Randomized Algorithms for Systems and Control: Theory and Applications

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- Additional documents, papers, etc, please consult

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- Questions may be sent to

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References

- R. Tempo, G. Calafiore and F. Dabbene, “Randomized Algorithms for Analysis and Control of Uncertain Systems,” Springer-Verlag, London, 2005
- R. Tempo and H. Ishii, “Monte Carlo and Las Vegas Randomized Algorithms for Systems and Control: An Introduction,” EJC, Vol. 13, pp. 189-203, 2007
- RACT: Randomized Algorithms Control Toolbox for Matlab *http://ract.sourceforge.net*

Theory and Applications

- Theory of randomized algorithms for control
- UAV applications

Overview

- Preliminaries
- Probabilistic Robustness Analysis and Synthesis
- Sequential Methods for Convex Problems
- Non-Sequential Methods
- A Posteriori Analysis
- RACT
- Systems and Control Applications

Preliminaries

Randomized Algorithms (RAs)

- Randomized algorithms are frequently used in many areas of engineering, computer science, physics, finance, optimization,…but their appearance in systems and control is mostly limited to Monte Carlo simulations…

- Main objective of this NATO LS: Introduction to rigorous study of RAs for uncertain systems and control, with specific UAV applications

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Randomized Algorithms (RAs)

- Computer science (RQS for sorting, data structuring)
- Robotics (motion and path planning problems)
- Mathematics of finance (path integrals)
- Bioinformatics (string matching problems)
- Distributed algorithms (PageRank in Google)
- Computer vision (computational geometry)

Uncertainty

- Uncertainty has been always a critical issue in control theory and applications
- First methods to deal with uncertainty were based on a stochastic approach
- Optimal control: LQG and Kalman filter
- Since early 80's alternative deterministic approach (worst-case or robust) has been proposed

Robustness

- Major stepping stone in 1981: Formulation of the $\mathcal{H}_\infty$ problem by George Zames
- Various "robust" methods to handle uncertainty now exist: Structured singular values, Kharitonov, optimization-based (LMI), *l*-one optimal control, quantitative feedback theory (QFT)

Robustness

- Late 80’s and early 90’s: Robust control theory became a well-assessed area
- Successful industrial applications in aerospace, chemical, electrical, mechanical engineering, …
- However, …

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Limitations of Robust Control - 1

- Researchers realized some drawbacks of robust control
- Consider uncertainty Δ bounded in a set $\mathcal{B}$ of radius ρ. Largest value of ρ such that the system is stable for all $\Delta \in \mathcal{B}$ is called (worst-case) robustness margin
- Conservatism: Worst case robustness margin may be small
- Discontinuity: Worst case robustness margin may be discontinuous wrt problem data

Limitations of Robust Control - 2

- Computational Complexity: Worst case robustness is often $\mathcal{NP}$-hard (not solvable in polynomial time unless $\mathcal{P}=\mathcal{NP}$)

- Various robustness problems are $\mathcal{NP}$-hard
  - static output feedback
  - structured singular value
  - stability of interval matrices

Different Paradigm Proposed

- New paradigm proposed is based on uncertainty randomization and leads to randomized algorithms for analysis and synthesis
- Within this setting a different notion of problem tractability is needed
- Objective: Breaking the curse of dimensionality[1]

[1] R. Bellman (1957)

Probability *and* Robustness

- The interplay of Probability and Robustness for control of uncertain systems
- Robustness: Deterministic uncertainty bounded
- Probability: Random uncertainty (pdf is known)
- Computation of the probability of performance
- Controller which stabilizes *most* uncertain systems

Probabilistic Robustness Analysis

Uncertain Systems

$M(s)$ System

Δ Uncertainty

- Δ belongs to a structured set $\mathcal{B}$
  - Parametric uncertainty q
  - Nonparametric uncertainty Δ_{np}
  - Mixed uncertainty

Worst Case Model

- Worst case model: Set membership uncertainty
- The uncertainty Δ is bounded in a set $\mathcal{B}$

$$\Delta \in \mathcal{B}$$

- Real parametric uncertainty $q=[q_1,\ldots, q_\ell] \in \mathbf{R}^l$

$$q_i \in [q_i^-, q_i^+]$$

- Nonparametric uncertainty

$$\{\Delta_{\mathrm{np}} \in \mathbf{R}^{n,n} : \|\Delta_{\mathrm{np}}\| \leq 1\}$$

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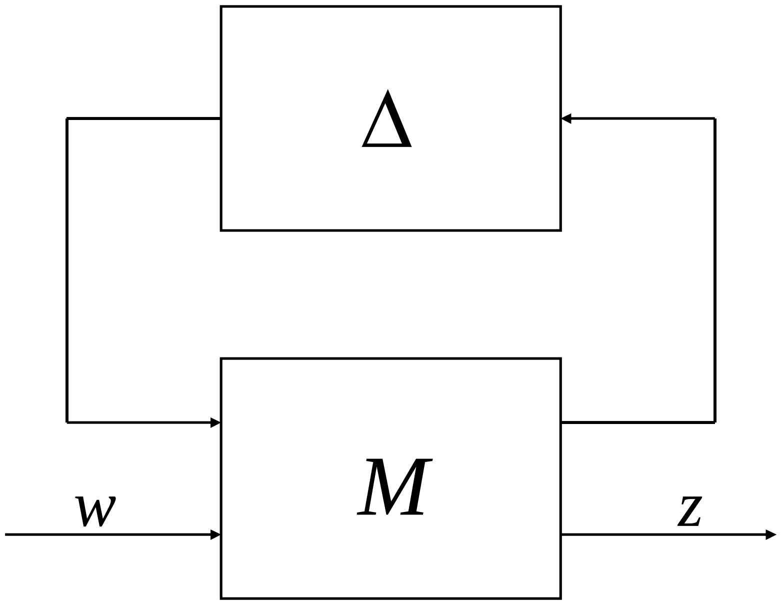
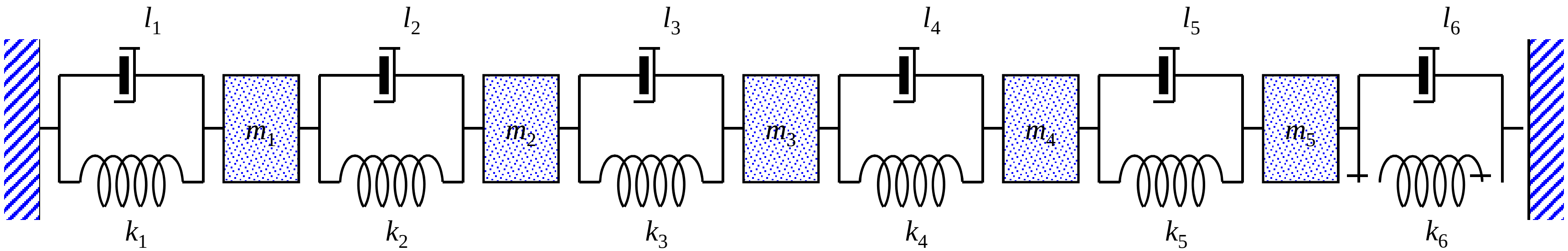
Robustness



- Uncertainty Δ is bounded in a structured set $\mathcal{B}$
- $z = F_u(M,\Delta)\, w$, where $F_u(M,\Delta)$ is the upper LFT

Example: Flexible Structure - 1

- Mass spring damper model
- Real parametric uncertainty affecting stiffness and damping
- Complex unmodeled dynamics (nonparametric)



Flexible Structure - 2

- M-Δ configuration for controlled system and study robustness

$$M(s) = C(sI - A)^{-1}B$$

$$\Delta = \begin{bmatrix} q_1 I_6 & 0 & 0 \\ 0 & q_2 I_6 & 0 \\ 0 & 0 & \Delta_{\mathrm{np}} \end{bmatrix}$$

$q_1, q_2 \in \mathbf{R}$

$\Delta_{\mathrm{np}} \in \mathbf{C}^{4,4}$

$\mathcal{B} = \{\Delta : \sigma(\Delta) < 1\}$

Probabilistic Model

- Probability density function associated to $\mathcal{B}$
- We assume that Δ is a random matrix (vector) with given density function and support $\mathcal{B}$
- Example: Δ is uniform in $\mathcal{B}$

Performance Function

- In classical robustness we guarantee that a certain performance requirement is attained for all $\Delta \in \mathcal{B}$
- This can be stated in terms of a performance function for analysis

$$J = J(\Delta)$$

- Example: $\mathcal{H}_\infty$ performance

Example: $\mathcal{H}_\infty$ Performance

- Compute the $\mathcal{H}_\infty$ norm of the upper LFT $F_u(M,\Delta)$

$$J(\Delta) = \| F_u(M, \Delta)\|_\infty$$

- For given $\gamma > 0$, check if

$$J(\Delta) \leq \gamma$$

for all $\Delta \in \mathcal{B}$

Probability of Performance

- Given a performance level γ, we define the probability of performance

$$\text{Prob}\{J(\Delta) \leq \gamma\}$$

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Measure of Violation and Reliability

- We define the measure of violation

$$V = 1 - \mathrm{Prob}\{J(\Delta) \leq \gamma\} = \mathrm{Prob}\{J(\Delta) > \gamma\}$$

- Probability of performance is also denoted as reliability

$$R = \mathrm{Prob}\{J(\Delta) \leq \gamma\} = 1 - V$$

Probabilistic Estimates

- Computing V and R requires to solve a difficult integration problem
- We use randomized algorithms to determine a probabilistic estimate of V and R

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Randomized Algorithm: Definition

- Randomized Algorithm (RA): An algorithm that makes random choices during its execution to produce a result
- Example of a “random choice” is a coin toss

heads or *tails*

Randomized Algorithm: Definition

- Randomized Algorithm (RA): An algorithm that makes random choices during its execution to produce a result

- For hybrid systems, “random choices” could be switching between different states or logical operations
- For uncertain systems, “random choices” require (vector or matrix) random sample generation

Monte Carlo Randomized Algorithm

- Monte Carlo Randomized Algorithm: A randomized algorithm that may produce incorrect results, but with bounded error probability

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Las Vegas Randomized Algorithm

- Las Vegas Randomized Algorithm: A randomized algorithm that always produces correct results, the only variation from one run to another is the running time

Monte Carlo Experiment

- We draw N i.i.d. random samples of Δ according to the given probability measure

$$\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(N)} \in \mathcal{B}$$

- The multisample within $\mathcal{B}$ is

$$\Delta^{1,\ldots,N} = \{\Delta^{(1)}, \ldots, \Delta^{(N)}\}$$

- We evaluate

$$J(\Delta^{(1)}), J(\Delta^{(2)}), \ldots, J(\Delta^{(N)})$$

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Estimated Probability of Reliability

- We construct the estimated probability of reliability

$$\hat{R}_N = \frac{1}{N}\sum_{i=1}^{N} \mathbf{I}\left(J(\Delta^{(i)})\right)$$

where $\mathbf{I}(\cdot)$ denotes the indicator function

$$\mathbf{I}\left(J(\Delta^{(i)})\right) = \begin{cases} 1 & \text{if } J(\Delta^{(i)}) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Sample Complexity

- We need to compute the size of the Monte Carlo experiment (sample complexity)
- This requires to introduce probabilistic accuracy $\varepsilon \in (0,1)$ and confidence $\delta \in (0,1)$
- Given ε, $\delta \in (0,1)$, we want to determine N such that the probability event

$$\left| R - \hat{R}_N \right| \leq \varepsilon$$

holds with probability at least 1- δ

Chernoff Bound[1]

■ Chernoff Bound

Given ε, δ ∈(0,1), if

$$N \geq N_{\mathrm{ch}} = \left\lceil \frac{\log\frac{2}{\delta}}{2\varepsilon^2} \right\rceil$$

then the probability inequality

$$\left| R - \hat{R}_N \right| \leq \varepsilon$$

holds with probability at least 1- δ

[1] H. Chernoff (1952)

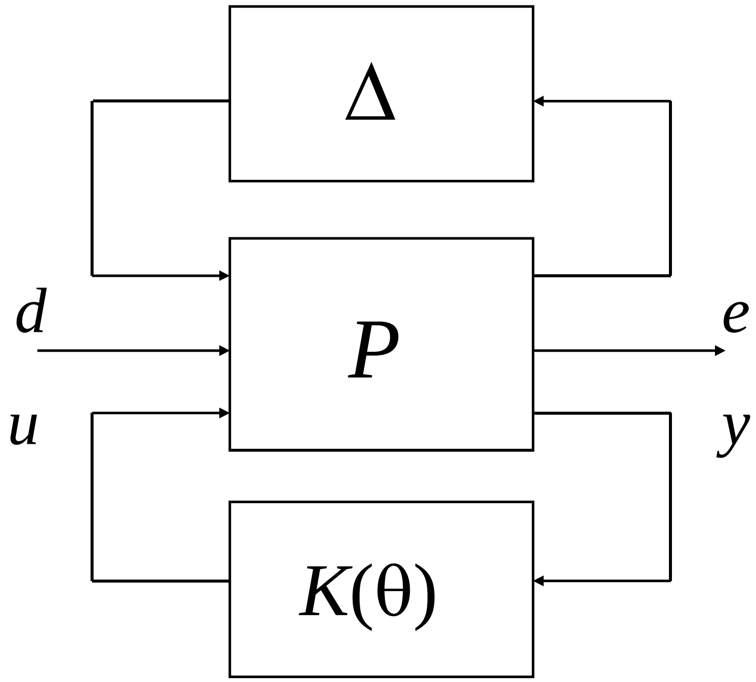
Remarks

- Chernoff bound improves upon other bounds such as the Law of Large Numbers (Bernoulli)
- Dependence is logarithmic on $1/\delta$ and quadratic on $1/\varepsilon$
- Sample size is independent on the number of controller and uncertain parameters

ε	0.1%	0.1%	0.5%	0.5%
$1-\delta$	99.9%	99.5%	99.9%	99.5%
N	$3.9 \cdot 10^6$	$3.0 \cdot 10^6$	$1.6 \cdot 10^6$	$1.2 \cdot 10^5$

Probabilistic Robust Synthesis

Synthesis Paradigm



- Design the parameterized controller $K(\theta)$ to guarantee stability and performance

Synthesis Performance Function

- Parameterized controller $K(\theta)$
- We replace $J(\Delta)$ with a synthesis performance function representing system constraints

$$J = J(\theta, \Delta)$$

where $\theta \in \Theta$ represents the controller parameters to be determined and Θ is their bounding set

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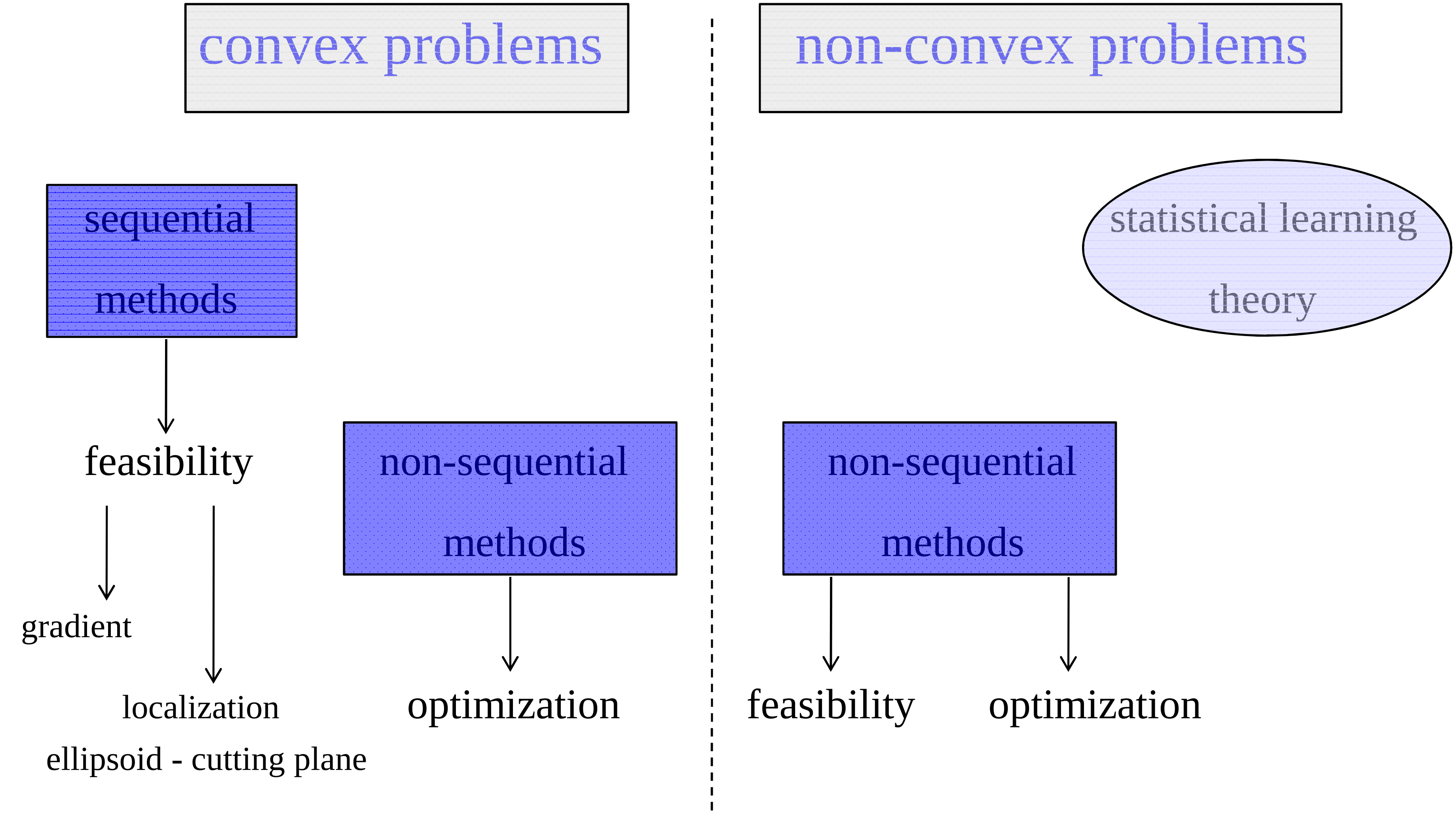
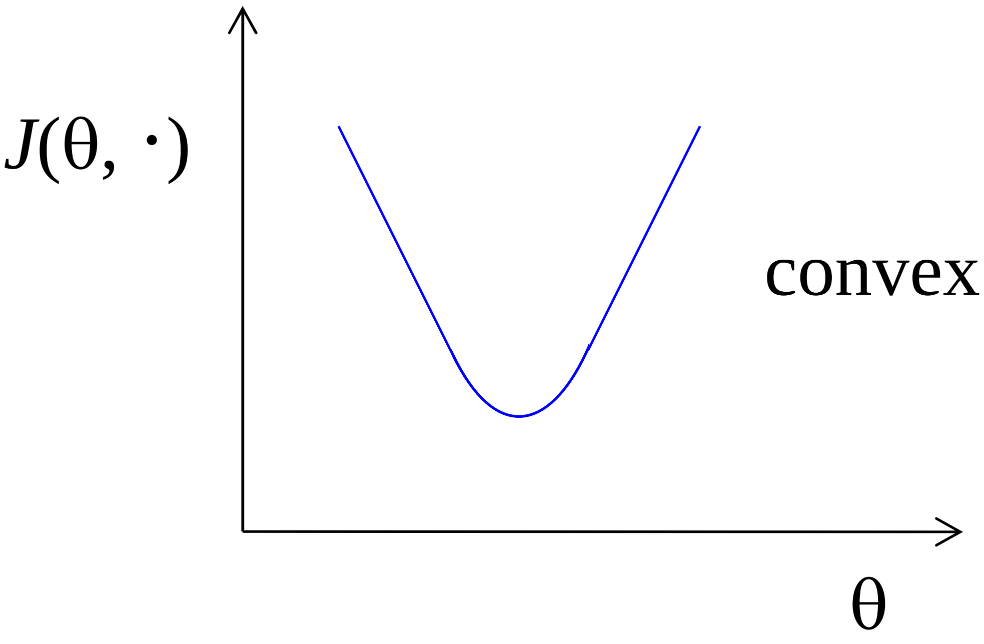
Probabilistic Design Methods: The Big Picture



Quadratic Performance and Convexity

Convexity Assumption

- Convexity Assumption: The function $J(\theta, \Delta)$ is convex in θ for any fixed value of $\Delta \in \mathcal{B}$



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Convex Functions and LQ Regulators

- Examples of convex functions arise when considering various control problems, such as design of LQ regulators
- This is illustrated by means of an application example for control of lateral motion of an aircraft

Example: Control of Lateral Motion of Aircraft[1]

- Multivariable example for the design of a controller for the lateral motion of an aircraft.
- The model consists of four states and two inputs

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where A and B are given by

[1] R. Tempo, G. Calafiore and F. Dabbene (2005)

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State Space Matrices

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & L_p & L_\beta & L_r \\ g/V & 0 & Y_\beta & -1 \\ N_{\dot\beta}\left(g/V\right) & N_p & N_\beta + N_{\dot\beta} Y_\beta & N_r - N_{\dot\beta} \end{bmatrix}$$

$$B=\begin{bmatrix} 0 & 0 \\ 0 & L_{\delta a} \\ Y_{\delta r} & 0 \\ N_{\delta r} + N_{\dot\beta} Y_{\delta r} & N_{\delta a} \end{bmatrix}$$

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State Variables and Control Inputs

- State variables
  - x_1 bank angle
  - x_2 derivative of bank angle
  - x_3 sideslip angle
  - x_4 jaw rate
- Control inputs
  - u_1 rudder deflection
  - u_2 aileron deflection

Uncertain Parameters

- Each parameter value is perturbed by a relative uncertainty equal to 10% around its nominal value $\bar{\Delta}_i$
- The uncertainty vector (parametric uncertainty)

$$\Delta = [\Delta_1, \Delta_2, \ldots, \Delta_{13}]^T$$

varies in an hyperrectangle centered at the nominal value

$$\mathcal{B} = \{\Delta : \Delta_i \in [0.90\bar{\Delta}_i, 1.10\,\bar{\Delta}_i],\ i=1,\ldots,13\}$$

- We have uncertain matrices $A(\Delta)$ and $B(\Delta)$

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Parameter Nominal Values

L_p=-2.93	L_β=-4.75	L_r=0.78	g/V=0.086	Y_β=-0.11
$N_{\dot\beta}$=0.1	N_p=-0.042	N_β=2.601	N_r=-0.29	$L_{\delta a}$=-3.91
$Y_{\delta r}$=0.035	$N_{\delta r}$=-2.5335	$N_{\delta a}$=0.31		

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Quadratic Performance Function

- We design a state feedback controller $u = Kx$ that robustly stabilizes the system guaranteeing a decay rate $\alpha > 0$
- Define the quadratic performance function

$$\Phi_{QP}(P, W, \Delta) = A(\Delta)P + PA^T(\Delta) + B(\Delta)W^T + WB^T(\Delta) + 2\alpha P$$

where $P = P^T > 0$ and W are matrices of suitable dimensions

Sufficient Condition

- A sufficient condition for the existence of a controller K is to find $P=P^T > 0$ and W such that

$$\Phi_{QP}(P, W, \Delta) \leq 0$$

is satisfied for all $\Delta \in \mathcal{B}$

- Equivalently we find (common) solutions $P=P^T > 0$ and W of the quadratic cost function

$$\Phi_{QP}(P, W, \Delta) \leq 0$$

for all $\Delta \in \mathcal{B}$

Control Gain

- A control gain which robustly guarantees the decay rate α for all $\Delta \in \mathcal{B}$ is given by

$$K = W^{\mathrm{T}} P^{-1}$$

- This problem can be reformulated in terms of linear matrix inequalities (LMIs)
- The controller is parameterized as $K=K(\theta)$, where

$$\theta = \{P, W\}$$

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Linear Matrix Inequalities (LMIs)

$\mathcal{B}$

- This quadratic constrained problem can be written in the general setting of LMIs
- Find θ such that

$$F(\theta, \Delta) \leq 0$$

for all $\Delta \in \mathcal{B}$ where

$$F(\theta, \Delta) = F_0(\Delta) + \theta_1 F_1(\Delta) + \ldots + \theta_n F_n(\Delta)$$

and $F_i(\Delta)$ are real symmetric matrices depending (nonlinearly) on Δ

Performance Function

- To rewrite an LMI in terms of a performance function $J(\theta, \Delta)$ we set

$$J(\theta, \Delta) = \lambda_{\max} F(\theta, \Delta)$$

where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of $(\cdot)$

Multiobjective Design Problems

- To consider scalar-valued constraints is without loss of generality
- Multiobjective design problems can be easily handled
- Multiple constraints of the form

$$J_1(\theta, \Delta) \leq 0, \ldots, \; J_n(\theta, \Delta) \leq 0$$

can be reduced to a single scalar-valued constraint setting

$$J(\theta, \Delta) = \max_i J_i(\theta, \Delta)$$

Sequential Methods for Convex Problems

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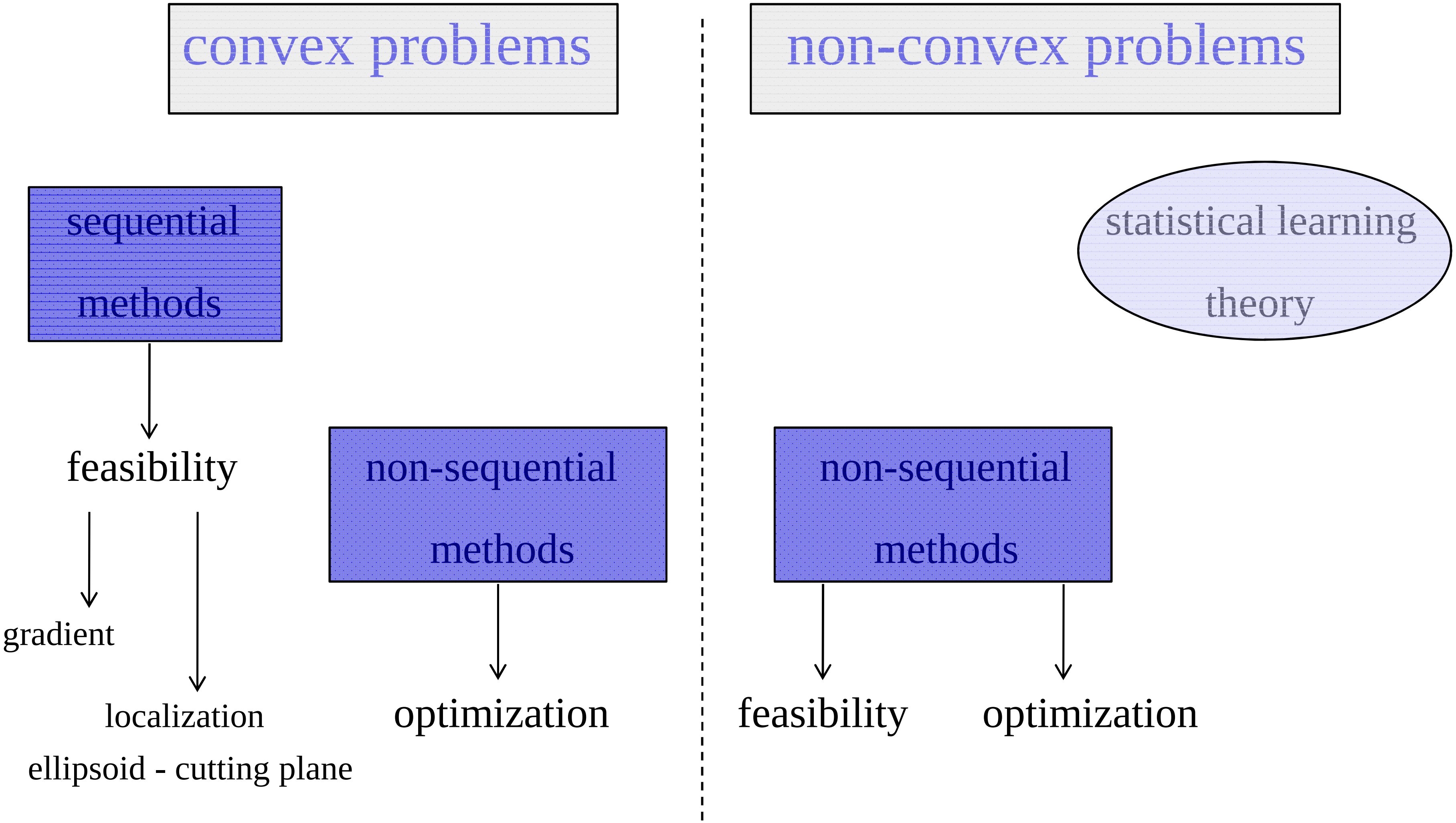
Probabilistic Design Methods: The Big Picture



Sequential Methods for Design

- We study randomized sequential methods for finding a probabilistic feasible solution θ
- That is we determine θ satisfying the uncertain inequality

$$J(\theta, \Delta) \leq 0$$

with some probability

Definition of r-feasibility

- r-feasibility: For given $r>0$, we say that $J(\theta, \Delta) \leq 0$ is r-feasible if the solution set

$$\mathcal{S} = \{\theta : J(\theta, \Delta) \leq 0 \text{ for all } \Delta \in \mathcal{B}\}$$

contains a (full-dimensional) ball of radius r

Performance Function

- Let Δ be a random vector distributed according to a probability measure
- Given probabilistic accuracy $\varepsilon \in (0,1)$, we search for $P=P^{\mathrm{T}} > 0$ and W such that

$$\mathrm{Prob}\{\Delta \in \mathcal{B}:\ \Phi_{QP}(P, W, \Delta) \leq 0\} > 1 - \varepsilon$$

- Defining the performance function

$$J(P, W, \Delta) = \lambda_{\max} \Phi_{QP}(P, W, \Delta)$$

the problem is to find $P=P^{\mathrm{T}} > 0$ and W such that

$$\mathrm{Prob}\{\Delta \in \mathcal{B}:\ J(P, W, \Delta) \leq 0\} > 1 - \varepsilon$$

Probability of Violation

- The probability of violation of the controller θ is

$$V(\theta) = \text{Prob}\{\Delta \in \mathcal{B}:\ J(\theta, \Delta) > 0\}$$

- We want to find θ such that the probability of violation is small

$$V(\theta) < \varepsilon$$

- If such θ exists in the feasible set $\mathcal{S}$, then we have a probabilistic feasible solution (probabilistic robust design)

Controller Reliability

- Given accuracy $\varepsilon \in (0,1)$, probabilistic robust design requires finding controller parameters θ such that the controller reliability

$$R(\theta) = 1 - V(\theta)$$

is at least $1 - \varepsilon$

Sequential Methods for Design

- Randomized sequential algorithms for finding a probabilistic feasible solution θ are based on two fundamental ingredients

i) Oracle checking probabilistic feasibility of a candidate solution

ii) Update rule exploiting convexity to construct a new candidate solution based on the oracle outcome

Meta-Algorithm

1. Initialization: set $k = 0$ and choose an initial solution θ_0

2. Oracle: Oracle returns *true* if θ_k is a probabilistic feasible controller and Exit returning $\theta_{seq} = \theta_k$

   Otherwise, the Oracle returns *false* and a violation certificate

3. Update Rule: Construct θ_{k+1} based on θ_k and on Δ_k

4. Outer iteration: Set $k=k+1$ and Goto 2

Probabilistic Oracle

- Oracle is the randomized part of the algorithm and decides probabilistic feasibility of the current solution
- We generate N_k i.i.d. samples of Δ within $\mathcal{B}$ (multisample)

$$\Delta^{(1)}, \ldots , \Delta^{(N_k)} \in \mathcal{B}$$

- The candidate solution θ_k is probabilistic feasible if

$$J(\theta_k, \Delta^{(i)}) \leq 0$$

for all $i = 1, \ldots , N_k$

- Otherwise if $J(\theta_k, \Delta^{(i)}) > 0$ we set $\Delta_k = \Delta^{(i)}$

Oracle (Inner) Iterations

- Consider the multisample size[1]

$$N_k \geq N_{\text{oracle}} = \left\lceil \frac{\log \frac{\pi^2 (k+1)^2}{6\delta}}{\log \frac{1}{1-\varepsilon}} \right\rceil$$

where ε , $\delta \in (0,1)$ are accuracy and confidence

- N_k is the number of Oracle (inner) iterations

[1] Y. Oishi (2007)

Algorithm Oracle

- Input: θ_k, N_k
- Output: feasibility (true/false), violation certificate Δ_k

- for i =1, ... , N_k , draw a sample $\Delta^{(i)}$

- Randomized test

- if $J(\theta_k, \Delta^{(i)}) > 0$, set $\Delta_k = \Delta^{(i)}$, feasibility = false
- exit and return Δ_k
- end if
- end for

Update Rule: Gradient Method

- We assume that the subgradient $\partial_k(\theta)$ of $J(\theta,\Delta)$ is computable at Δ_k
- If $J(\theta, \Delta_k)$ is differentiable at θ, then $\partial_k(\theta)$ is the gradient of $J(\theta, \Delta)$

Gradient Step and Stepsize

- Update rule is a classical gradient step

$$\theta_{k+1} = \begin{cases} \theta_k - \eta_k \dfrac{\partial_k(\theta_k)}{\left\|\partial_k(\theta_k)\right\|} & \text{if } \partial_k(\theta_k) \neq 0 \\ \theta_k & \text{otherwise} \end{cases}$$

- Let $r > 0$, then the stepsize η_k is given by

$$\eta_k = \begin{cases} \dfrac{J(\theta_k, \Delta_k)}{\left\|\partial_k(\theta_k)\right\|} + r & \text{if } \partial_k(\theta_k) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Algorithm Update Rule (Gradient)

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- Input: θ_k, Δ_k
- Output: θ_{k+1}

- compute the subgradient $\partial_k(\theta)$ of $J(\theta, \Delta_k)$
- compute the stepsize $\eta_k = \begin{cases} \dfrac{J(\theta_k, \Delta_k)}{\|\partial_k(\theta_k)\|} + r & \text{if } \partial_k(\theta_k) \neq 0 \\ 0 & \text{otherwise} \end{cases}$
- update

$$\theta_{k+1} = \begin{cases} \theta_k - \eta_k \dfrac{\partial_k(\theta_k)}{\|\partial_k(\theta_k)\|} & \text{if } \partial_k(\theta_k) \neq 0 \\ \theta_k & \text{otherwise} \end{cases}$$

Outer Iterations

- Define

$$N_{\text{outer}} = \left\lceil \frac{R^2}{r^2} \right\rceil$$

where R is the distance between the initial solution θ_0 and the center of a ball of radius r contained in the solution set $\mathcal{S}$

- r is imposed by the desired radius of feasibility
- If R is unknown, then we replace it with an upper bound which can be easily estimated

Algorithm Sequential Design

- Input: $\varepsilon, \delta \in (0,1)$, N_{outer}
- Output: θ_{seq}
- choose θ_0, set $k=0$ and feasibility=false
- Outer iteration
- while feasibility = false and $k < N_{\text{outer}}$
- determine multisample size N_k
- invoke Oracle obtaining feasibility (true/false) and Δ_k
- if feasibility = false then compute θ_{k+1} using Update Rule
- else set $\theta_{\text{seq}} = \theta_k$
- set $k = k + 1$
- end while

Probabilistic Properties of Sequential Design

- Theorem[1]

  Let Convexity Assumption hold and let $\varepsilon, \delta \in (0,1)$

- If Algorithm Sequential Design terminates at some outer iteration $k < N_{outer}$ returning θ_{seq}, then the probability that $V(\theta_{seq}) > \varepsilon$ is at most δ

- If Algorithm Sequential Design reaches the outer iteration N_{outer}, then the problem is not r-feasible

[1] F. Dabbene and R. Tempo (2008)

Remark: Successful/Unsuccessful Exit

- The first situation corresponds to a successful exit: The algorithms returns a probabilistic controller θ_{seq}
- The second situation corresponds to an unsuccessful exit: No solution has been found in N_{outer} iterations
- In this case we have a certificate of violation Δ_k returned by the Oracle showing that the problem is not r-feasible

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Aircraft Example Revisited: Sequential Methods

- Setting α =0.5, we look for a probabilistic solution to the uncertain LMI

$$P=P^{T}>0 \qquad \Phi_{QP}(P, W, \Delta) \leq 0$$

where the quadratic performance function is given by

$$\Phi_{QP}(P, W, \Delta) = A(\Delta)P + PA^{T}(\Delta) + B(\Delta)W^{T} + WB^{T}(\Delta)+2\alpha P$$

- Letting ε =0.01 and $\delta = 10^{-6}$, the sequential algorithm is guaranteed to return (with 99.9999% probability) a solution P, W such that quadratic performance holds with 99% probability

Numerical Results

- Algorithm terminated after k = 28 (outer) iterations
- Quadratic performance was checked by the Oracle for

$$N_k = 2{,}029$$

uncertainty samples

- We obtained …

P_{seq} and W_{seq}

$$P_{seq} = \begin{bmatrix} 0.3075 & -0.3164 & -0.0973 & -0.0188 \\ -0.3164 & 0.5822 & -0.0703 & -0.0993 \\ -0.0973 & -0.0703 & 0.2277 & 0.2661 \\ -0.0188 & -0.0993 & 0.2661 & 0.7100 \end{bmatrix}$$

$$W_{seq} = \begin{bmatrix} -0.0191 & 0.2733 \\ -0.0920 & 0.4325 \\ 0.0803 & -0.3821 \\ 0.4496 & -0.2032 \end{bmatrix}$$

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Probabilistic Controller K_{seq}

- Probabilistic controller $K = W^{T}P^{-1}$ is given by

$$K_{seq} = \begin{bmatrix} -2.9781 & -1.9139 & -3.2831 & 1.5169 \\ 7.3922 & 5.1010 & 4.1401 & -0.9284 \end{bmatrix}$$

- With an a-posteriori analysis we will check if K_{seq} is a robust controller and its probabilistic properties

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Non-Sequential Methods for Convex Problems

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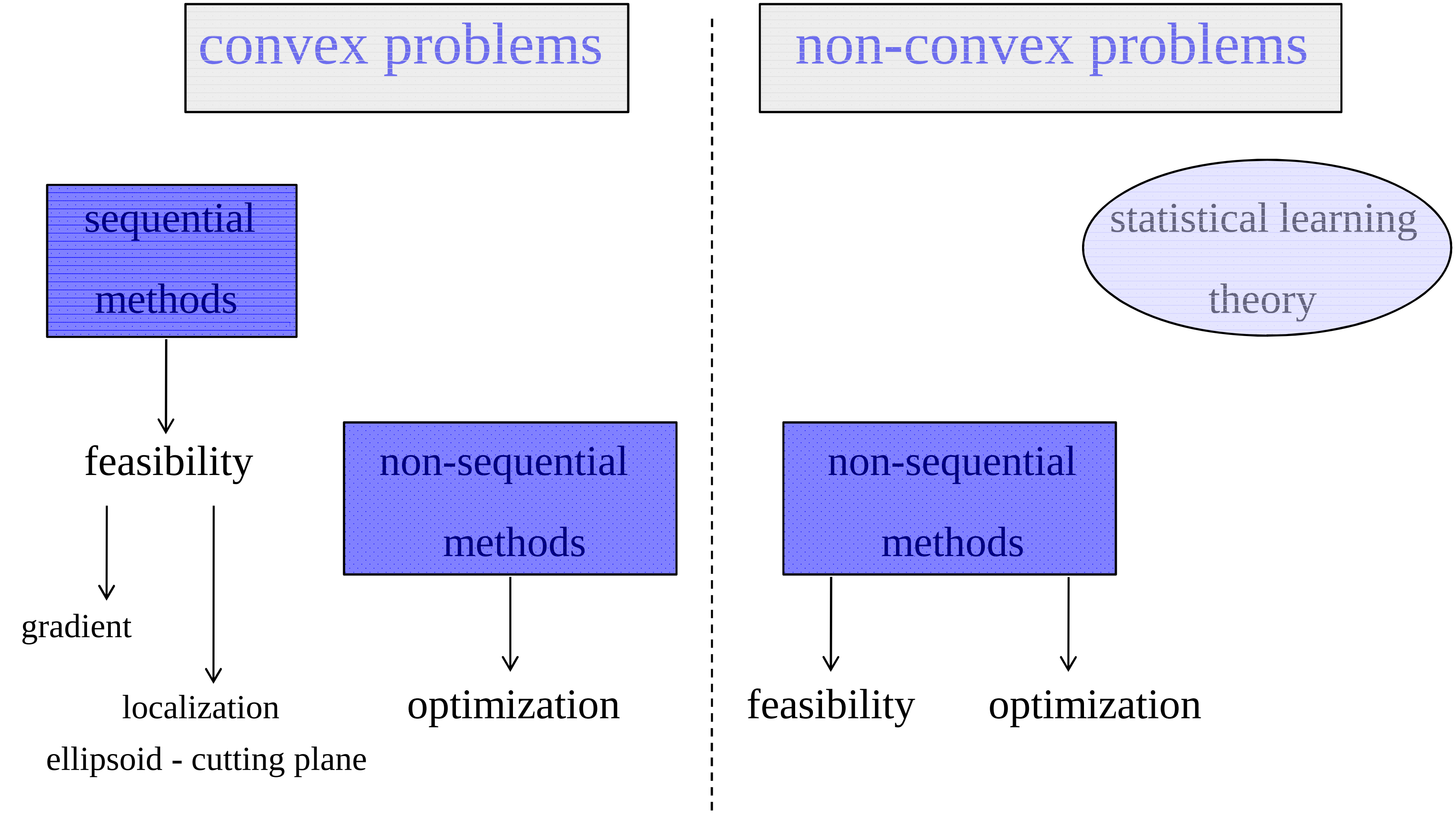
Probabilistic Design Methods: The Big Picture



Convexity Assumption

- Convexity Assumption: The function $J(\theta, \Delta)$ is convex in θ for any fixed value of $\Delta \in \mathcal{B}$

Scenario Approach

- Non-sequential method which provides a one-shot solution for general uncertain convex problems
- Randomization of $\Delta \in \mathcal{B}$ and solution of a single convex optimization problem
- Derivation of a formula involving sample size, number of controller parameters, probabilistic accuracy and confidence
- Explicit computation of the sample complexity

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Convex Semi-Infinite Optimization

- Semi-infinite optimization problem

$$\min_{\theta \in \mathbf{R}^n} c^T \theta \quad \text{subject to } J(\theta, \Delta) \leq 0 \quad \text{for all } \Delta \in \mathcal{B}$$

where $J(\theta, \Delta) \leq 0$ is convex in θ for all $\Delta \in \mathcal{B}$ and n is the number of design parameters

Scenario Problem

- We construct a scenario problem using randomization
- Taking i.i.d. random samples $\Delta^{(i)}$, $i = 1, \ldots, N$, we construct the sampled constraints

$$J(\theta, \Delta^{(i)}) \leq 0, \quad i = 1, \ldots, N$$

and form the scenario optimization problem (convex problem)

$$\theta_{\text{scen}} = \arg\min_{\theta \in \mathbf{R}^n} c^T \theta \quad \text{subject to} \quad J(\theta, \Delta^{(i)}) \leq 0, \quad i = 1, \ldots, N$$

Convex Scenario Design

- Theorem[1]

Let Convexity Assumption hold. Suppose that $N \geq n$ and $\varepsilon, \delta \in (0,1)$ satisfy the inequality

$$\binom{N}{n}(1-\varepsilon)^{N-n} \leq \delta$$

then, the probability that

$$V(\theta_{scen}) = \text{Prob}\{\Delta \in \mathcal{B}: \ J(\theta_{scen}, \Delta) > 0\} > \varepsilon$$

is at most δ

[1] G. Calafiore and M. Campi (2005)

Remarks

- We have considered the case when the scenario problem admits a feasible solution and this solution is unique
- Clearly, if the scenario problem is unfeasible, then also the original semi-infinite convex problem is unfeasible
- The assumption on uniqueness of the solution can be relaxed in most practical cases

Sample Complexity

- Computing the minimum value of N such that

$$\binom{N}{n}(1-\varepsilon)^{N-n} \leq \delta$$

holds is immediate (given ε, δ and n, is a one-parameter problem)

- A different issue is to derive the sample complexity which is an *explicit* relation of the form

$$N = N(\varepsilon, \delta, n)$$

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Sample Complexity of the Scenario Problem

- Sample complexity can be computed for the scenario problem
- In[1] it has been proven that the relation

$$\binom{N}{n}(1-\varepsilon)^{N-n} \leq \delta$$

holds if

$$N \geq N_{scen}(\varepsilon,\delta,n) = \left\lceil \frac{2}{\varepsilon}\log\left(\frac{1}{2\delta}\right) + 2n + \frac{2}{\varepsilon}\log(4) \right\rceil$$

[1] T. Alamo, R. Tempo and E.F. Camacho (2007)

Algorithm Scenario Design

- Input: ε, δ, n
- Output: θ_{scen}

- compute the sample size $N_{\text{scen}}(\varepsilon, \delta, n)$
- draw $N \geq N_{\text{scen}}(\varepsilon, \delta, n)$ i.i.d. samples $\Delta^{(i)}$
- solve the convex optimization problem

$$\theta_{\text{scen}} = \arg \min_{\theta \in \mathbf{R}^n} c^T \theta \quad \text{subject to} \quad J(\theta, \Delta^{(i)}) \leq 0, \; i = 1, \ldots, N$$

Aircraft Example Revisited: Scenario Design

- The objective is to determine a probabilistic solution to the optimization problem

$$\min_{P,\,W} \quad \mathrm{Tr}\, P \quad \text{subject to} \quad P=P^{\mathrm{T}} > 0,\ \Phi_{\mathrm{QP}}(P, W, \Delta) \leq 0$$

where $\mathrm{Tr}(\cdot)$ denotes the trace of $(\cdot)$

- Setting ε =0.01 and $\delta=10^{-6}$, we compute the sample complexity for n=18 obtaining

$$N_{\mathrm{scen}} = 7{,}652$$

- Hence we need to solve a convex optimization problem with 7,652 constraints and 18 design variables

P_{scen} and W_{scen}

$$P_{scen} = \begin{bmatrix} 0.1445 & -0.0728 & 0.0035 & 0.0085 \\ -0.0728 & 0.2192 & -0.0078 & -0.0174 \\ 0.0035 & -0.0078 & 0.1375 & 0.0604 \\ 0.0085 & -0.0174 & 0.0604 & 0.1975 \end{bmatrix}$$

$$W_{scen} = \begin{bmatrix} 0.0109 & 0.0908 \\ 7.2929 & 3.4846 \\ 0.0439 & -0.0565 \\ 0.6087 & -3.9182 \end{bmatrix}$$

Probabilistic Controller K_{scen}

- Probabilistic controller $K = W^{T} P^{-1}$ is equal to

$$K_{\text{scen}} = \begin{bmatrix} 20.0816 & 40.3852 & -0.4946 & 5.9234 \\ 10.7941 & 18.1058 & 9.8937 & -21.7363 \end{bmatrix}$$

Non-Sequential Methods for Non-Convex Problems

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Probabilistic Design Methods: The Big Picture

convex problems

sequential methods

feasibility

gradient

localization

ellipsoid - cutting plane

non-sequential methods

optimization

non-convex problems

statistical learning theory

non-sequential methods

feasibility

optimization

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Statistical Learning Theory for Control Design of Uncertain Systems

- Statistical learning theory is a branch of the theory of empirical processes
- Significant results have been obtained in various areas, including neural networks, system identification, pattern recognition, …
- We study statistical learning theory for control design of uncertain systems

Statistical Learning Theory

- Main objective is to derive uniform convergence laws (for all controller parameters) and the sample complexity
- This leads to a powerful methodology for control synthesis (feasibility and optimization) which is not based upon a convexity assumption on the controller parameters
- The sample complexity is significantly larger than that derived in the convex case

Controller Reliability

- Recall that the reliability for the controller $K(\theta)$ is

$$R(\theta) = \mathrm{Prob}\{\Delta \in \mathcal{B}:\ J(\theta, \Delta) \leq 0\} = 1 - V(\theta)$$

- Computing $R(\theta)$ requires to solve a difficult integration problem
- For fixed θ we compute a probabilistic estimate of reliability setting a simple Monte Carlo experiment

Monte Carlo Experiment

- We take N i.i.d. random samples of Δ according to the given probability measure

$$\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(N)} \in \mathcal{B}$$

- We evaluate

$$J(\theta, \Delta^{(1)}), J(\theta, \Delta^{(2)}), \ldots, J(\theta, \Delta^{(N)})$$

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Estimated Probability of Reliability

- Given controller parameters θ, we construct a probabilistic estimated of reliability

$$\hat{R}_N(\theta) = \frac{1}{N}\sum_{\mathrm{i}=1}^{N} \mathbf{I}\left(J(\theta, \Delta^{(i)})\right)$$

where $\mathbf{I}\,(\cdot)$ denotes the indicator function

$$\mathbf{I}\left(\mathrm{J}(\theta, \Delta^{(\mathrm{i})})\right) = \begin{cases} 1 & \text{if } J(\theta, \Delta^{(\mathrm{i})}) \le 0 \\ 0 & \text{otherwise} \end{cases}$$

Law of Large Numbers

- Monte Carlo analysis (Law of Large Numbers) studies the sample complexity such that for *fixed* θ the probability inequality

$$\left| R(\theta) - \hat{R}_N(\theta) \right| \leq \varepsilon$$

holds with probability at least 1- δ

Uniform Convergence Law

- Statistical learning theory studies the sample complexity such that the probability inequality

$$\left| R(\theta) - \hat{R}_N(\theta) \right| \leq \varepsilon$$

holds uniformly for all θ with probability at least 1- δ

Optimization of Non-Convex Problems

Constrained Feedback Design with Uncertainty

- The objective is to minimize an objective function $c(\theta)$ subject to the performance constraint

$$J(\theta, \Delta) \leq 0$$

- The problem is formulated in terms of a binary performance function

Binary Performance Function g

- We introduce the performance function g

  $g: \Theta \times \mathcal{B} \rightarrow \{0,1\}$

  which is a binary measurable function defined as

$$g(\theta, \Delta) = \begin{cases} 0 & \text{if } J(\theta, \Delta) \leq 0 \\ 1 & \text{otherwise} \end{cases}$$

Binary Probability of Violation

- Given $\theta \in \mathbf{R}^n$, the binary probability of violation for the function $g(\theta, \Delta)$ is defined as

$$V_g(\theta) = \mathrm{Prob}\{\Delta \in \mathcal{B}\colon g(\theta, \Delta) = 1\}$$

Binary Optimization Problem

- Semi-Infinite Optimization Problem: Find the optimal solution of the problem

$$\min_{\theta \in \mathbf{R}^n} \; c(\theta) \quad \text{subject to } g(\theta, \Delta) = 0 \text{ for all } \Delta \in \mathcal{B}$$

where $c\colon \Theta \rightarrow \mathbf{R}$ is a measurable function

Randomized Non-Convex Optimization Problem

- Generate N i.i.d. samples (multisample) within $\mathcal{B}$

$$\Delta^{1,\ldots,N} = \{\Delta^{(1)}, \ldots , \Delta^{(N)}\}$$

according to a given probability measure

- Compute a (local) solution of the non-convex optimization problem

$$\theta_{\mathrm{ncon}} = \arg\min_{\theta\in\mathbf{R}^n} c(\theta) \;\; \text{subject to} \;\; g(\theta,\Delta^{(i)}) = 0, \;\; i = 1,\ldots, N$$

Boolean Binary Function g

- The function g: $\mathbf{R}^n \times \mathcal{B} \rightarrow \{0,1\}$ is (γ, m)-Boolean binary if for fixed Δ can be written as a Boolean expression consisting of m polynomials in the variables θ_i, $i=1,\dots,n$

$$\beta_1(\theta, \Delta), \dots, \beta_m(\theta, \Delta)$$

and the degree with respect to θ_i of all these polynomials is no larger than γ

- Example: For fixed Δ take $m=1$ and

$$g = \beta_1(\theta) = 3 + 2\,\theta_1^2 - 5\,\theta_2^4\,\theta_3 + \dots + 4\,\theta_1^2\,\theta_2\,\theta_4^7 \qquad \gamma = 7$$

Non-Convex Learning Based Design

- Theorem[1]

Let $g(\theta, \Delta)$ be (γ, m)-Boolean. Given $\varepsilon \in (0,0.14)$ and $\delta \in (0,1)$, if

$$N \geq N_{\mathrm{ncon}}(\varepsilon,\delta,n) = \left\lceil \frac{1}{\varepsilon}\left(4.1\log\left(\frac{21.64}{\delta}\right) + 36n\log_2 \max\left\{\frac{2}{\varepsilon}, 4e\gamma m\right\}\right)\right\rceil$$

where e is the Euler number, then the probability that

$$V_g(\theta_{\mathrm{ncon}}) = \mathrm{Prob}\{\Delta \in \mathcal{B}: g(\theta_{\mathrm{ncon}}, \Delta) = 1\} > \varepsilon$$

is at most δ

[1] T. Alamo, R. Tempo and E.F. Camacho (2007)

Comments - 1

- The function *g* is a Boolean expression consisting of polynomials; constraints and objective function are non-convex
- Sample complexity result holds for any suboptimal (local) solution
- We can use linearization algorithms to obtain a local solution (no need to compute a global solution)
- The approach consists of uncertainty randomization and deterministic optimization in controller space
- We avoid randomization of controller parameters

Empirical Mean of Violation

- Given N i.i.d. samples within $\mathcal{B}$

$$\Delta^{1,\ldots,N} = \{\Delta^{(1)}, \ldots , \Delta^{(N)}\}$$

the empirical mean of violation is equal to

$$\hat{V}_g(\theta) = \frac{1}{N}\sum_{i=1}^{N} g\left(\theta, \Delta^{(i)}\right)$$

- Since g is a binary function

$$\hat{V}_g(\theta) \in [0, 1]$$

Randomized Optimization Problem

- Recall that the randomized optimization problem is given by

$$\theta_{\text{ncon}} = \arg\min_{\theta \in \mathbf{R}^n} c(\theta) \text{ subject to } g(\theta, \Delta^{(i)}) = 0, \; i = 1, \ldots, N$$

- This problem is equivalent to

$$\theta_{\text{ncon}} = \arg\min_{\theta \in \mathbf{R}^n} c(\theta) \text{ subject to } \hat{V}_g(\theta) = 0$$

Comments

- Solving the original semi-infinite optimization problem is extremely difficult given the infinite number of constraints
- Using the concept of empirical mean, the optimization problem has only one constraint with a finite sum (for fixed θ)
- Develop a strategy to solve semi-infinite optimization problems such that the empirical mean of violation is zero

Algorithm Non-Convex Design

- Input: ε, δ, n
- Output: θ_{ncon}

- compute the sample size $N_{\text{ncon}}(\varepsilon, \delta, n)$
- draw $N \geq N_{\text{ncon}}(\varepsilon, \delta, n)$ i.i.d. samples $\Delta^{(i)}$
- compute (local) solution of the non-convex problem

$$\theta_{\text{ncon}} = \arg\min_{\theta \in \mathbf{R}^n} c(\theta) \quad \text{subject to} \quad \hat{V}_g(\theta) = 0$$

Aircraft Example Revisited: Learning Design

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- In this example we consider Hurwitz stability instead of quadratic stability (the problem is non-convex)
- The objective is to determine a controller K that computes a probabilistic solution to the optimization problem

$$\min_{\alpha,K}(-\alpha) \text{ subject to } \big(A(\Delta) + B(\Delta)K + \alpha \mathrm{I}\big) \text{ Hurwitz for all } \Delta \in \mathcal{B}$$

$$K_{ij} \in [-\overline{K}_{ij}, \overline{K}_{ij}]$$

Bounds on the Gain Matrix

- The matrix $\overline{K}$ is given by

$$\overline{K} = \begin{bmatrix} 5 & 0.5 & 5 & 5 \\ 5 & 2 & 20 & 1 \end{bmatrix}$$

Sample Complexity

- By means of tedious computations involving reformulation of Hurwitz stability in terms of polynomial Boolean functions we obtain

$$n = 9, \gamma = 10, m = 20$$

- Setting $\varepsilon = 0.01$ and $\delta = 10^{-6}$ the sample complexity can be easily derived

$$N_{\mathrm{ncon}}(\varepsilon, \delta, n) = 366{,}130$$

Probabilistic Controller K_{ncon}

- Probabilistic controller for Hurwitz stability is given by

$$K_{\text{ncon}} = \begin{bmatrix} 0.8622 & 0.2714 & -5.0000 & 2.7269 \\ 5.0000 & 1.4299 & 3.9328 & -1.0000 \end{bmatrix}$$

$\alpha = 3.7285$

- We notice that three gains are saturated, i.e. they are equal to the prespecified bound on the gain matrix

A Posteriori Analysis

A Posteriori Analysis

- When a probabilistic controller K_{prob} has been design with one of the previous methods, we need to verify its performance and address the following questions:

1. Is K_{prob} a robust controller (in the classical sense)?
2. What is the probabilistic performance of K_{prob}?

A Posteriori Deterministic Analysis

Worst-Case Performance

- Deterministic (or worst-case) analysis provides the radius of deterministic performance ρ_{wc}
- The radius ρ_{wc} is the largest value of $\rho > 0$ for which the constraint

$$J(\theta, \Delta) \leq 0$$

is robustly satisfied for all $\Delta \in \mathcal{B}_\rho = \{\Delta \in \rho\mathcal{B}\}$

Aircraft Example Revisited: Worst-Case Analysis

- Consider the previous aircraft example and study the dependence of $A(\Delta)$ and $B(\Delta)$ on uncertain parameters

$$\Delta = [\Delta_1, \Delta_2, \ldots, \Delta_l]^T$$

restricted in the hyperrectangle $\mathcal{B}_\rho$

- We notice that $A(\Delta)$ and $B(\Delta)$ depend multiaffinely on Δ

A function f: $\mathbf{R}^l \rightarrow \mathbf{R}$ is multiaffine if the condition holds: If all components $\Delta_1, \ldots, \Delta_l$ except one are fixed, then f is affine

Multiaffine Dependence

$$A(\Delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \Delta_1 & \Delta_2 & \Delta_3 \\ \Delta_4 & 0 & \Delta_5 & -1 \\ \Delta_4\Delta_6 & \Delta_7 & \Delta_8 + \Delta_5\Delta_6 & \Delta_9 - \Delta_6 \end{bmatrix}$$

$$B(\Delta) = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_{10} \\ \Delta_{11} & 0 \\ \Delta_{12} + \Delta_6\Delta_{11} & \Delta_{13} \end{bmatrix}$$

Quadratic Performance and Vertices - 1

- For fixed ρ quadratic performance of state space uncertain systems affected by multiaffine uncertainty is equivalent to quadratic performance of the vertex set $\mathcal{B}_\rho$

Quadratic Performance and Vertices - 2

- Recall that

$$\Phi_{QP}(P, W, \Delta) = A(\Delta)P + PA^T(\Delta) + B(\Delta)W^T + WB^T(\Delta) + 2\alpha P$$

- Then, given P_{seq} and W_{seq}

$$\Phi_{QP}(P_{seq}, W_{seq}, \Delta) \leq 0 \text{ for all } \Delta \in \mathcal{B}_\rho$$

if and only if

$$\Phi_{QP}(P_{seq}, W_{seq}, \Delta_v^i) \leq 0 \text{ for all } i = 1, \ldots, 2^l$$

where Δ_v^i represents the i-th vertex of $\mathcal{B}_\rho$

Line Search for Radius Computation

- Computing the worst-case radius requires to solve a one-dimensional problem in the variable ρ and check if $\Phi_{QP}(P_{seq}, W_{seq}, \Delta_v{}^i) \leq 0$ for all vertices of $\mathcal{B}_\rho$
- This problem can be solved using bisection, but an exponential number of vertices of $\mathcal{B}_\rho$ should be considered (8,192 vertices in this case)

Worst-Case Radius of Performance

- Performing this analysis for P_{seq} and W_{seq} we compute the worst-case radius of performance

$$\rho_{wc} = 0.12$$

- Hence robust quadratic performance is guaranteed for all $\Delta \in \mathcal{B}_\rho$, $\rho = [0,0.12]$

A Posteriori Probabilistic Analysis

Controller Reliability

- Recall that the reliability for the controller $K(\theta)$ is

$$R(\theta) = \mathrm{Prob}\{\Delta \in \mathcal{B}:\ J(\theta, \Delta) \leq 0\}$$

- Take $\theta_{seq} = \{P_{seq}, W_{seq}\}$
- Computing $R(\theta_{seq})$ for fixed θ_{seq} requires to solve a difficult integration problem
- We determine an estimate of this probability setting a simple Monte Carlo experiment

Monte Carlo Experiment

- We take N i.i.d. random samples of Δ according to the given probability measure

$$\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(N)} \in \mathcal{B}$$

- We evaluate

$$J(\theta_{\mathrm{seq}}, \Delta^{(1)}), J(\theta_{\mathrm{seq}}, \Delta^{(2)}), \ldots, J(\theta_{\mathrm{seq}}, \Delta^{(N)})$$

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Estimated Probability of Reliability

- Given controller θ_{seq}, we construct the estimated probability of reliability

$$\hat{R}_N(\theta_{\text{seq}}) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{I}\left(J(\theta_{\text{seq}}, \Delta^{(i)})\right)$$

where $\mathbf{I}\,(\cdot)$ denotes the indicator function

$$\mathbf{I}\left(J(\theta_{\text{seq}}, \Delta^{(\mathrm{i})})\right) = \begin{cases} 1 & \text{if } J(\theta_{\text{seq}}, \Delta^{(\mathrm{i})}) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

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Sample Complexity

- We need to compute the size of the Monte Carlo experiment (sample complexity)
- To this end, given ε, δ ∈(0,1), we need to determine the sample complexity *N* such that the probability event

$$\left| R(\theta_{\mathrm{seq}}) - \hat{R}_N(\theta_{\mathrm{seq}}) \right| \leq \varepsilon$$

  holds with probability at least 1- δ
- Sample complexity is provided by the Chernoff Bound

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Probability Degradation Function

- The next step is to study how the estimated probability $\hat{R}_N(\theta_{\mathrm{seq}})$ degrades as a function of the radius ρ
- This is called the probability degradation function
- We can compare this function with the worst-case radius ρ_{wc} to provide additional information for the control designer

Algorithm Probabilistic Analysis

- Input: ε, δ, θ_{seq}
- Output: $\hat{R}_N(\theta_{\text{seq}})$

- compute the sample size $N_{\text{ch}}(\varepsilon, \delta)$
- draw $N \geq N_{\text{ch}}(\varepsilon, \delta)$ i.i.d. samples $\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(N)}$
- return

$$\hat{R}_N(\theta_{\text{seq}}) = \frac{1}{N} \sum_{\text{i}=1}^{N} \mathbf{I}\Big(J(\theta_{\text{seq}}, \Delta^{(i)})\Big)$$

Numerical Results - 1

- Taking ε=0.005, $\delta=10^{-6}$, by means of the Chernoff bound we obtain N_{ch} =290,174
- Then, we estimate the probability degradation function for 100 equispaced values of ρ in the range [0.12,0.5]
- For each grid point the estimated probability of reliability (or performance) is computed by means of Algorithm Probabilistic Analysis

Numerical Results - 2

- For each grid point ρ, the inequality

$$\left| R(\theta_{\text{seq}}) - \hat{R}_N(\theta_{\text{seq}}) \right| \leq 0.005$$

holds with probability at least 0.999999

- The probability degradation function is now shown

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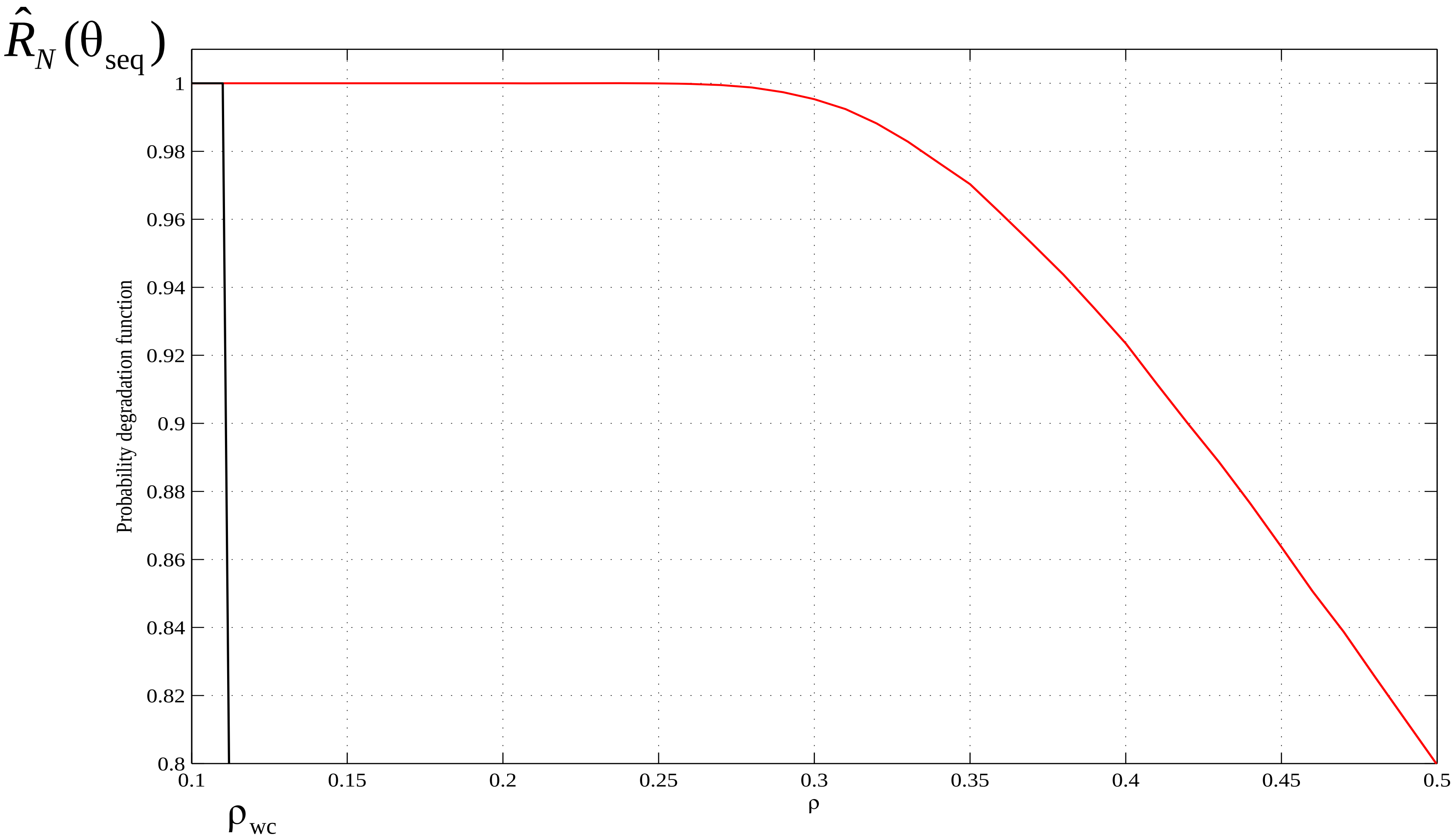
Probability Degradation Function



Comments

- We observe that if a 2% loss of probabilistic performance is tolerated, then the performance margin may be increased by 270% with respect to its deterministic counterpart ρ_{wc}
- For ρ=0.34, the estimated probability of performance is 0.98
- Notice that the estimated probability $\hat{R}_N(\theta_{seq})$ is equal to one up to $\rho = 0.26$

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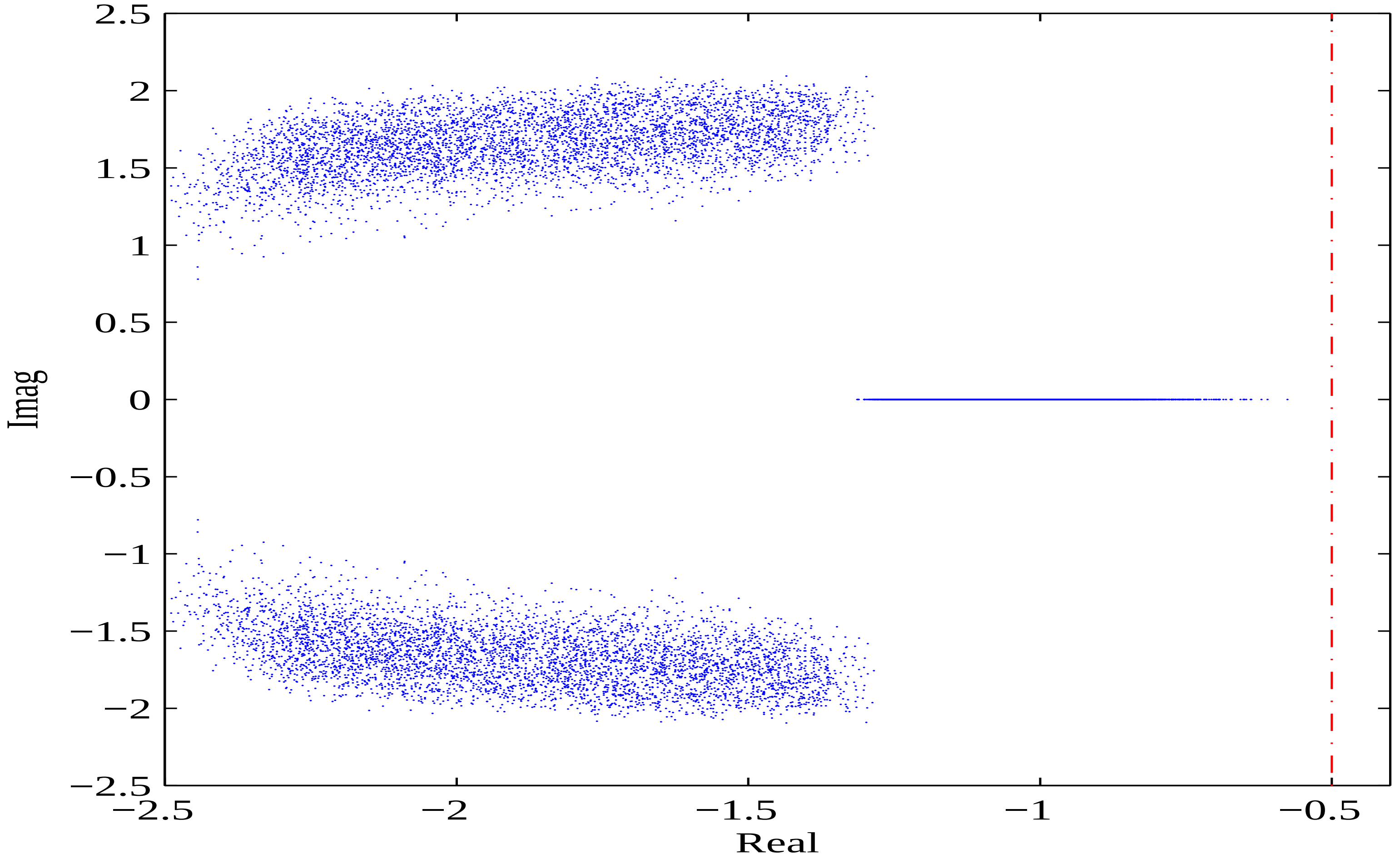
Closed-Loop Eigenvalues for ρ = 0.34



RACT

Randomized Algorithms Control Toolbox

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RACT

- RACT: Randomized Algorithms Control Toolbox for Matlab
- RACT has been developed at IEIIT-CNR and at the Institute for Control Sciences-RAS, based on a bilateral international project
- Members of the project

Andrey Tremba (Main Developer and Maintainer)
Giuseppe Calafiore
Fabrizio Dabbene
Elena Gryazina
Boris Polyak (Co-Principal Investigator)
Pavel Shcherbakov
Roberto Tempo (Co-Principal Investigator)

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RACT

- Main features
- Define a variety of uncertain objects: scalar, vector and matrix uncertainties, with different pdfs
- Easy and fast sampling of uncertain objects of almost any type
- Sequential randomized algorithms for feasibility of uncertain LMIs using stochastic gradient and localization methods (ellipsoid or cutting plane)
- Non-sequential randomized algorithms for optimization of convex problems

RACT

- Under construction
- Non-sequential randomized algorithms for feasibility and optimization of non-convex problems

RACT

- RACT: Randomized Algorithms Control Toolbox for Matlab

http://ract.sourceforge.net

Systems and Control Applications

Systems and Control Applications - 1

- Aerospace control: Applications of randomized strategies for the design of control algorithms for lateral and longitudinal control of aircrafts (e.g. F-16)[1,2]
- Flexible and truss structures: Probabilistic robustness of systems with bounded random uncertainty affecting sensors and actuators[3,4]
- Model (in)validation: Computationally efficient algorithm for robust performance in the presence of structured uncertainty[5]

[1] C.I. Marrison and R.F. Stengel.(1998)
[2] B. Lu and F. Wu (2006)
[3] G. Calafiore, F. Dabbene and R. Tempo (2000)
[4] G.C. Calafiore and F. Dabbene (2008)
[5] M.Sznaier, C.M. Lagoa, and M.C. Mazzaro (2007)

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Systems and Control Applications - 2

- Adaptive control: Methodology for the design of cautious adaptive controllers based on two-step procedure with controller tuning[1]
- Switched systems: Randomized algorithms for synthesis of multimodal systems with state-dependent switching[2]
- Network control: Congestion control of high-speed communication networks using different topologies[3]
- Automotive: Randomization-based approaches for model validation of advanced driver assistance systems[4]

[1] M.C. Campi and M. Prandini (2003)
[2] H. Ishii, T. Basar and R. Tempo (2005)
[3] T. Alpcan, T. Basar and R. Tempo (2005)
[4] O.J. Gietelink, B. De Schutter, and M. Verhaegen (2005)

Systems and Control Applications - 3

- Model predictive control (MPC): Sequential methods (ellipsoid-based) to design robustly stable finite horizon MPC schemes[1]
- Fault detection and isolation: Risk-adjusted randomization approach for robust simultaneous fault detection and isolation of MIMO systems[2]
- Circuits and embedded systems: Performance subject to uncertain components introduced during the manufacturing process[3-4]

[1] S. Kanev and M. Verhaegen (2006)
[2] W. Ma, M.Sznaier and C.M. Lagoa (2007)
[3] C. Lagoa, F. Dabbene and R. Tempo (2008)
[4] C. Alippi (2002)

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Systems and Control Applications - 4

- Unmanned aerial vehicles (UAV): Robust and randomized control design of a mini-UAV[1]

[1] L. Lorefice, B. Pralio and R. Tempo (2007)

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Control Design of a Mini-UAV

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Italian National Project for Fire Prevention

- This activity is supported by the Italian Ministry for Research within the National Project

Study and development of a real-time land control and monitoring system for fire prevention

- Five research groups are involved together with a government agency for fire surveillance and patrol located in Sicily
- The aerial platform is based on the MicroHawk configuration, developed at the Aerospace Engineering Department, Politecnico di Torino, Italy

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MH1000 Platform - 1

- Platform features
  - wingspan 3.28 ft (1 m)
  - total weight 3.3 lb (1.5 kg)

MH1000 Platform - 2

- Main on-board equipment
  - various sensors and two cameras (color and infrared)
- DC motor
- Remote piloting and autonomous flight
- Flight endurance of about 40 min
- Flight envelope
  - min/max velocity: 33 ft/s (10 m/s) – 66 ft/s (17 m/s)
  - average velocity: 43 ft/s (14 m/s)

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Flight Envelope (Limits)

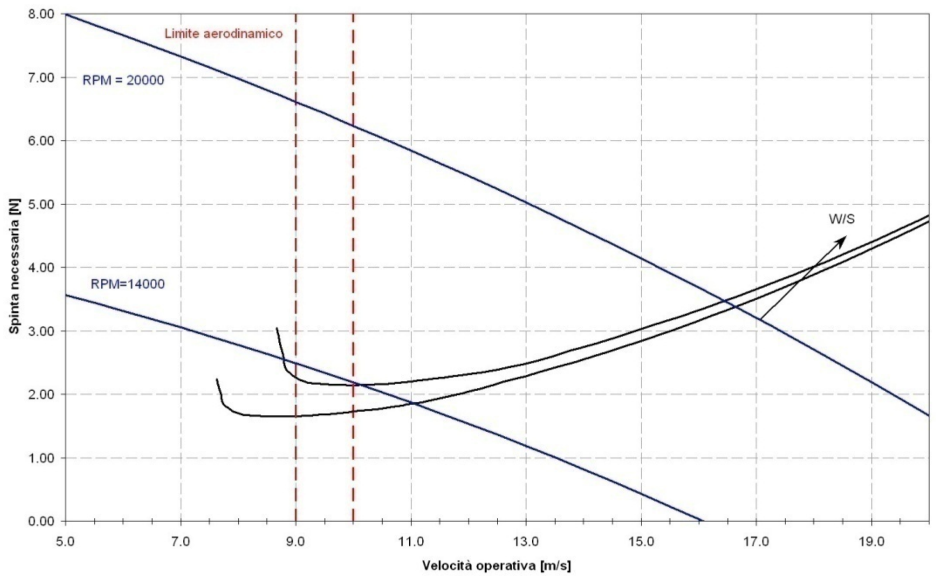
Wing loading effect → total weight

Aerodynamic constraint (red) → minimum flight speed (stall effect)

Propeller sizing effect

Propulsive constraint (blu) → maximum flight speed

velocity: 33 ft/s (10 m/s) – 66 ft/s (17 m/s)



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Basic on-board Systems

DC motor: Hacker B20-15L (4:1)

- weight: 58 g
- dimensions: Ø 20 x 40 mm
- Kv: 3700 rpm/volt

controller: Hacker Master Series 18-B-Flight

- weight: 21 g
- dimensions: 33 X 23 X 7 mm
- current drain: 18 A

battery: Kokam 2000HD (3x)

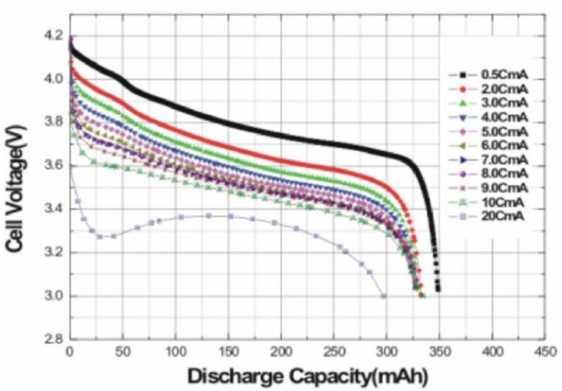
- weight: 160 g
- dimensions: 79 X 42 X 25 mm
- capacity: 2000 mAh

receiver: Schulze Alpha840W

- weight: 13.5 g
- dimensions: 52 X 21 X 13 mm
- 8 channels

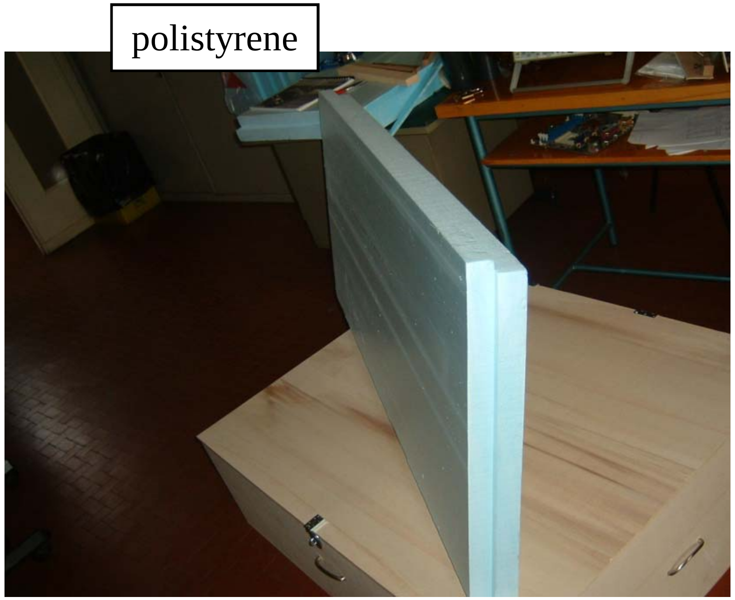
servo: Graupner C1081 (2x)

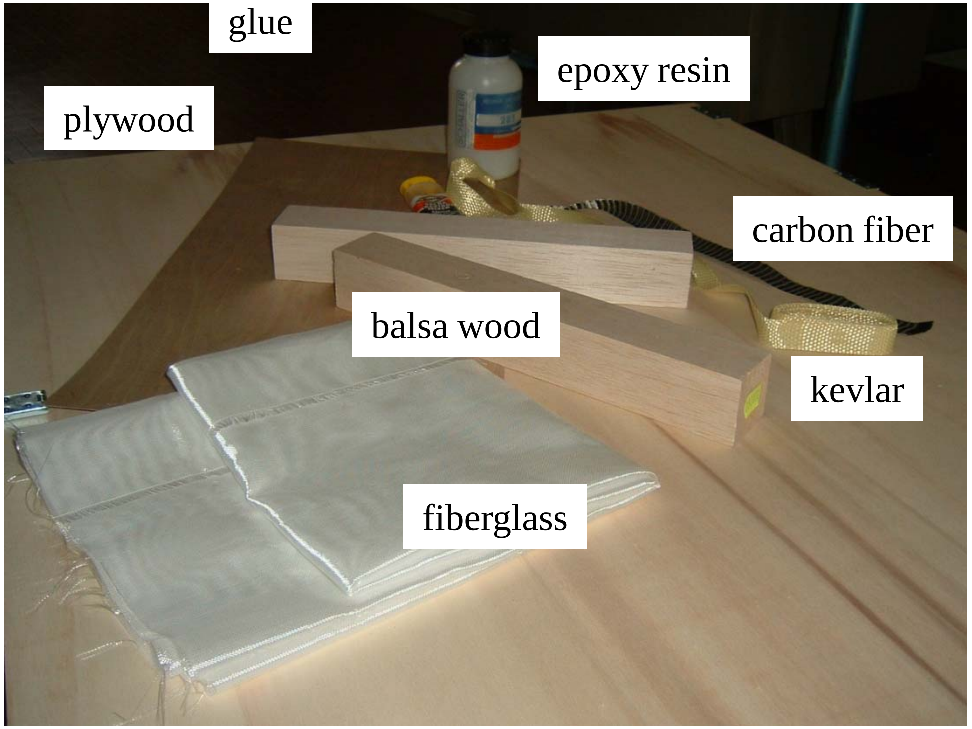
- weight: 13 g
- dimensions: 23 X 9 X 21 mm
- torque: 12 Ncm



Prototype Manufacturing - 1

raw material

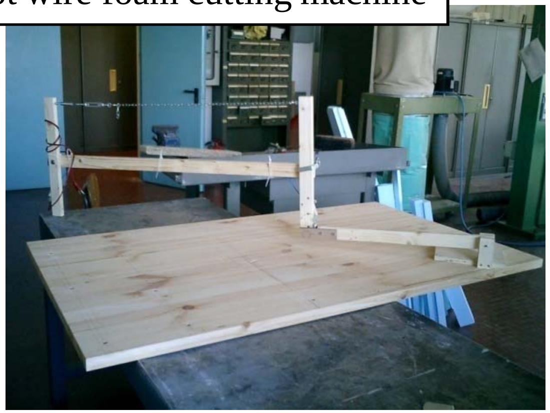




Prototype Manufacturing - 2

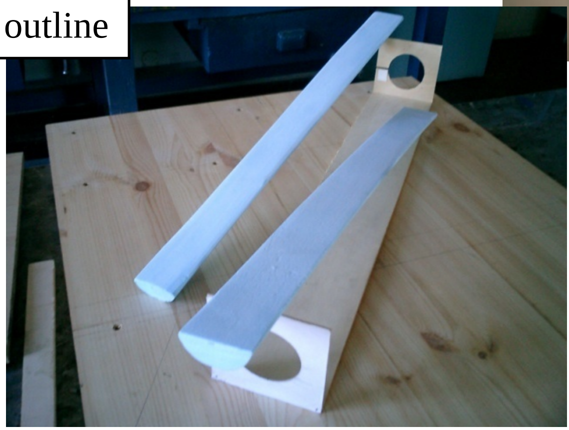
working instruments

hot wire foam cutting machine



lifting surfaces outline

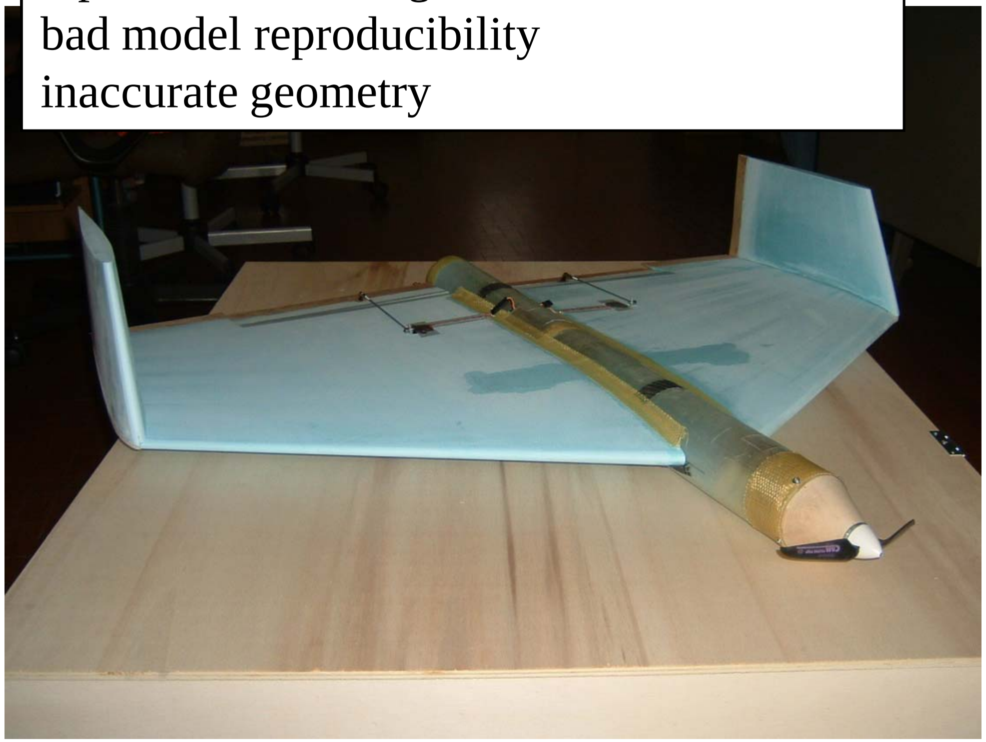
slide outline

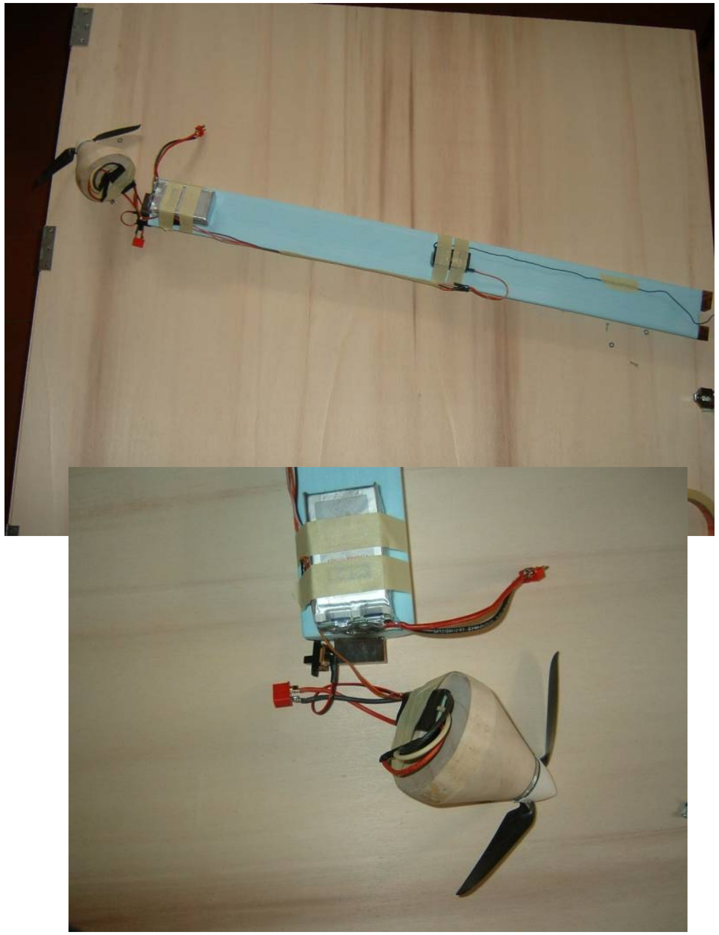


fuselage reference

Prototype Manufacturing - 3

easy construction
rapid manufacturing
bad model reproducibility
inaccurate geometry





State Space Model

- State space formulation obtained by linearization of the full (12 coupled nonlinear ODE) model

$$\dot{x}(t) = A(\Delta)\, x(t) + B(\Delta)\, u(t)$$

$$u(t) = -K\, x(t)$$

where $x = [V, \alpha, q, \theta]^T$ (V flight speed, α angle of attack, q and θ pitch rate and angle), Δ uncertainty

- Consider longitudinal plane dynamics stabilization
- Control u is the symmetrical elevon deflection

Uncertainty Description - 1

- We consider structured parameter uncertainties affecting plant and flight conditions, and aerodynamic database
- Uncertainty vector $\Delta = [\Delta_1,\ldots, \Delta_{16}]$ where $\Delta_i \in [\Delta_i^-, \Delta_i^+]$
- Key point: There is no explicit relation between state space matrices A and B and uncertainty Δ
- This is due to the fact that state space system is obtained through linearization and off-line flight simulator
- The only techniques which could be used in this case are simulation-based which lead to randomized algorithms

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Uncertainty Description - 2

- We consider random uncertainty $\Delta = [\Delta_1,..., \Delta_{16}]^T$
- The pdf is either uniform (for plant and flight conditions) or truncated Gaussian (for aerodynamic database uncertainties)
- Flight conditions uncertainties need to take into account large variations on physical parameters
- Uncertainties for aerodynamic data are related to experimental measurement or round-off errors

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Plant and Flight Condition Uncertainties

parameter	pdf	$\bar{\Delta}_i$	%	Δ_i^-	Δ_i^+	#
flight speed [ft/s]	$\mathcal{U}$	42.65	± 15	36.25	49.05	1
altitude [ft]	$\mathcal{U}$	164.04	± 100	0	328.08	2
mass [lb]	$\mathcal{U}$	3.31	± 10	2.98	3.64	3
wingspan [ft]	$\mathcal{U}$	3.28	± 5	3.12	3.44	4
mean aero chord [ft]	$\mathcal{U}$	1.75	± 5	1.67	1.85	5
wing surface [ft^2]	$\mathcal{U}$	5.61	± 10	5.06	6.18	6
moment of inertia [lb ft^2]	$\mathcal{U}$	1.34	± 10	1.21	1.48	7

Aerodynamic Database Uncertainties

parameter	pdf	$\bar{\Delta}_i$	σ_i	#
C_X [-]	$\mathcal{G}$	-0.01215	0.00040	8
C_Z [-]	$\mathcal{G}$	-0.30651	0.00500	9
C_m [-]	$\mathcal{G}$	-0.02401	0.00040	10
C_{Xq} [rad^{-1}]	$\mathcal{G}$	-0.20435	0.00650	11
C_{Zq} [rad^{-1}]	$\mathcal{G}$	-1.49462	0.05000	12
C_{mq} [rad^{-1}]	$\mathcal{G}$	-0.76882	0.01000	13
C_X [rad^{-1}]	$\mathcal{G}$	-0.17072	0.00540	14
C_Z [rad^{-1}]	$\mathcal{G}$	-1.41136	0.02200	15
C_m [rad^{-1}]	$\mathcal{G}$	-0.94853	0.01500	16

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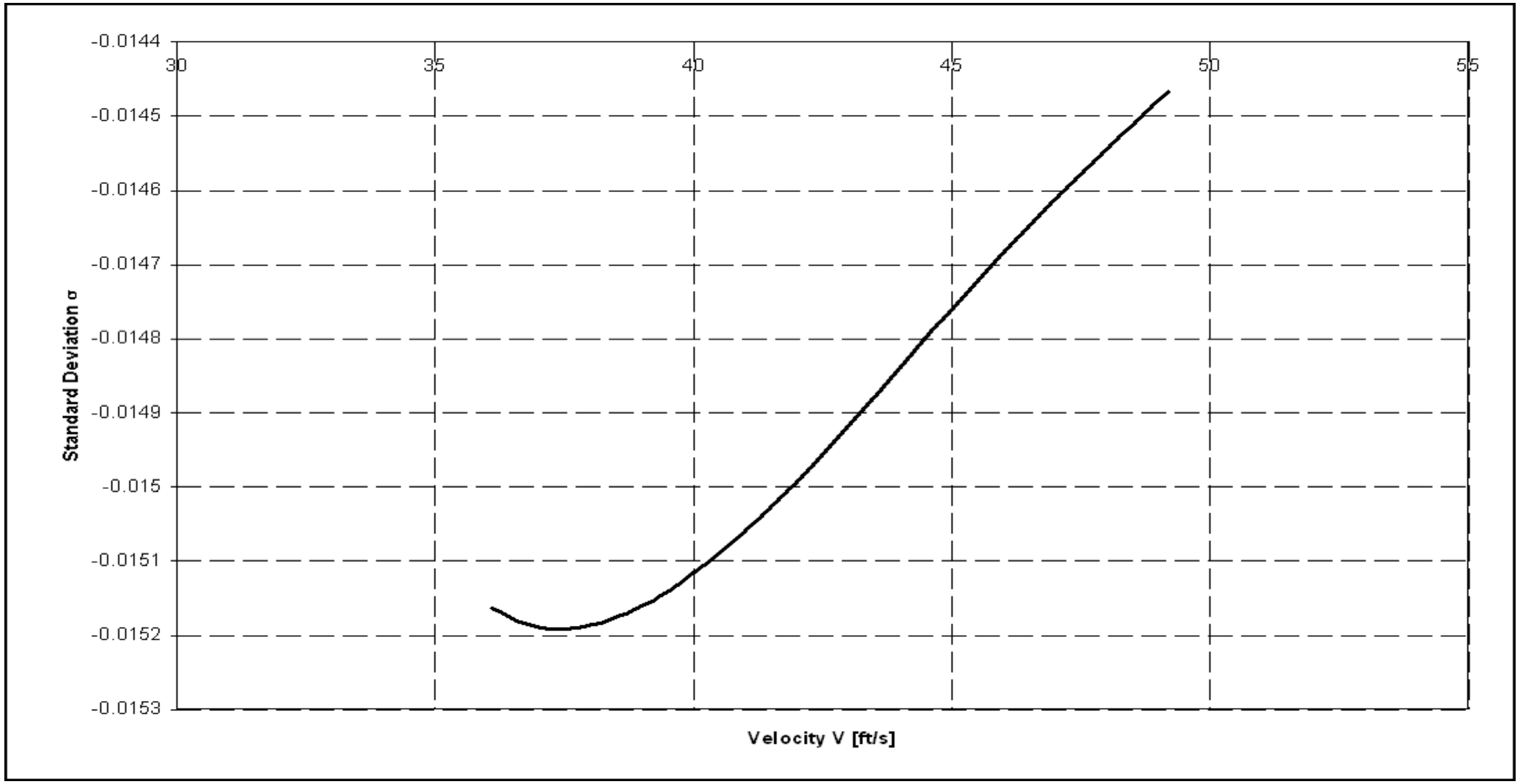
Standard Deviation and Velocity



Standard deviation is experimentally computed from the velocity

Critical Parameters and Matrices

- We select flight speed (Δ_1) and take off mass (Δ_3) as critical parameters
- Flight speed is taken as critical parameter to optimize gain scheduling issues
- Take off mass is a key parameter in mission profile definition
- We define critical matrices

$$A_c^1 \quad A_c^2 \quad A_c^3 \quad A_c^4 \qquad B_c^1 \quad B_c^2 \quad B_c^3 \quad B_c^4$$

- They are constructed setting Δ_1, Δ_3 to their extreme values; the remaining Δ_i are set to their nominal values

Phase 1: Random Gain Synthesis (RGS)

- Critical parameters are flight speed and take off mass
- Specification property

$$\mathcal{S}_1 = \{K\text{: } A_c - B_c K \text{ satisfies the specs below}\}$$

$$\omega_{SP} \in [4.0, 6.0] \text{ rad/s} \qquad \zeta_{SP} \in [0.5, 0.9] \qquad \omega_{PH} \in [1.0, 1.5] \text{ rad/s}$$

$$\zeta_{PH} \in [0.1, 0.3] \qquad \Delta\omega_{SP} < \pm\, 45\% \qquad \Delta\omega_{PH} < \pm\, 20\%$$

where ω and ζ are undamped natural frequency and damping ratio of the characteristic modes; $_{SP}$ and $_{PH}$ denote short period and phugoid mode

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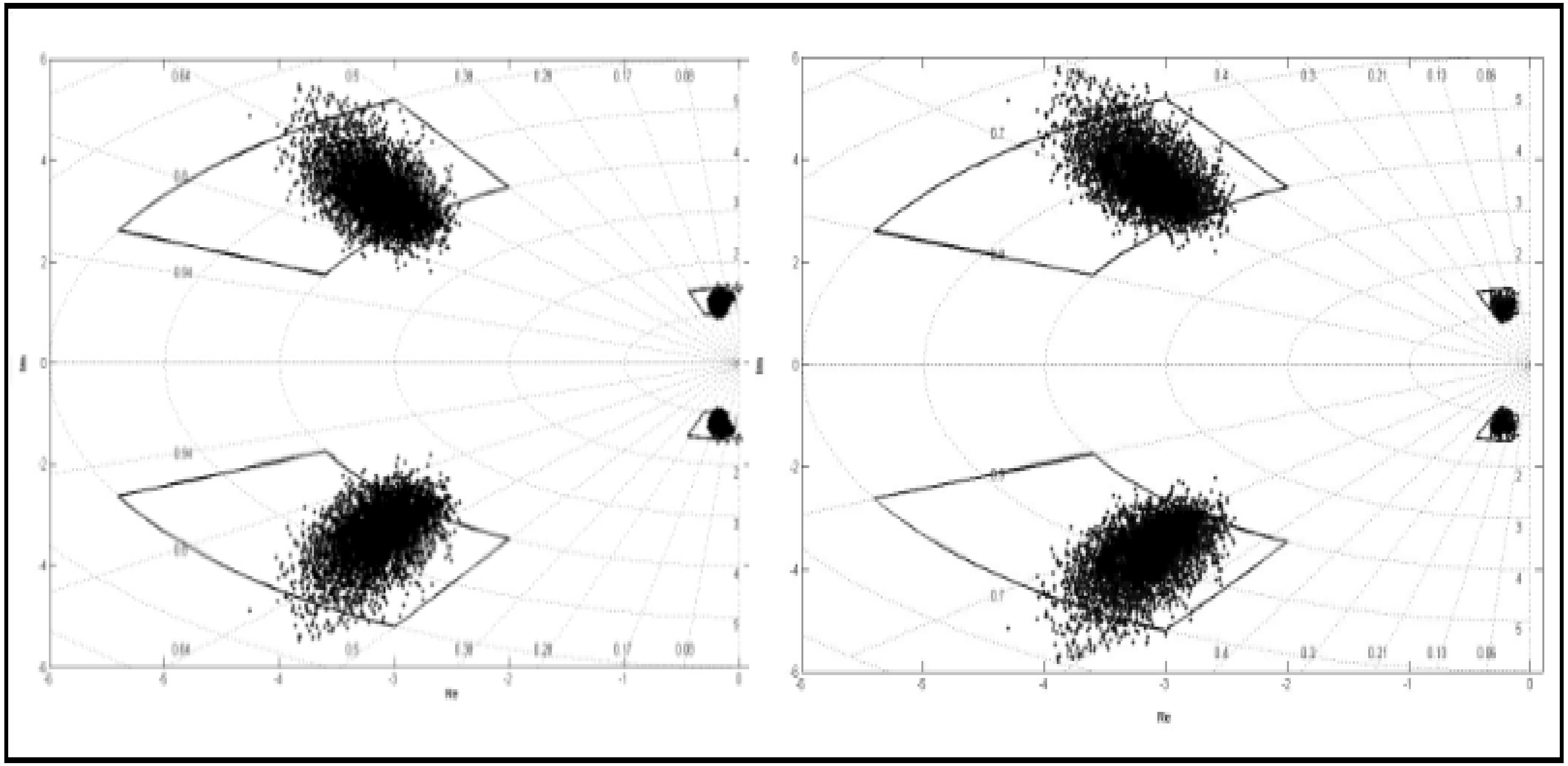
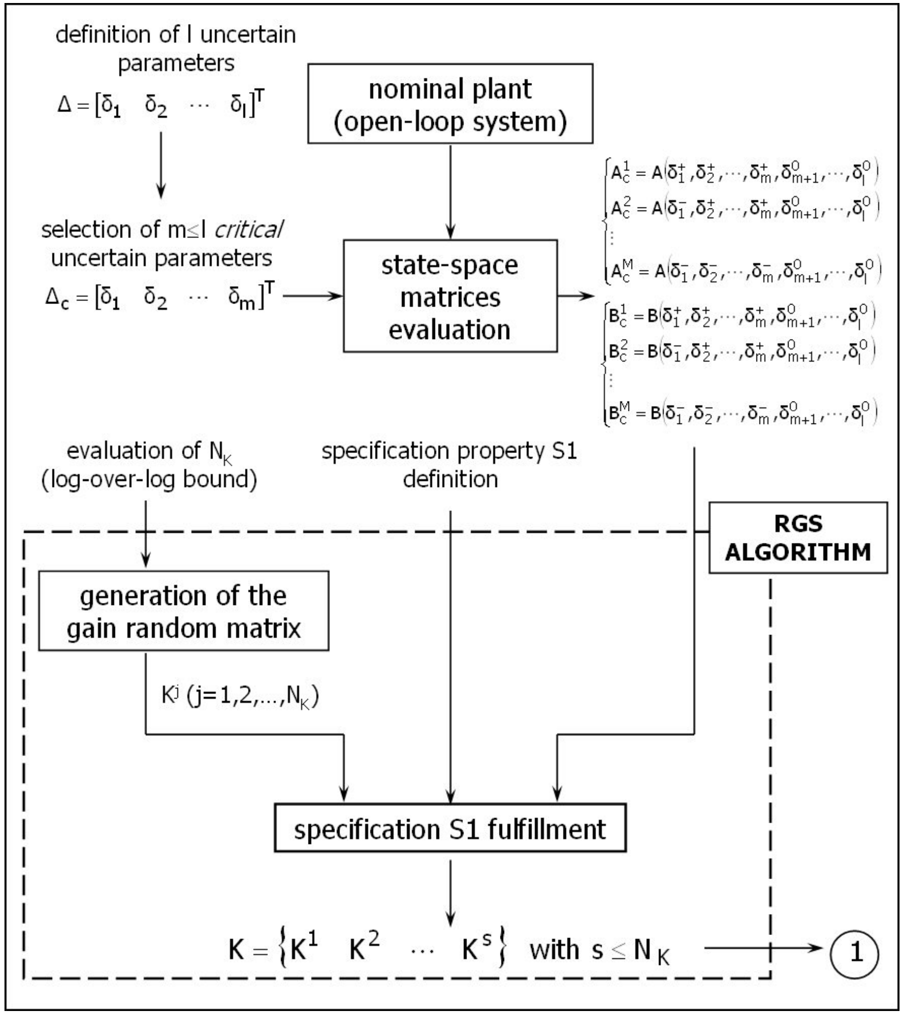
Specs in the Complex Plane



Randomized Algorithm 1 (RGS)

- Uniform pdf for controller gains *K* in given intervals
- Accuracy and confidence

  $\varepsilon = 4 \cdot 10^{-5}$ and $\delta = 3 \cdot 10^{-4}$
- Number of random samples is computed with "Log-over-Log" Bound obtaining $N = 200{,}000$
- We obtained $s = 5$ gains K^i satisfying specification property S_1



Random Gain Set

gain set	K_V	K_α	K_q	K_θ
K^1	0.00044023	0.09465000	0.01577400	-0.00473510
K^2	0.00021450	0.09581200	0.01555500	-0.00323510
K^3	0.00054999	0.09430800	0.01548200	-0.00486340
K^4	0.00010855	0.09183200	0.01530000	-0.00404380
K^5	0.00039238	0.09482700	0.01609300	-0.00417340

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Phase 2: Random Stability Robustness Analysis (RSRA)

- Take $K_{\text{rand}} = K^i$ obtained in Phase 1
- Randomize Δ according to the given pdf and take N random samples Δ^i
- Specification property

$$\mathcal{S}_2 = \{\Delta: A(\Delta) - B(\Delta)\ \ K_{\text{rand}}\ \ \text{satisfies the specs of } \mathcal{S}_1\}$$

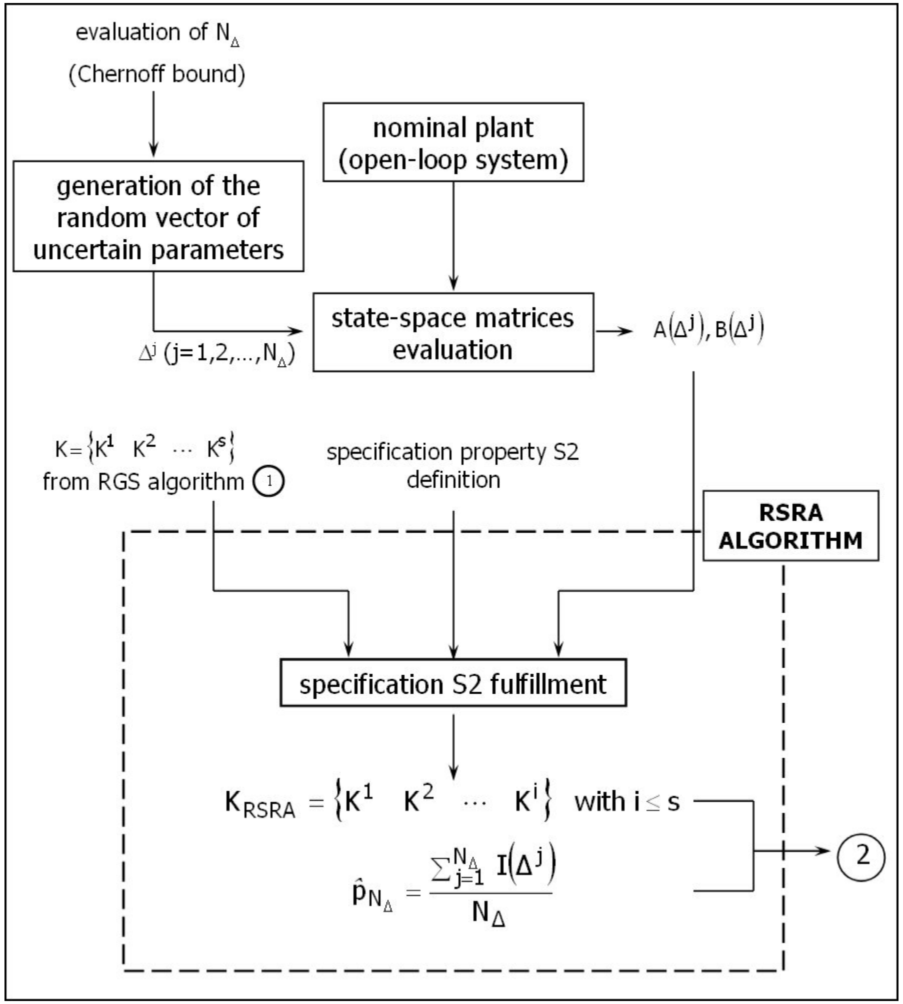
- Computation of the empirical probability of stability

Randomized Algorithm 2 (RSRA)

- Take K_{rand} from Phase 1
- Accuracy and confidence

$$\varepsilon = \delta = 0.0145$$

- Number of random samples is computed with Chernoff Bound obtaining N =5,000
- Empirical probability is computed



Empirical Probability of Stability for Phase 2

gain set	empirical probability
K^1	88.56%
K^2	90.60%
K^3	89.31%
K^4	93.86%
K^5	85.14%

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Probability Degradation Function

- Flight condition uncertainties are multiplied by the radius $\rho > 0$ keeping the nominal value constant

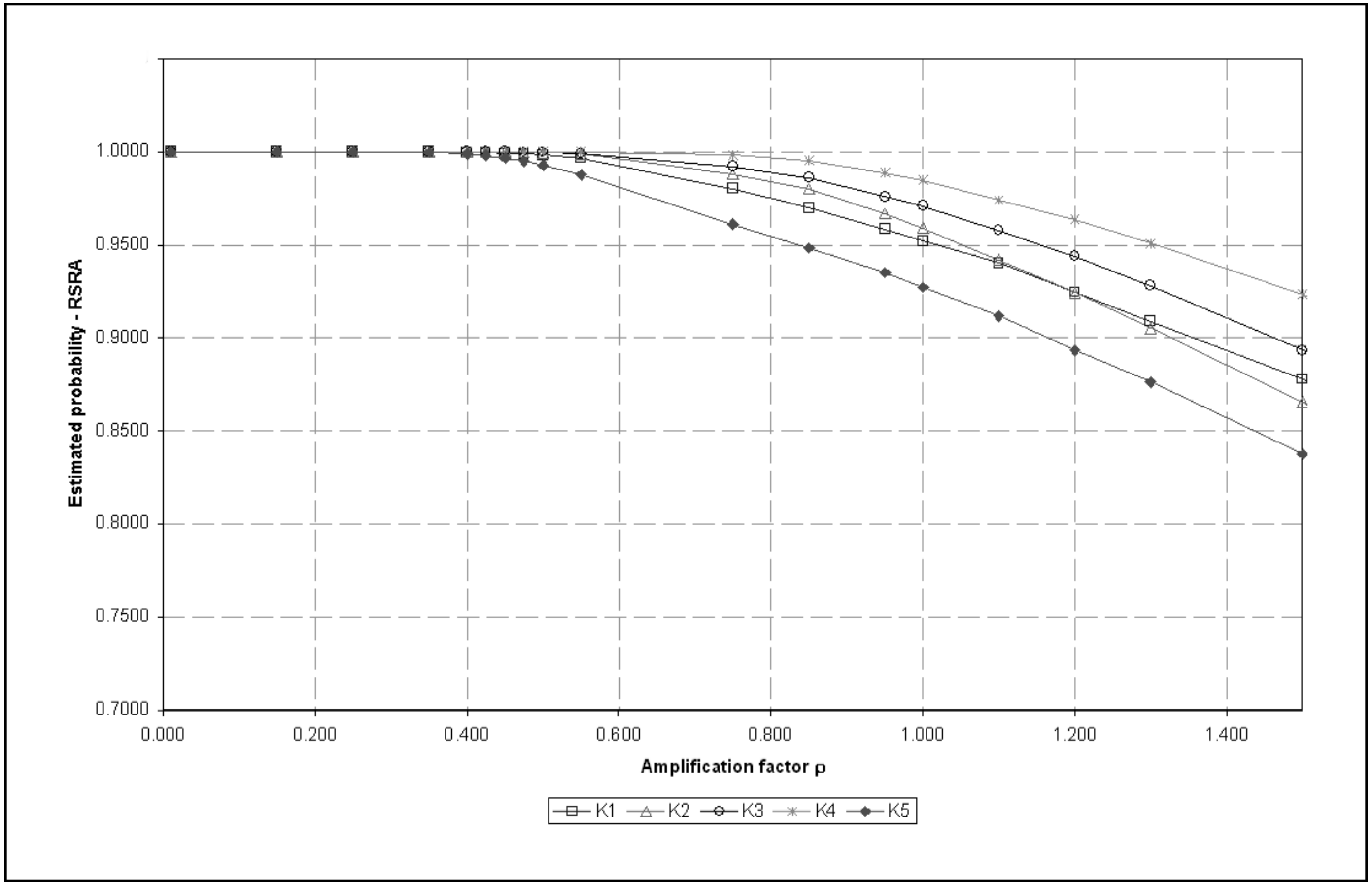
$$\Delta_i \in \rho\,[\Delta_i^-, \Delta_i^+] \qquad \text{for } i = 1, 2, \ldots, 7$$

- No uncertainty affects the aerodynamic database, i.e.

$$\Delta_i = \bar{\Delta}_i \qquad \text{for } i = 8, 9, \ldots, 16$$

- For fixed $\rho \in [0,1.5]$ we compute the empirical probability for different gain sets K^i
- The plot empirical probability vs ρ is the probability degradation function

Probability Degradation Function for Phase 2



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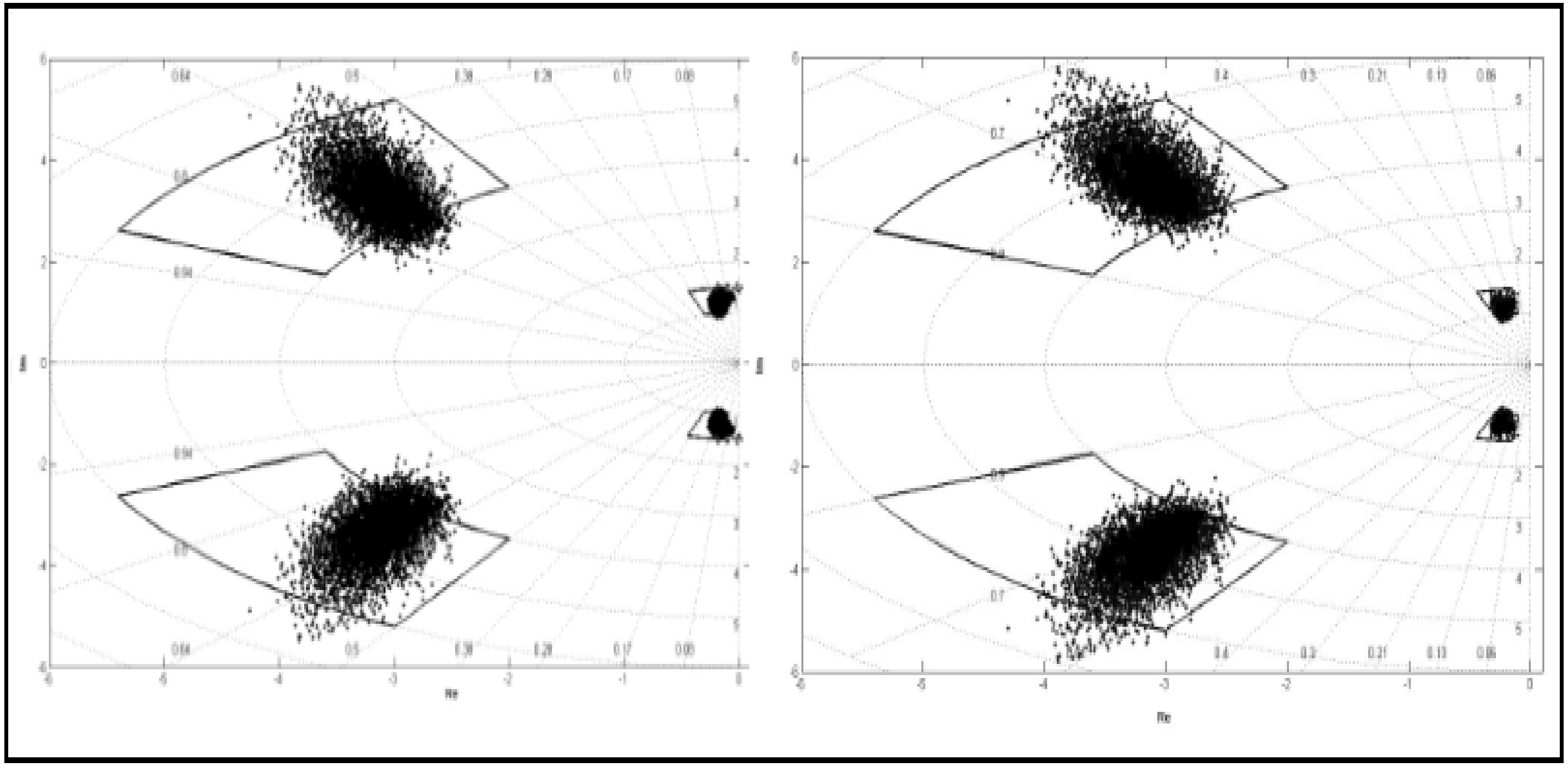
Root Locus Plot for Phase 2



Root locus for K^2 (left) and K^4 (right)

Phase 3: Random Performance Robustness Analysis (RPRA)

- This phase is similar to Phase 2, but military specs are considered (bandwidth criterion)
- Specification property

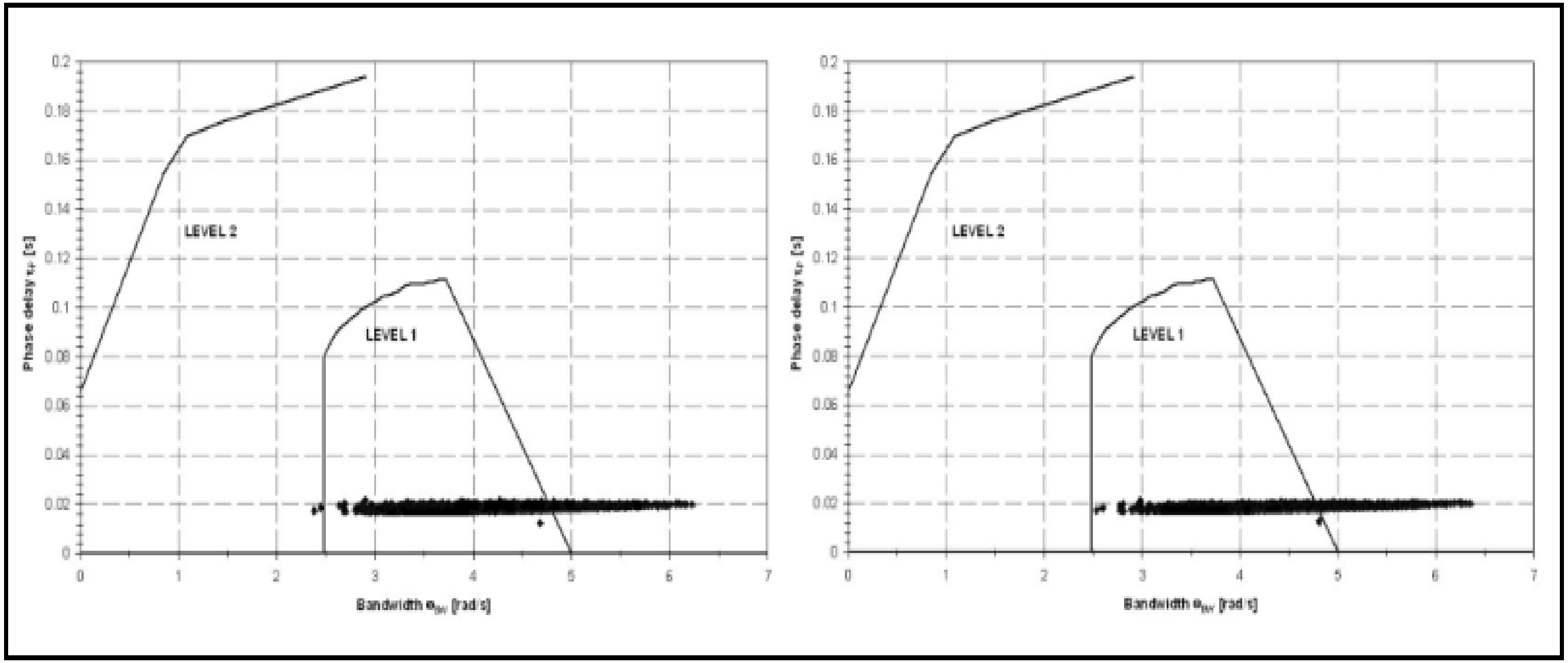
$$S_3 = \{\Delta: A(\Delta) - B(\Delta)\ K_{\text{rand}} \text{ satisfies the specs below}\}$$

$$\omega_{BW} \in [2.5, 5.0] \text{ rad/s} \qquad \tau_P \in [0.0, 0.5] \text{ s}$$

where ω_{BW} and τ_P are bandwidth and phase delay of the frequency response

- Computation of the empirical probability that S_3 is satisfied

Bandwidth Criterion



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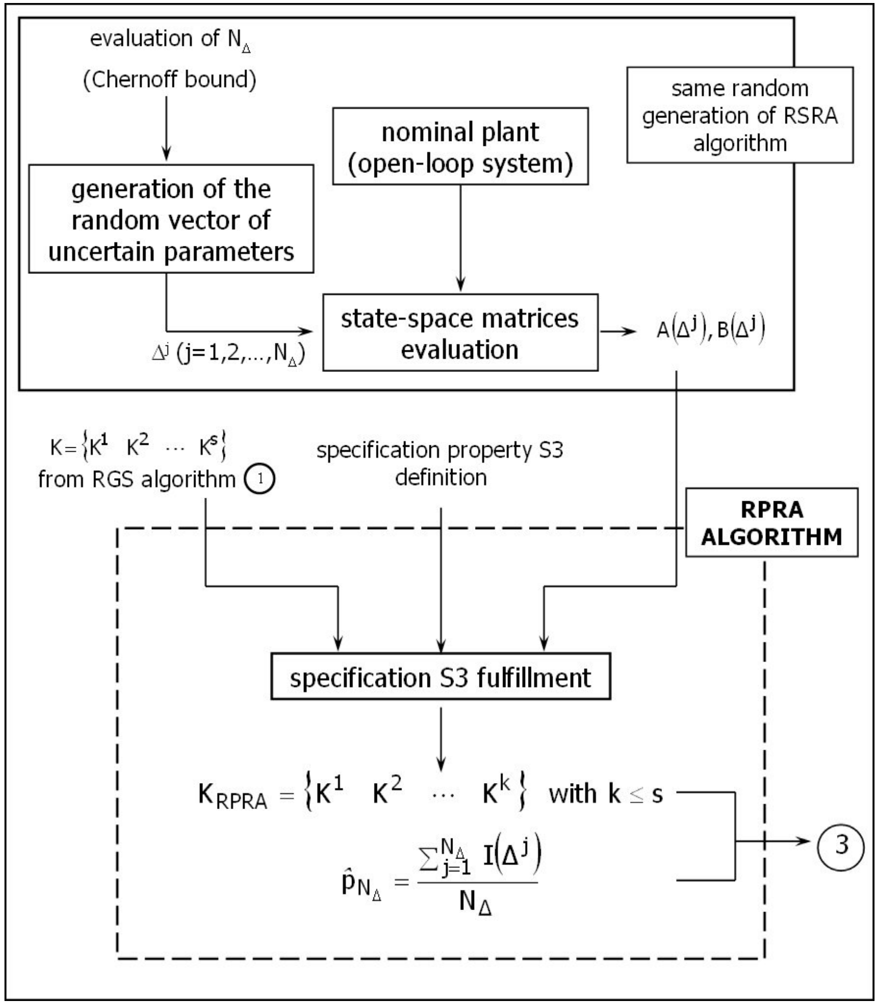
Randomized Algorithm 3 (RPRA)

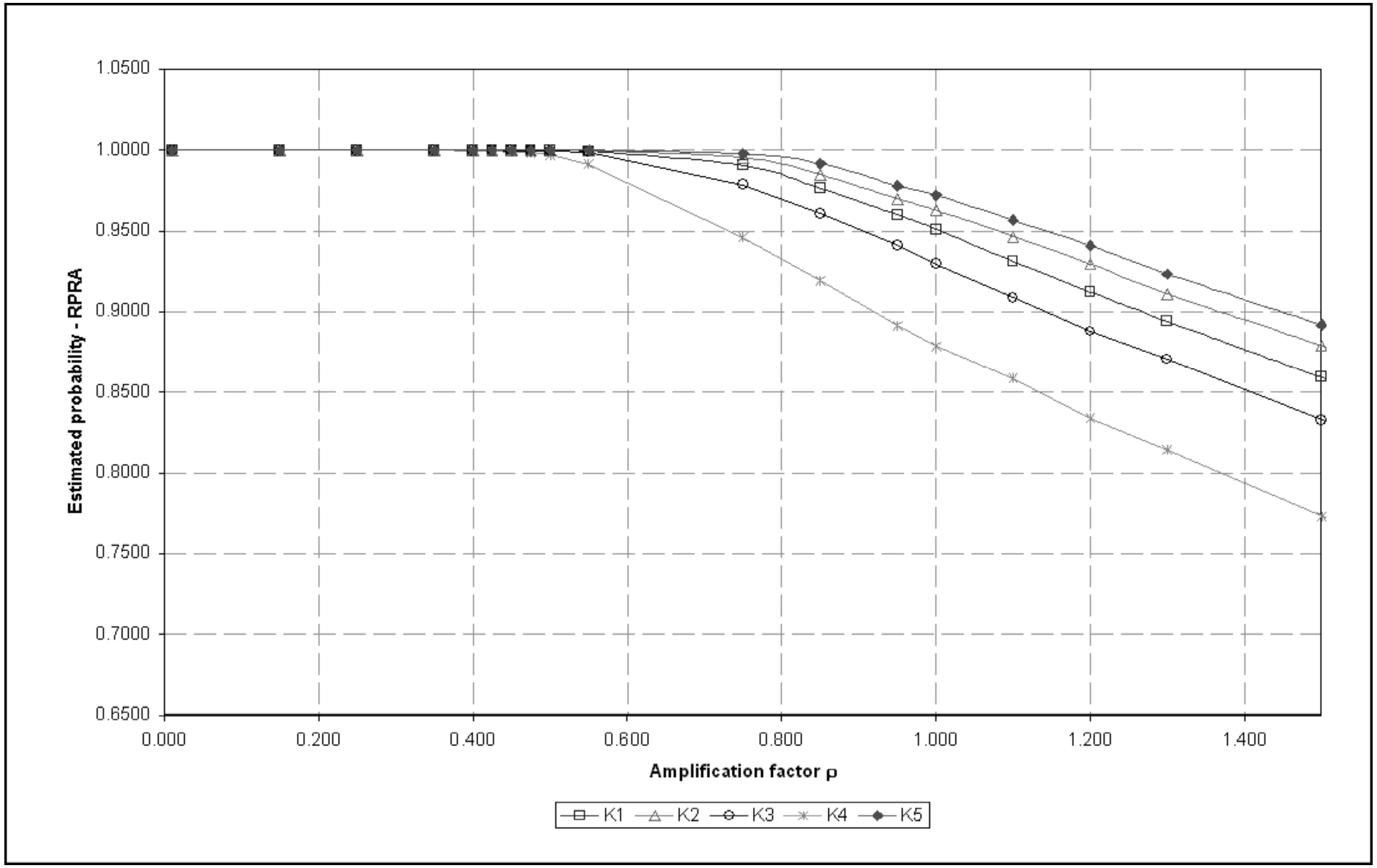
- Take K_{rand} from Phase 1
- Numer of random samples is computed with the Chernoff Bound obtaining N =5,000
- Empirical probability is computed



Empirical Probability of Performance for Phase 3

gain set	empirical probability
K^1	93.58%
K^2	95.16%
K^3	90.80%
K^4	84.78%
K^5	96.06%

Probability Degradation Function for Phase 3



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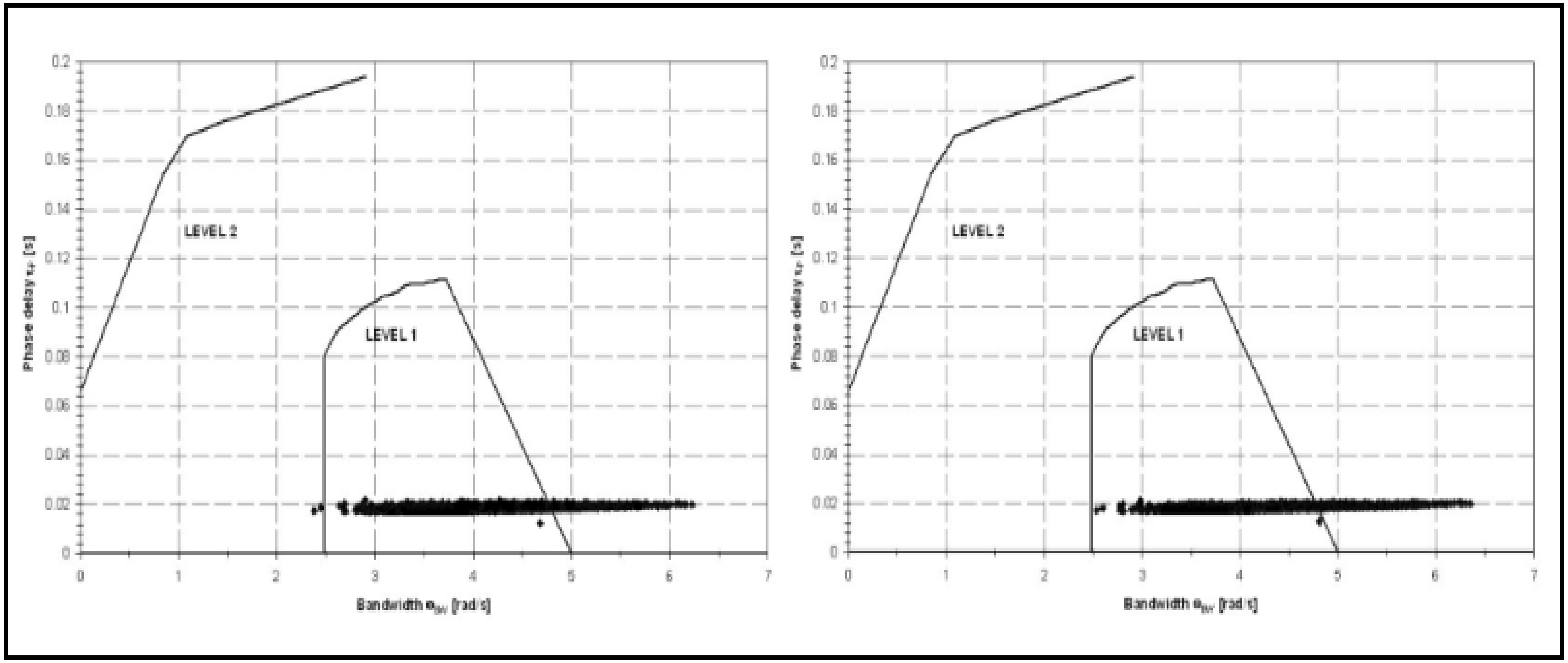
Bandwidth Criterion for Phase 3



Bandwidth criterion for K^1 (left) and K^3 (right)

Gain Selection

- Multi-objective criterion as a compromise between different specifications

Finally we selected gain K^1 as the best compromise between all the specs and criteria

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Conclusions: Flight Tests in Sicily - 1

- Evaluation of the payload carrying capabilities and autonomous flight performance
- Mission test involving altitude, velocity and heading changing was performed in Sicily
- Checking effectiveness of the control laws for longitudinal and lateral-directional dynamics
- Flight control design based on RAs for stabilization and guidance

Conclusions: Flight Tests in Sicily - 2

- Satisfactory response of MH1000
- Possible improvements by iterative design procedure
- Stability of the platform is crucial for the video quality and in the effectiveness of the surveillance and monitoring tasks

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Color Camera: Right Turn

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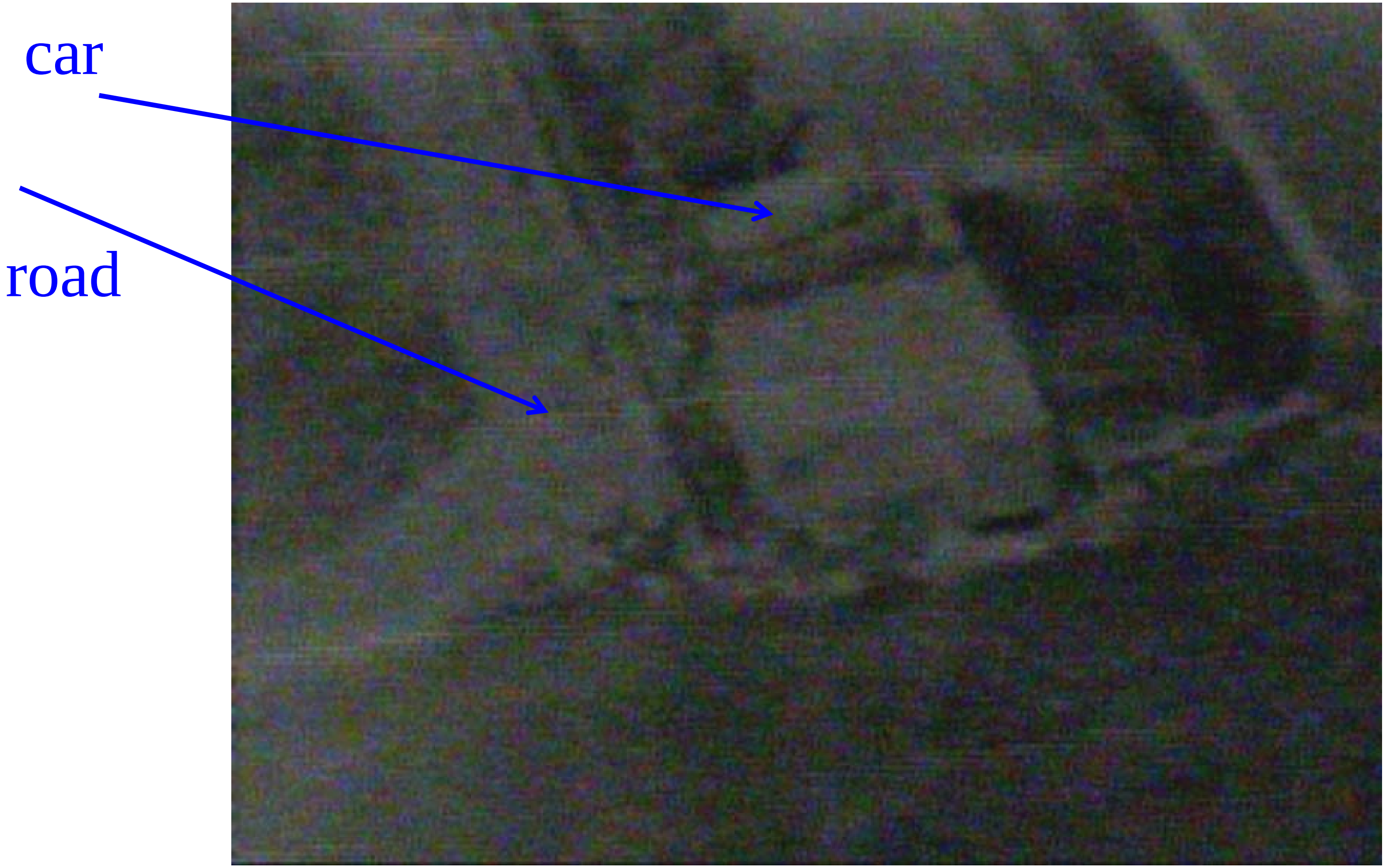
Color Camera: Landing Phase

Infrared Camera - 1

car

Infrared Camera - 1



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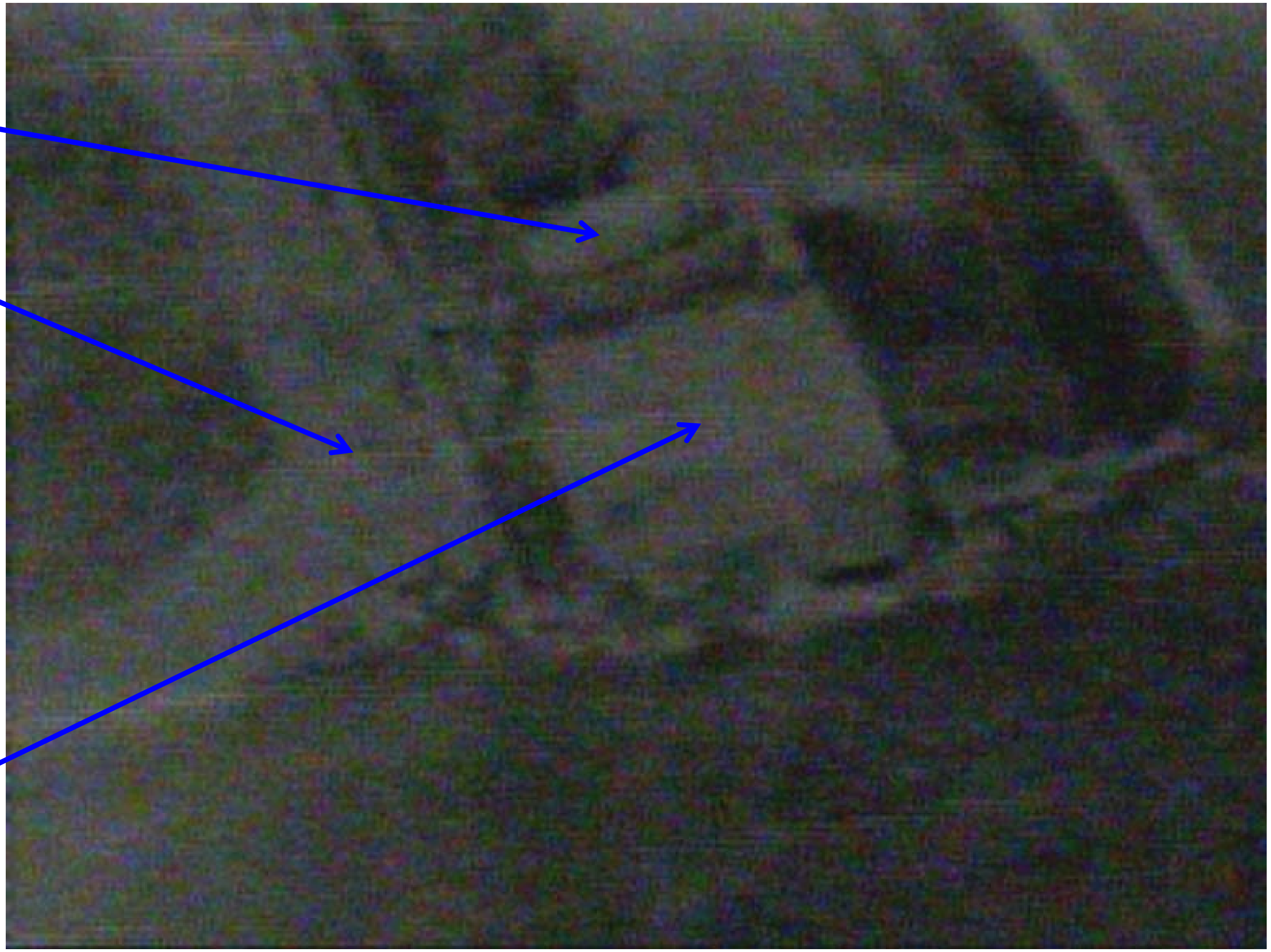
Infrared Camera - 1



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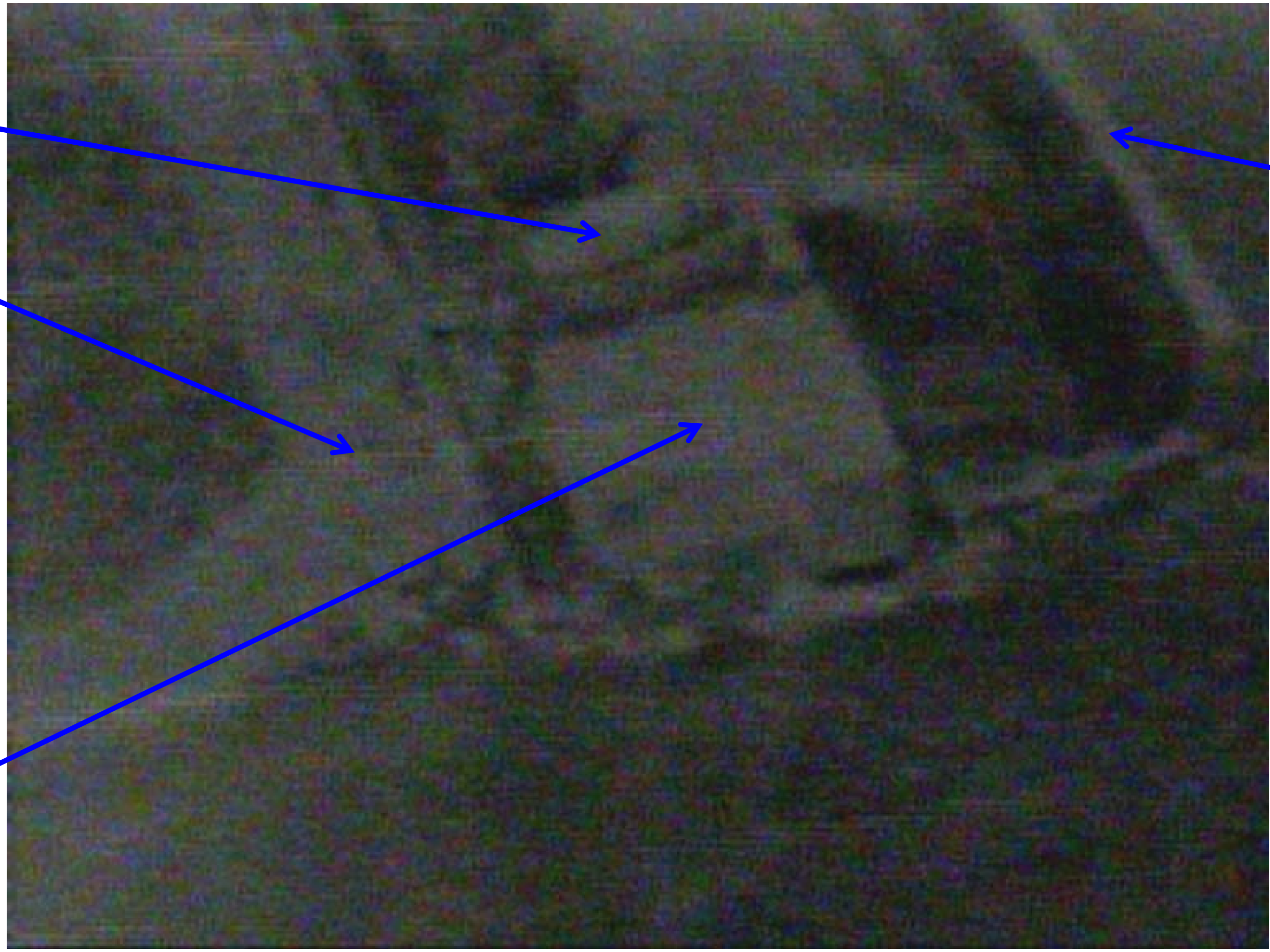
Infrared Camera - 1



Infrared Camera - 2

Infrared Camera - 3