{ij} \right] 
- \frac{m}{M} \Gamma_{lk} \times \left[ B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij} \right] \right\} \} \right\} \times \left[ B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij} \right]^{-1}
\]

Eq. (15) is valid for any combination of simply supported, clamped or free boundaries. The formula requires identifying an admissible basis and evaluating the boundary matrices given in Eq. (7). Selecting orthonormal polynomials as the basis requires numerical evaluation of the boundary matrices and, as a result, no simplification occurs in the form of Eq. (15). If free boundary conditions are excluded, then the following results holds \(E_{pm} = -c_{pm}\) and \(e_{pm} = -c_{pm}\). Eq. (15) then becomes

\[
\frac{\omega^2 \alpha^2 \rho h}{D} = \left[ A_{ij} b_{ij} + 2v^2 R^2 C_{ij} c_{ij} + 2(1-v)R^2 E_{ij} e_{ij} + R^4 B_{ij} a_{ij} + k_p a^2 R^2 \Gamma_{ij} \right] 
- \frac{1}{(B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij})^2} \times \left\{ \frac{\sum_{k \neq i} \sum_{l \neq j} \left\{ \left[ A_{ik} b_{ij} + 2v^2 R^2 C_{ik} c_{ij} + 2(1-v)R^2 E_{ik} e_{ij} + R^4 B_{ik} a_{ij} + k_p a^2 R^2 \Gamma_{ij} \right] 
- \frac{m}{M} \Gamma_{lk} \times \left[ B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij} \right] - \left[ \left[ A_{ik} b_{ij} + 2v^2 R^2 C_{ik} c_{ij} + 2(1-v)R^2 E_{ik} e_{ij} + R^4 B_{ik} a_{ij} + k_p a^2 R^2 \Gamma_{ij} \right] 
- \frac{m}{M} \Gamma_{lk} \times \left[ B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij} \right] \right\} \} \right\} \times \left[ B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij} \right]^{-1}
\]

and if both added discrete mass, \(m=0\), and elastic support stiffness, \(k_p=0\), Eq. (17) reduces to

\[
\frac{\omega^2 \alpha^2 \rho h}{D} = \left[ A_{ij} b_{ij} + 2v^2 R^2 C_{ij} c_{ij} + 2(1-v)R^2 E_{ij} e_{ij} + R^4 B_{ij} a_{ij} + k_p a^2 R^2 \Gamma_{ij} \right] 
- \frac{1}{(B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij})^2} \times \left\{ \frac{\sum_{k \neq i} \sum_{l \neq j} \left\{ \left[ A_{ik} b_{ij} + 2v^2 R^2 C_{ik} c_{ij} + 2(1-v)R^2 E_{ik} e_{ij} + R^4 B_{ik} a_{ij} + k_p a^2 R^2 \Gamma_{ij} \right] 
- \frac{m}{M} \Gamma_{lk} \times \left[ B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij} \right] - \left[ \left[ A_{ik} b_{ij} + 2v^2 R^2 C_{ik} c_{ij} + 2(1-v)R^2 E_{ik} e_{ij} + R^4 B_{ik} a_{ij} + k_p a^2 R^2 \Gamma_{ij} \right] 
- \frac{m}{M} \Gamma_{lk} \times \left[ B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij} \right] \right\} \} \right\} \times \left[ B_{ij} b_{ij} + \frac{m}{M} \Gamma_{ij} \right]^{-1}
\]

Specific results can now be evaluated for a variety of support conditions including simply support on all sides, simply supported on opposite sides, and clamped on all sides.

5. Simply supported plates (S–S–S–S)

The first plate to be considered is one containing any number of discrete masses and equal elastic supports such that the ith spring coordinates lie within \(0 < x_i < a\) and \(0 < y_i < b\). For a plate simply supported on all sides, the boundary condition in the \(x\) and \(y\)-direction is given by

\[
w = 0 \quad M_{xx} = 0 \quad x = 0, a \quad w = 0 \quad M_{yy} = 0 \quad y = 0, b
\]

Selecting orthonormal polynomials \([11]\) as the basis functions requires numerically evaluating the boundary matrices. These results are recalled when needed to evaluate Eq. (16). The
Choosing beam shape functions allows for an analytical evaluation of the boundary conditions which can be directly substituted into Eq. (16). To this end, select the basis function as

$$\phi_m (x) = \sqrt{2} \sin (\frac{\pi}{a} x) \quad \phi_m (y) = \sqrt{2} \sin (\frac{\pi y}{b})$$

Evaluating the boundary matrices given by Eq. (7) provides

$$A_{im} = i^2 \pi^2 \delta_{im} \quad a_{jn} = j^2 \pi^2 \delta_{jn}$$
$$B_{im} = \delta_{im} \quad b_{jn} = \delta_{jn}$$
$$C_{im} = -i^2 \pi^2 \delta_{im} \quad c_{jn} = -j^2 \pi^2 \delta_{jn}$$

and substituting these into Eq. (16) yields the desired approximate closed-form expression for simply supported boundary conditions on all sides.

$$\frac{\omega^2}{D} \left( \frac{a^4}{\pi^4} - \frac{2R^2 \pi^4}{a^4} + \frac{k_a a^2}{\pi^4 D} \right)$$

$$= \pi^4 \left[ i^4 + 2R^2 i^2 j^2 + R^4 j^4 + \frac{k_a a^2}{\pi^4 D} \Gamma_{ij} \right]$$

$$\times \left[ 1 + \frac{m}{M} \Gamma_{ij} \right]$$

$$- \frac{\pi^4}{(1 + (m/M) \Gamma_{ij})^2}$$

$$\times \sum_{k \neq l} \sum_{l 
eq j} \left\{ \left( i^4 + 2R^2 i^2 j^2 + R^4 j^4 + \frac{k_a a^2}{\pi^4 D} \Gamma_{il} \right) \right\}^2$$

$$\times \left[ \frac{m}{M} \Gamma_{ij} \right]$$

$$\times \left[ 1 + \frac{m}{M} \Gamma_{ij} \right]$$

$$\times \left[ \frac{k_a a^2}{\pi^4 D} \Gamma_{ijkl} \right]$$

$$\times \left[ 1 + \frac{m}{M} \Gamma_{ijkl} \right]$$

(21)

A simplification to Eq. (21) for the case of no added discrete masses produces

$$\frac{\omega^2}{D} \left( \frac{a^4}{\pi^4} - \frac{2R^2 \pi^4}{a^4} + \frac{k_a a^2}{\pi^4 D} \Gamma_{mm} \right)$$

$$-4 \pi^4 \sum_{k \neq l} \sum_{l \neq j} \left\{ \frac{(k_a a^2/\pi^4 D) \Gamma_{ijkl}}{a} \right\}$$

(22)

and if both added discrete mass $m=0$ and elastic support stiffness $k_a=0$, Eq. (22) reduces to the well known equation for the frequency of a simply supported, isotropic plate.

$$\frac{\omega^2}{D} a^4 \left[ i^4 + 2R^2 i^2 j^2 + R^4 j^4 \right]$$

(23)

6. Opposite edges simply supported

In a similar manner, an expression for a plate with two opposite edges simply supported in the $x$-direction and clamped in the $y$-direction requires identifying the following beam shape functions

$$\phi_m (x) = \sqrt{2} \sin (\frac{\pi n x}{a})$$

$$\phi_m (y) = \sqrt{2} \sin (\frac{\pi m y}{b})$$

and

$$\cos (\lambda_a) \cos (\lambda_b) = 1$$

$$\sigma_n = \frac{\cos (\lambda_a) - \cos (\lambda_b)}{\sin (\lambda_a) - \sin (\lambda_b)}$$

then the boundary matrices, Eq. (7) become

$$A_{im} = i^2 \pi^2 \delta_{im} \quad a_{jn} = j^2 \pi^2 \delta_{jn}$$
$$B_{im} = \delta_{im} \quad b_{jn} = \delta_{jn}$$
$$C_{im} = -i^2 \pi^2 \delta_{im} \quad c_{jn} = -j^2 \pi^2 \delta_{jn}$$

and Eq. (16) becomes

$$\frac{\omega^2}{D} a^4 \left[ i^4 + 2R^2 i^2 j^2 + R^4 j^4 + \frac{k_a a^2}{D} \Gamma_{ij} \right]$$

$$\times \left[ 1 + \frac{m}{M} \Gamma_{ij} \right]$$

$$- \frac{1}{(1 + (m/M) \Gamma_{ij})^2}$$

$$\times \sum_{k \neq l} \sum_{l 
eq j} \left\{ \left[ i^4 + 2R^2 i^2 j^2 + R^4 j^4 + \frac{k_a a^2}{\pi^4 D} \Gamma_{il} \right] \right\}^2$$

$$\times \left[ \frac{m}{M} \Gamma_{ij} \right]$$

$$\times \left[ 1 + \frac{m}{M} \Gamma_{ij} \right]$$

$$\times \left[ -2R^2 \frac{k_a a^2}{\pi^4 D} \Gamma_{ijkl} \right]$$

$$\times \left[ 1 + \frac{m}{M} \Gamma_{ijkl} \right]$$

$$\times \left( 1 + \frac{m}{M} \Gamma_{ijkl} \right)^{-1}$$

(24)

Plates which are simply supported in the $x$-direction and free in $y$-directions have boundary conditions given by

$$w = 0 \quad M_{xx} = 0 \quad x = 0, a$$

$$\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} = 0 \quad \frac{\partial^2 w}{\partial y^2} + (2 - v) \frac{\partial^2 w}{\partial x^2} = 0 \quad y = 0, b$$

and following beam shape functions of

$$\phi_m (x) = \sqrt{2} \sin (\frac{\pi n x}{a})$$

$$\phi_m (y) = \sqrt{2} \sin (\frac{\pi m y}{b})$$

are used to satisfy them. As in the case of a plate with clamped conditions along the $y$-direction, the coefficients defined above hold for this case of free boundary in along the $y$-direction. The boundary matrices give through Eq. (8) become

$$A_{im} = i^2 \pi^2 \delta_{im} \quad a_{jn} = j^2 \pi^2 \delta_{jn}$$
$$B_{im} = \delta_{im} \quad b_{jn} = \delta_{jn}$$
$$C_{im} = -i^2 \pi^2 \delta_{im} \quad c_{jn} = -j^2 \pi^2 \delta_{jn}$$

$$E_{im} = \frac{m \pi^2 \delta_{im}}{a} \quad e_{jn} = \frac{n \pi^2 \delta_{jn}}{b}$$

As before substituting the boundary matrices into Eq. (16) provides

$$\frac{\omega^2}{D} a^4 \left[ i^4 + 2R^2 i^2 j^2 + R^4 j^4 + \frac{k_a a^2}{\pi^4 D} \Gamma_{ij} \right]$$

$$\times \left[ 1 + \frac{m}{M} \Gamma_{ij} \right]$$

$$- \frac{1}{(1 + (m/M) \Gamma_{ij})^2}$$

$$\times \sum_{k \neq l} \sum_{l 
eq j} \left\{ \left[ i^4 + 2R^2 i^2 j^2 + R^4 j^4 + \frac{k_a a^2}{\pi^4 D} \Gamma_{il} \right] \right\}^2$$

$$\times \left[ \frac{m}{M} \Gamma_{ij} \right]$$

$$\times \left[ 1 + \frac{m}{M} \Gamma_{ij} \right]$$

$$\times \left[ -2R^2 \frac{k_a a^2}{\pi^4 D} \Gamma_{ijkl} \right]$$

$$\times \left[ 1 + \frac{m}{M} \Gamma_{ijkl} \right]$$

$$\times \left( 1 + \frac{m}{M} \Gamma_{ijkl} \right)^{-1}$$

(25)
functions instead as the basis functions. Eq. (16) was used to generate the approximate closed-form results. Configuration A without any added discrete masses was used to generate results in all the tables. In general the closed-form expression accurately predicts the fundamental frequency for all boundary conditions using either type of basis function. The largest percent difference in the two results occurs for the SFSP boundary condition. Although negligible, these are 0.5% and 0.73% using the orthogonal polynomials and beam shape functions, respectively.

Tables 4–8 present the first five frequencies computed by both methods. Again both orthogonal polynomials and beam shape functions and Table 3 presents the same data using beam shape functions.

\[
\begin{align*}
\times \left[ b_y + m \frac{m} {\ell} \omega^2 \right]^{-1} \sum \sum \left[ \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 \right] \\
\times \left[ \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 \right] \\
\times \left[ \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 \right] \\
\times \left[ \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 \right] \\
\times \left[ \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 \right] \\
\times \left[ \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 - 2 \times \left( \frac{\ell}{2} \right)^2 \right]
\end{align*}
\]

(25)

For all other combination of boundary conditions, clamped in the x-direction and free in y-direction for instance, Eq. (16) will be used since no simplification occurs after evaluating it with the boundary matrices.

7. Results

In this section, numerical results are presented for the derived approximate closed-form results and compared to results generated using the Rayleigh–Ritz method for the elastically supported plate. Under consideration is a square plate 30 in × 30 in × 0.5 in with Young’s modulus E and Poisson’s ratio ν of 10 × 10^6 psi and 0.315, respectively. Each of the fours springs has the same stiffness of 130 lb/in. Choice of spring placement and configuration is infinite. Three possible plate configurations are shown in Fig. 1 with corresponding spring coordinates given in Table 1. Configuration A has springs placed along the main dialogues, B has springs centered in all the tables. In general the closed-form expression accurately predicts the fundamental frequency for all boundary conditions using either type of basis function. The largest percent difference in the two results occurs for the SFSP boundary condition. Although negligible, these are 0.5% and 0.73% using the orthogonal polynomials and beam shape functions, respectively.

Table 2 presents convergence results for Ritz and closed-form expression using orthogonal polynomials.

<table>
<thead>
<tr>
<th>Spring</th>
<th>Configuration A</th>
<th>Configuration B</th>
<th>Configuration C</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>Y</td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>1</td>
<td>7.5</td>
<td>7.5</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>22.5</td>
<td>7.5</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>22.5</td>
<td>22.5</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>7.5</td>
<td>22.5</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 3 presents convergence results for Ritz and closed-form expression using beam shape functions.

<table>
<thead>
<tr>
<th>N</th>
<th>Ritz Quad</th>
<th>Ritz Quad</th>
<th>Ritz Quad</th>
<th>Ritz Quad</th>
<th>Ritz Quad</th>
<th>Ritz Quad</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>3.149</td>
<td>5.730</td>
<td>5.730</td>
<td>4.609</td>
<td>4.609</td>
</tr>
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<td>3.148</td>
<td>3.148</td>
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<td>5.728</td>
<td>4.608</td>
<td>4.608</td>
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<tr>
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<td>3.147</td>
<td>3.147</td>
<td>5.727</td>
<td>5.727</td>
<td>4.608</td>
<td>4.608</td>
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<tr>
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<td>3.147</td>
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<td>5.727</td>
<td>4.608</td>
<td>4.608</td>
</tr>
<tr>
<td>9</td>
<td>3.147</td>
<td>3.147</td>
<td>5.727</td>
<td>5.727</td>
<td>4.608</td>
<td>4.608</td>
</tr>
<tr>
<td>11</td>
<td>3.147</td>
<td>3.147</td>
<td>5.727</td>
<td>5.727</td>
<td>4.608</td>
<td>4.608</td>
</tr>
</tbody>
</table>

Fig. 1. Spring locations.
the lowest numerical result. In order to compare frequencies

\[ N \]

configuration A. The number of terms taken in the displacement
functions are used as basis functions. All results are generated for
configuration A. The number of terms taken in the displacement
functions are used as basis functions. All results are generated for

\[ \text{Table 4} \]

Comparison of normalized Ritz frequencies with closed-formed approximate frequencies for SSSS boundary condition (Hertz).

<table>
<thead>
<tr>
<th>( j )</th>
<th>( k_j )</th>
<th>( k_{\bar{j}} )</th>
<th>( k_{\bar{k}} )</th>
<th>( k_{\bar{y}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poly</td>
<td>Beam</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.147</td>
<td>3.158</td>
<td>3.147</td>
<td>3.158</td>
</tr>
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<td>2</td>
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<td>7.867</td>
<td>7.254</td>
<td>7.867</td>
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<tr>
<td>3</td>
<td>7.859</td>
<td>7.867</td>
<td>9.424</td>
<td>15.71</td>
</tr>
<tr>
<td>5</td>
<td>15.71</td>
<td>15.71</td>
<td>–</td>
<td>40.84</td>
</tr>
</tbody>
</table>

\[ \text{Table 5} \]

Comparison of normalized Ritz frequency with quadratic approximate equation for CCCC boundary condition, \( N=11, k=1612.5 \).

<table>
<thead>
<tr>
<th>( j )</th>
<th>( k_j )</th>
<th>( \bar{k}_{j1} )</th>
<th>( \bar{k}_{j2} )</th>
<th>( \bar{k}_{j3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poly</td>
<td>Beam</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5.727</td>
<td>5.732</td>
<td>5.727</td>
<td>5.731</td>
</tr>
<tr>
<td>2</td>
<td>11.68</td>
<td>11.85</td>
<td>11.67</td>
<td>11.85</td>
</tr>
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<td>3</td>
<td>11.68</td>
<td>11.85</td>
<td>20.89</td>
<td>20.99</td>
</tr>
<tr>
<td>4</td>
<td>17.22</td>
<td>17.58</td>
<td>32.92</td>
<td>33.52</td>
</tr>
<tr>
<td>5</td>
<td>21.94</td>
<td>20.94</td>
<td>46.83</td>
<td>49.20</td>
</tr>
</tbody>
</table>

\[ \text{Table 6} \]

Comparison of normalized Ritz frequency with quadratic approximate equation for SCSC boundary condition.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( k_j )</th>
<th>( \bar{k}_{j1} )</th>
<th>( \bar{k}_{j2} )</th>
<th>( \bar{k}_{j3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poly</td>
<td>Beam</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4.608</td>
<td>4.616</td>
<td>4.608</td>
<td>4.616</td>
</tr>
<tr>
<td>2</td>
<td>8.713</td>
<td>8.722</td>
<td>11.03</td>
<td>11.21</td>
</tr>
<tr>
<td>3</td>
<td>11.03</td>
<td>11.21</td>
<td>20.24</td>
<td>20.55</td>
</tr>
<tr>
<td>4</td>
<td>15.05</td>
<td>15.26</td>
<td>32.57</td>
<td>33.17</td>
</tr>
<tr>
<td>5</td>
<td>16.26</td>
<td>16.27</td>
<td>46.53</td>
<td>48.91</td>
</tr>
</tbody>
</table>

\[ \text{Table 7} \]

Comparison of normalized Ritz frequency with quadratic approximate equation for SFSF boundary condition using beam shape functions.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( k_j )</th>
<th>( \bar{k}_{j1} )</th>
<th>( \bar{k}_{j2} )</th>
<th>( \bar{k}_{j3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poly</td>
<td>Beam</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.729</td>
<td>2.749</td>
<td>7.363</td>
<td>15.17</td>
</tr>
<tr>
<td>2</td>
<td>7.352</td>
<td>7.204</td>
<td>11.48</td>
<td>18.98</td>
</tr>
<tr>
<td>3</td>
<td>7.545</td>
<td>10.25</td>
<td>13.80</td>
<td>20.67</td>
</tr>
<tr>
<td>4</td>
<td>12.06</td>
<td>17.61</td>
<td>20.32</td>
<td>25.65</td>
</tr>
<tr>
<td>5</td>
<td>14.27</td>
<td>22.20</td>
<td>24.61</td>
<td>29.63</td>
</tr>
</tbody>
</table>

\[ \text{Table 8} \]

Comparison of normalized Ritz frequency with quadratic approximate equation for CFCF boundary condition using beam shape functions.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( k_j )</th>
<th>( \bar{k}_{j1} )</th>
<th>( \bar{k}_{j2} )</th>
<th>( \bar{k}_{j3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poly</td>
<td>Beam</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4.29</td>
<td>4.30</td>
<td>10.81</td>
<td>20.09</td>
</tr>
<tr>
<td>2</td>
<td>8.41</td>
<td>8.06</td>
<td>14.21</td>
<td>23.30</td>
</tr>
<tr>
<td>3</td>
<td>10.80</td>
<td>10.87</td>
<td>16.16</td>
<td>24.70</td>
</tr>
<tr>
<td>4</td>
<td>14.76</td>
<td>17.94</td>
<td>21.92</td>
<td>28.88</td>
</tr>
<tr>
<td>5</td>
<td>14.84</td>
<td>22.46</td>
<td>25.97</td>
<td>32.53</td>
</tr>
</tbody>
</table>

functions are used as basis functions. All results are generated for
configuration A. The number of terms taken in the displacement expansion corresponds to \( N=11 \). Frequencies from the Ritz analysis are ordered beginning with the fundamental frequency, the lowest numerical result. In order to compare frequencies

\[ \text{Table 4} \]

Comparison of normalized Ritz frequencies with closed-formed approximate frequencies for SFSF boundary condition (Hertz).

<table>
<thead>
<tr>
<th>( j )</th>
<th>( k_j )</th>
<th>( k_{\bar{j}} )</th>
<th>( k_{\bar{k}} )</th>
<th>( k_{\bar{y}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poly</td>
<td>Beam</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4.608</td>
<td>4.616</td>
<td>4.608</td>
<td>4.616</td>
</tr>
<tr>
<td>2</td>
<td>8.713</td>
<td>8.722</td>
<td>11.03</td>
<td>11.21</td>
</tr>
<tr>
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<td>11.03</td>
<td>11.21</td>
<td>20.24</td>
<td>20.55</td>
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<tr>
<td>4</td>
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<td>16.27</td>
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<td>48.91</td>
</tr>
</tbody>
</table>

\[ \text{Table 5} \]

Comparison of normalized Ritz frequency with quadratic approximate equation for CCCC boundary condition, \( N=11, k=1612.5 \).

<table>
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<tr>
<th>( j )</th>
<th>( k_j )</th>
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<tr>
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<td></td>
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<tr>
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<td>7.363</td>
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<td>7.204</td>
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<td>12.06</td>
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<td>5</td>
<td>14.27</td>
<td>22.20</td>
<td>24.61</td>
<td>29.63</td>
</tr>
</tbody>
</table>

\[ \text{Table 6} \]

Comparison of normalized Ritz frequency with quadratic approximate equation for SCSC boundary condition.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( k_j )</th>
<th>( \bar{k}_{j1} )</th>
<th>( \bar{k}_{j2} )</th>
<th>( \bar{k}_{j3} )</th>
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<tbody>
<tr>
<td>Poly</td>
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<tr>
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<td>12.06</td>
<td>17.61</td>
<td>20.32</td>
<td>25.65</td>
</tr>
<tr>
<td>5</td>
<td>14.27</td>
<td>22.20</td>
<td>24.61</td>
<td>29.63</td>
</tr>
</tbody>
</table>

8. Conclusion

In this paper approximate closed-form formulas were developed to determine the vibration of elastically supported plates. A combination of edge support conditions were investigated including simply supported, clamped and free. Both orthogonal

Please cite this article as: Joseph Watkins R, Barton O Jr. Characterizing the vibration of an elastically point supported rectangular plate using eigensensitivity analysis. Thin Walled Struct (2010), doi:10.1016/j.tws.2009.11.005
polynomials and beam shape functions were used as admissible basis functions. When compared to Rayleigh–Ritz results, the computed fundamental frequency, for all boundary conditions, using the approximate closed-form expressions are excellent. The accuracy of higher frequencies, computed using the approximated closed-form expression is dependent upon boundary condition and choice of basis functions. A maximum percent difference of 0.06% occurs using beam shape functions while 12.9% occurs using the orthogonal polynomials for the SCSC boundary condition. For other mixed boundary conditions, which includes a free edge, percent differences in the fifth mode of 4.8% and 8.9% occurs for the SFSF and CFCF boundary conditions, respectively.

References

1 Altekin M. Free vibration and buckling of super-elliptical plates resting on symmetrically distributed point-supported plates. Thin-Walled Structures 2008;46:1066–86.