High-order finite element methods (also known as spectral/hp element methods) using either the continuous Galerkin or discontinuous Galerkin formulation have reached a level of sophistication such that they are now commonly applied to a diverse set of real-life engineering problems. Visualization of computed results is often used as a means of understanding and evaluating the numerical approximation of the mathematical model, and it provides a means of "closing the loop" – that is, of critically evaluating the computational results for refinement of the model and/or numerics or for interpretation of the physical world. Visualizations of high-order finite element results which do not respect the a priori knowledge of how the data were
ABSTRACT

High-order finite element methods (also known as spectral/hp element methods) using either the continuous Galerkin or discontinuous Galerkin formulation have reached a level of sophistication such that they are now commonly applied to a diverse set of real-life engineering problems. Visualization of computed results is often used as a means of understanding and evaluating the numerical approximation of the mathematical model, and it provides a means of “closing the loop” – that is, of critically evaluating the computational results for refinement of the model and/or numerics or for interpretation of the physical world. Visualizations of high-order finite element results which do not respect the a priori knowledge of how the data were produced and which do not provide a quantification of the visual error produced undermine the scientific process just described. The goals of this effort are to define, investigate, and address the technical obstacles inherent in visualization of data derived from high-order numerical methods and to develop algorithms and software solutions that can be employed by the high-order simulation community.

List of papers submitted or published that acknowledge ARO support during this reporting period. List the papers, including journal references, in the following categories:

(a) Papers published in peer-reviewed journals (N/A for none)


Number of Papers published in peer-reviewed journals: 6.00

(b) Papers published in non-peer-reviewed journals or in conference proceedings (N/A for none)

Number of Papers published in non peer-reviewed journals: 0.00

(c) Presentations
1) Intelligent Visualization and Simulation Lab, University of Kaiserslautern, Germany. Presented a talk entitled “Visualization of High-Order Finite Element Methods”, June 2008.


4) Center of Complex Systems and Visualization, University of Bremen, Germany. Presented a talk entitled “Particle Systems for Efficient and Accurate High-Order Finite Element Visualization”, March 2007.

5) Sean Curtis, Robert M. Kirby and Jennifer K. Ryan, “Accuracy Enhancing Filtering With Application To Visualization”. Presented at the 7th World Congress on Computational Mechanics, July 2006.


Number of Presentations: 6.00

Non Peer-Reviewed Conference Proceeding publications (other than abstracts):

Number of Non Peer-Reviewed Conference Proceeding publications (other than abstracts): 0

Peer-Reviewed Conference Proceeding publications (other than abstracts):

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(d) Manuscripts

Number of Manuscripts: 0.00

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**Student Metrics**

This section only applies to graduating undergraduates supported by this agreement in this reporting period

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The number of undergraduates funded by this agreement who graduated during this period with a degree in science, mathematics, engineering, or technology fields: ...... 0.00

The number of undergraduates funded by your agreement who graduated during this period and will continue to pursue a graduate or Ph.D. degree in science, mathematics, engineering, or technology fields: ...... 0.00

Number of graduating undergraduates who achieved a 3.5 GPA to 4.0 (4.0 max scale): ...... 0.00

Number of graduating undergraduates funded by a DoD funded Center of Excellence grant for Education, Research and Engineering: ...... 0.00

The number of undergraduates funded by your agreement who graduated during this period and intend to work for the Department of Defense: ...... 0.00

The number of undergraduates funded by your agreement who graduated during this period and will receive scholarships or fellowships for further studies in science, mathematics, engineering or technology fields: ...... 0.00

**Names of Personnel receiving masters degrees**

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Inventions (DD882)
Abstract: High-order finite element methods (also known as spectral/hp element methods) using either the continuous Galerkin or discontinuous Galerkin formulation have reached a level of sophistication such that they are now commonly applied to a diverse set of real-life engineering problems. Visualization of computed results is often used as a means of understanding and evaluating the numerical approximation of the mathematical model, and it provides a means of “closing the loop” – that is, of critically evaluating the computational results for refinement of the model and/or numerics or for interpretation of the physical world. Visualizations of high-order finite element results which do not respect the a priori knowledge of how the data were produced and which do not provide a quantification of the visual error produced undermine the scientific process just described. The goals of this effort are to define, investigate, and address the technical obstacles inherent in visualization of data derived from high-order numerical methods and to develop algorithms and software solutions that can be employed by the high-order simulation community.

Statement of Problem Studies

The goals of this effort are to define, investigate, and address the technical obstacles inherent in visualization of data derived from high-order numerical methods and to develop algorithms and software solutions that can be employed by the high-order simulation community.

Summary of Results

In this section, we first present the motivating work (published in [R1] by one of the investigators) for our previous ARO grant and then provide a summary of results as a consequence of ARO funding. To summarize – six peer-reviewed journal articles have been published or accepted for publication: four articles targeting the visualization community [J1, J2, J4, J5] and two targeting the computational mathematics community [J3, J6].
Isosurface Visualization

Our initial work on high-order finite element visualization was motivated by the work of Nelson and Kirby [R1], in which they presented an algorithm for ray-casting high-order, spectral/hp elements. Their method uses a world-space approximation of the composition of the coordinate transformation and the reference space basis functions. It assumes multi-linear mappings (linear element boundaries in world space), and includes a quantification of the approximation and root-finding error. They show that the image-space method compares favorably with marching cubes in compute time when the tolerances on surface position are sufficiently high. Figure 1 provides an example of the type of visualizations produced by their work. The marching cubes image (left) was generated by sampling the finite element volume on a rectilinear grid of spacing $h$ using a marching cubes algorithm to provide a tessellated isosurface, and rendering the triangular isosurface using ray-casting (since the marching cubes result is a triangular mesh, the ray-casting can be done exactly as done in [R2]). For the marching cubes image presented, a grid spacing of $h=0.015$ (yielding 4,705,274 voxels) was used. For the high-order ray-traced image (right), mapping inversion error of $10^{-8}$ and 11th order projected polynomials were used. These parameters were chosen such that the spectral/hp element evaluation time and rendering time was nearly identical to generate the two images. The root-mean-square error for the marching cubes image is 0.0158; the root-mean-square error for the ray-traced image is 3.5e-11. The images look very similar, however the root-mean-square error difference between the images is significant. We should also point out that the file size for the marching cubes representation is over an order of magnitude larger than the high-order representation.

Figure 1: Marching cubes image with $h=0.015$ corresponding to 4,705,274 voxels (left) and ray-traced solution using 11th order projected polynomials (right) for isosurface of pressure at $C = 0.0$ chosen such that the spectral/hp element data evaluation and rendering time is nearly identical (on the order of 200 seconds). The root-mean-square error for the marching cubes image is 0.0158; the root-mean-square error for the ray-traced image is 3.5e-11.
Although the ray-casting methodology provided a “pixel exact” visualization of the isosurface, it did so in what is referred to as “image space”. This implies that even after a researcher found the isosurface of interest which they wanted to examine, each rotation, translation or zoom into the image required approximately the same amount of rendering time as each pixel's color has to be recomputed.

The classic way to attempt to solve this issue is to render things in “object space” – that is, to generate objects (triangles, for instance) on the isosurface so that once an isosurface is found and an object is created, its rendering can be done quickly. In [J1] we proposed visualizing isosurfaces in high-order finite element datasets with a particle system as a means of solving this problem. We presented a framework that allows particles to sample an isosurface in reference space, avoiding the costly inverse mapping of positions from world space when evaluating the basis functions. The distribution of particles across the reference space isosurface is controlled by geometric information from the world space isosurface, such as the surface gradient and curvature. The resulting particle distributions can be distributed evenly or adapted to accommodate world-space surface features. This provides compact, efficient, and accurate isosurface representations of these challenging data sets. In Figure 2 we present a visualization of an isosurface of pressure within an incompressible fluid flow field.

Figure 2: An isosurface of a finite element fluid simulation pressure field sampled with a particle system. The color indicates the relative direction of the surface normal at the particle (blue indicates outward and red indicates inward).

When one employs objects to mark or denote an isosurface, one faces the challenge of knowing how many objects to use and how densely to pack them. A sparse packing of the objects can miss critical features of the isosurface. A dense packing can be very inefficient (especially when the density is much higher than is needed). In [J2], we describe a method for constructing isosurface triangulations of sampled, volumetric, three-dimensional scalar fields that attempts to tackle this sampling density problem. The resulting meshes consist of triangles that are of consistently high quality, making them well suited for accurate interpolation of scalar and vector-valued quantities, as required for numerous applications in visualization and numerical simulation. The proposed method does not rely on a local construction or adjustment of triangles as is done, for instance, in advancing wavefront or adaptive refinement methods. Instead, a system of dynamic particles optimally samples an implicit function such that the
particles' relative positions can produce a topologically correct Delaunay triangulation. Thus, the proposed method relies on a *global* placement of triangle vertices. The main contributions of this work was the integration of dynamic particles systems with surface sampling theory and PDE-based methods for controlling the local variability of particle densities, as well as detailing a practical method that accommodates Delaunay sampling requirements to generate sparse sets of points for the production of high-quality tessellations. In [J5] we extended this work to handle surfaces that come as a consequence of multi-material interfaces.

- **Streamline Integration**

A quick search of both the visualization and the application domain literature demonstrates that streamlines are a popular visualization tool, second only to isosurfaces. The bias toward using streamlines is in part explained by studies that show streamlines to be effective visual representations for elucidating the salient features of the vector fields [R3]. Furthermore, streamlines as a visual representation are appealing because they are applicable for both two-dimensional and three-dimensional fields [R4]. It was for this reason that we invested time considering how streamlining would be impacted by high-order finite element data.

Streamline integration is often accomplished through the application of ordinary differential equation (ODE) integrators such as predictor-corrector or Runge-Kutta schemes. The foundation for the development of these schemes is the use of Taylor series for building numerical approximations of the solution of the ODE of interest. Taylor series can be further used to elucidate the error characteristics of the derived scheme. All schemes employed for streamline integration that are built using such an approach exhibit error characteristics which are predicated on the smoothness of the field through which the streamline is being integrated.

Low-order and high-order finite volume and finite element fields are among the most common types of fluid flow simulation datasets available. Streamlining is commonly applied to these datasets. The property of these fields which challenges classic streamline integration using Taylor series based approximations is that finite volume fields are piecewise discontinuous and finite element fields are only $C^0$ continuous. Hence one of the limiting factors of streamline accuracy and integration efficiency is the lack of smoothness at the inter-element level of finite volume and finite element data.

Adaptive error control techniques are often used to ameliorate the challenge posed by inter-element discontinuities. To paraphrase a classic work on the subject of solving ODEs with discontinuities [R5], one must (1) detect, (2) determine the order, size and location of, and (3) judiciously “pass over” discontinuities for effective error control. Such an approach has been effectively employed within the visualization community for overcoming the challenges posed by discontinuous data at the cost of increased number of evaluations of the field data. The number of evaluations of the field increases drastically with every discontinuity that is encountered [R5]. Thus if one requires a particular error tolerance and employs such methods for error control when integrating a
streamline through a finite volume or finite element dataset, a large amount of the computational work involved is due to handling inter-element discontinuities and not the intra-element integration. We demonstrate this in Figure 3 where one can see that the number of streamline sampling steps goes up drastically each time a streamline attempts to traverse over an element boundary.

As the root of the difficulties is the discontinuous nature of the data, one could speculate that if one were to filter the data in such a way that it was no longer discontinuous, streamline integration could then be made more efficient. The caveat that arises when one is interested in simulation and visualization error control is how does one select a filter that does not destroy the formal accuracy of the simulation data through which the streamlines are to be integrated? Recent mathematical advances [R6, R7] have shown that such filters can be constructed for high-order finite element and discontinuous Galerkin (high-order finite volume) data on uniform quadrilateral and hexahedral meshes. These filters are such that they have the provable quality that they increase the level of smoothness of the field without destroying the accuracy in the case that the “true solution” that the simulation is approximating is smooth. In fact, in many cases, these filters can increase the accuracy of the solution.

As part of our work, we investigated the use of such filters applied to discontinuous data prior to streamline integration, and found that they can drastically improve the computational efficiency of the integration process. We currently have two published papers on this topic [J3, J4] (one presenting this work to the visualization community, and one paper presenting new computational mathematics work which came as a consequence of this study). We also have an accepted paper in which we have adapted this idea to be more computationally efficient [J6]. We proposed a new technique that uses a one-dimensional convolution kernel to introduce continuity between elements, and increase smoothness while not introducing additional error in the solution. Furthermore, this one-dimensional implementation is the same regardless of the dimension of the
simulation data. This in turn will aid in accomplishing the goals of visualization of data over more complex geometries while still improving the smoothness of the field and not compromising the accuracy of the data.

**Supported Talks and Publications**

**Journal Publications**


**Invited Talks**

Intelligent Visualization and Simulation Lab, University of Kaiserslautern, Germany. Presented a talk entitled “Visualization of High-Order Finite Element Methods”, June 2008.


Center of Complex Systems and Visualization, University of Bremen, Germany. Presented a talk entitled “Particle Systems for Efficient and Accurate High-Order Finite Element Visualization”, March 2007.

Talks

Sean Curtis, Robert M. Kirby and Jennifer K. Ryan, “Accuracy Enhancing Filtering With Application To Visualization”. Presented at the 7th World Congress on Computational Mechanics, July 2006.


Supported Individuals

Professor Robert M. Kirby (PI) – Utah

Mr. Robert Haimes (MIT subcontract PI) – MIT

Sarah Geneser (Spring/Summer 2006, Summer 2008) – Utah
   Completed a PhD in Computer Science, Spring 2008, University of Utah

Miriah Meyer (Fall 2006 – Spring 2008) – Utah
   Completed a PhD in Computer Science, Spring 2008, University of Utah

David Walfisch (Fall 2005 – Spring 2008), MIT
   Completed a MS in Aeronautics, Spring 2008, MIT

References


MEMORANDUM OF TRANSMITTAL

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P.O. Box 12211
Research Triangle Park, NC 27709-2211

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CONTRACT/GRANT NUMBER: W911NF-05-C-0395

REPORT TITLE: Visualization of High-Order Finite Element methods

is forwarded for your information.

SUBMITTED FOR PUBLICATION TO (applicable only if report is manuscript):

Sincerely,

Enclosure 3
**REPORT OF INVENTIONS AND SUBCONTRACTS**

(Pursuant to "Patent Rights" Contract Clause) (See instructions on back)

The public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing the burden, to the Department of Defense, Executive Services Directorate (8000-0095). Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

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### 1. NAME OF CONTRACTOR/SUBCONTRACTOR
- Mike Kirby

### 2. CONTRACT NUMBER
- W911NF-05-1-0395

### 3. TYPE OF REPORT
- **X** INTERIM

### 4. REPORTING PERIOD
- **X** FROM 20050630 TO 20080731

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### SECTION III - CERTIFICATION

I certify that the reporting party has procedures for prompt identification and timely disclosure of "Subject Inventions," that such procedures have been followed and that all "Subject Inventions" have been reported.

- **X** SMALL BUSINESS

- **_** NONPROFIT ORGANIZATION

**DD FORM 882, JUL 2005**