Mathematical Perspectives on the Federal Thrift Savings Plan (TSP)

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LTC Scott T. Nestler, PhD
Assistant Professor and Research Analyst
Operations Research Center
U.S. Military Academy at West Point
Mathematical Perspectives on the Federal Thrift Savings Plan (TSP)

Operations Research Center U.S. Military Academy at West Point

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<th>Other Authors</th>
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<td>Nestler, Scott T.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Principal Author's Organization:</td>
<td>U.S. Military Academy at West Point</td>
<td></td>
</tr>
<tr>
<td>Complete mailing address:</td>
<td>211C Barry Rd, West Point, NY 10996</td>
<td></td>
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Releasing Official’s Title: MAJ, ORCEN Analyst
Printed name: Teague, Ed
Organization: U.S Military Academy at West Point
Complete mailing address: Mahan Hall, Bldg. 752, Rm 306, West Point, NY 10996
Phone: (845) 938-5661
Fax: (845) 938-5665
Date: 01 MAY 2008

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Disclaimer

• **You will not receive any personal financial advice during this talk, as I am not officially qualified or certified to do so.**

• **However, my presentation is intended to get you to think mathematically about one of the retirement savings options available to many of you.**
Questions for Consideration

- Why might I be more risk tolerant than I currently believe?
- What are the L (Lifecycle) funds? How are they constructed? Why might they be of interest (or not) to me?
- What if stock and index fund returns are not normally distributed, as is commonly assumed?
- How does the choice of reward and risk measures affect optimal TSP portfolios?
Big Picture on Saving $$$

“There is no scholarship for retirement!”
-Unknown

• Spouse’s 401(k) with matching funds
• Roth IRA (for Soldier/civilian and spouse)
• Thrift Savings Plan (TSP)
• Spouse’s 401(k) without matching funds
• Coverdale Educational Savings Accounts
• 529 Tuition Plans (prepaid or savings)

(ordering of these depends on tax considerations)
Thrift Savings Plan (TSP) Overview

• Largest defined contribution retirement savings and investment plan
  – 3.7 million participants
  – $210 billion in assets
• 401(k) equivalent for government employees and uniformed service members
• 5 non-traded core funds
• Can rebalance daily with no direct costs
## Core TSP Funds

<table>
<thead>
<tr>
<th>Fund</th>
<th>Description</th>
<th>Assets*</th>
<th>Mean Return#</th>
<th>Standard Deviation#</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>short-term, specially issued Treasury securities</td>
<td>$66.6B (39.2%)</td>
<td>6.4%</td>
<td>1.5%</td>
</tr>
<tr>
<td>F</td>
<td>tracks Lehman Brothers U.S. Aggregate (LBA) Index</td>
<td>$10.2B (6.0%)</td>
<td>7.3%</td>
<td>5.6%</td>
</tr>
<tr>
<td>C</td>
<td>tracks S&amp;P 500 Index</td>
<td>$66.7B (39.3%)</td>
<td>13.0%</td>
<td>17.9%</td>
</tr>
<tr>
<td>S</td>
<td>tracks Dow Jones Wilshire 4500 Completion Index</td>
<td>$13.7B (8.1%)</td>
<td>13.3%</td>
<td>19.9%</td>
</tr>
<tr>
<td>I</td>
<td>tracks MSCI EAFE (Europe, Australia, Far East) Index</td>
<td>$12.6B (7.4%)</td>
<td>7.8%</td>
<td>18.7%</td>
</tr>
</tbody>
</table>

* As of Dec 31, 2005  # For the period 1988-2005
Returns / Investment Horizon

• Returns
  – Arithmetic:
    \[ r_{i,t} = \frac{S_{i,t} - S_{i,t-1}}{S_{i,t-1}} \]
  – Log:
    \[ r_{i,t} = \ln(S_{i,t}) - \ln(S_{i,t-1}) = \ln \left( \frac{S_{i,t}}{S_{i,t-1}} \right) \]

• Investment Horizon: 20 years
  – Point of military (not ultimate) retirement
  – System encourages 20 year careers
  – Employment options vary greatly
  – Can move TSP assets to other plans
Daily Returns Time Series

- G Fund
- F Fund
- C Fund
- S Fund
- I Fund

Daily Returns Distributions

- G fund appears approximately Gaussian
- C, S, I, and F funds are more peaked with heavy tails
- Goodness of Fit testing at common levels of significance rejects Normal for F, C, and S funds, even with batched means
Mean-Variance Portfolio Optimization

*(Markowitz, 1952)*

Let $X_i =$ fraction of funds invested in asset $i$

$\bar{R}_i =$ expected return of asset $i$

$\bar{R}_p =$ expected return of portfolio $p$

$\sigma_i^2 =$ variance of return of asset $i$

$\sigma_{jk} =$ covariance of return of asset $j$ with asset $k$

Minimize

$$\sigma_p^2 = \sum_{j=1}^{N} X_j^2 \sigma_j^2 + \sum_{j=1}^{N} \sum_{k=1, k \neq j}^{N} X_j X_k \sigma_{jk}$$

Subject to:

$$\sum_{i=1}^{N} X_i = 1$$

$$\bar{R}_p = \sum_{i=1}^{N} X_i \bar{R}_i$$

$$X_i \geq 0, \quad i = 1, \ldots, N$$
**L (Lifecycle) Funds**

- Invest in 5 core TSP funds based on time horizon to provide highest possible rate of return for risk taken.
- Over time, investments shift away from stocks and into bonds.
- L Funds are great, but …
Reward-Risk Profile of TSP Funds
VG-ICA Factor Model
(Madan & Yen, 2004)

\[(R - \mu) = XB + \varepsilon\]
\[D = XB + \varepsilon\]

1. Use Independent Component Analysis (ICA) on asset returns $D$ to identify underlying factors $X$

2. Fit the Variance Gamma (VG) distribution to each retained factor by MLE

3. Use Expected Utility to determine optimal portfolio of VG-ICA factors; convert back to optimal portfolio of assets (TSP funds)
Independent Component Analysis (ICA)

- Principal Component Analysis (PCA)
  - Focus on finding **uncorrelated** components in Gaussian data
  - Maximizes explained variance
  - Uses second-order statistics

- Factor Analysis
  - Essentially PCA with extra terms to model noise

- ICA
  - Focus on **independent** and non-Gaussian components
  - Maximizes non-Gaussianity (to maximize information)
  - Uses higher-order statistics
Another ICA Example

Original Signals ($s$)  Mixed Signals ($x$)  ICA source estimates ($y$)

A (mixing Matrix)  W (de-mixing Matrix)
ICA versus PCA

- **Principal Component Analysis (PCA)** finds:
  - directions of maximal variance in Gaussian data (second-order statistics).
  - directions of maximal variance in non-Gaussian data (second-order statistics).

- **Independent Component Analysis (ICA)** finds directions of maximal independence in non-Gaussian data (higher-order statistics).
How Many ICs to Keep?

Dropping more than one IC reduces fit on at least one fund

The first four have excess kurtosis

→ Keep 4 Independent Components (ICs)
VG Process and Distribution

• Pure jump process with two representations
  – Time-changed Brownian motion (Madan & Seneta, 1990)
    \[ X_{VG}(t; \nu, \theta, \sigma) = b(\gamma(t;1,\nu), \theta, \sigma) \]
  – Difference of 2 Gamma processes (Madan, Carr & Chang, 1998)
    \[ X_{VG}(t) = G_p(t) - G_n(t) \]

• Parameters: \( \sigma \) controls spread
  \( \nu \) affects kurtosis
  \( \theta \) impacts skewness

• Density Function (Madan, Carr & Chang, 1998)
  \[ h(z) = \frac{2 \exp(\theta x / \sigma^2)}{\nu^{1/\nu} \sqrt{2\pi \sigma} \Gamma(\frac{1}{\nu})} \left( \frac{x^2}{2\sigma^2 / \nu + \theta^2} \right)^{\frac{1}{2\nu} - \frac{1}{4}} K_{\frac{1}{\nu} - \frac{1}{2}} \left( \frac{1}{\sigma^2} \sqrt{x^2 (2\sigma^2 / \nu + \theta^2)} \right) \]
  with \( x = z - \theta \) where \( z = \ln(S(t) / S(t - 1)) \)
Examples of VG Distributions

Effect of $\nu$

Effect of $\theta$

[Graphs showing the effect of $\nu$ and $\theta$ on VG distributions]
Fitting VG by MLE

- Given observed IID data $X_1, X_2, \ldots, X_n$, define the likelihood function as:

  $$L(\theta) = f(\theta)(X_1) f(\theta)(X_2) \cdots f(\theta)(X_n)$$

- The MLE (maximum likelihood estimator) $\hat{\theta}$ maximizes $L(\theta)$ over all permissible values of $\theta$.
- Actually, maximizing the log likelihood function $\ln(L(\theta))$ is easier.
- For the VG distribution with three parameters, this becomes:

  $$l(\sigma, \nu, \theta) = \ln L(\sigma, \nu, \theta) = \sum_i f(\sigma, \nu, \theta)(X_i)$$

  (using pdf given before from Madan, Carr, and Chang, 1998)
Comparison of Fitted VG and Normal(0,1)

IC1

IC2

IC3

IC4

Data

Normal(0,1)

Fitted VG
### Fitted VG Parameters / Chi-Square Statistics

<table>
<thead>
<tr>
<th>IC#</th>
<th>Fitted VG Parameters - Daily (Annualized)</th>
<th>$\chi^2$ Test Statistic (p-values) ($\chi^2_{01,17} = 33.41$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma$</td>
<td>$\nu$</td>
</tr>
<tr>
<td>IC1</td>
<td>0.933</td>
<td>0.969</td>
</tr>
<tr>
<td></td>
<td>(14.814)</td>
<td>(.00385)</td>
</tr>
<tr>
<td>IC2</td>
<td>0.980</td>
<td>0.820</td>
</tr>
<tr>
<td></td>
<td>(15.558)</td>
<td>(.00326)</td>
</tr>
<tr>
<td>IC3</td>
<td>0.989</td>
<td>0.586</td>
</tr>
<tr>
<td></td>
<td>(15.703)</td>
<td>(.00232)</td>
</tr>
<tr>
<td>IC4</td>
<td>0.991</td>
<td>0.468</td>
</tr>
<tr>
<td></td>
<td>(15.739)</td>
<td>(.00186)</td>
</tr>
</tbody>
</table>

- Some excess kurtosis and slight negative skewness in each IC
- VG fits much better than Normal distribution
Utility Theory & Risk Aversion

*Utility* - a measure of relative satisfaction obtained

*Risk Aversion* - concave utility function, as shown below

Utility (Wealth)

Utility (Wealth) vs Wealth diagram with points $W_L$, $W_H$, $V_L$, $V_H$, and $M$.
Aside on Risk Aversion/Tolerance
(Jennings & Reichenstein, 2001)

- Pensions considered when planning retirement income….. but NOT when calculating asset allocation
- Pensions and investment portfolio generate retirement funds; why not consider both in total portfolio?
- Many similarities between inflation-indexed Treasury bonds (TIPS) and military retirement
  - Linked to Consumer Price Index (CPI)
  - Backed by federal government
- Suggest treating after-tax present value as a “pseudo-bond” in total portfolio
- Discounting can be at recent TIPS rates (3%-5%) or higher personal discount rate (18+%)  
- Results in more aggressive (risk tolerant) portfolio in active investments than would otherwise result
NPV of Military Retirement

(Jennings & Reichenstein, 2001)

<table>
<thead>
<tr>
<th>Rank at Retirement</th>
<th>Years of Service</th>
<th>After-Tax NPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>LTC</td>
<td>20</td>
<td>$726,674</td>
</tr>
<tr>
<td>LTC</td>
<td>22</td>
<td>$802,690</td>
</tr>
<tr>
<td>COL</td>
<td>24</td>
<td>$994,468</td>
</tr>
<tr>
<td>COL</td>
<td>26</td>
<td>$1,096,490</td>
</tr>
<tr>
<td>COL</td>
<td>28</td>
<td>$1,166,125</td>
</tr>
<tr>
<td>COL</td>
<td>30</td>
<td>$1,205,255</td>
</tr>
</tbody>
</table>

Assumptions:

- Officer currently at 18 years of service
- 28% tax bracket
- 4% TIPS rate / inflation
“Pseudo-Bond” Example
(Nestler, 2007)

Desired/Current Financial Portfolio

40% Bonds

60% Stocks

Bonds

Stocks

Resulting Expanded Portfolio

Military or Government Pension

75% Bonds and “Pseudo-Bonds”

25% Stocks

Bonds

Stocks
Negative Exponential Utility

\[ U(w) = -e^{-cw}, \quad c > 0 \]

- Constant Absolute Risk Aversion (CARA) -- no “wealth effect”
- Computational tractability advantage over other (log, power) utility functions
- Analytical solution to maximization problem is available using Certainty Equivalent (CE)
- CE is well-known for Normal and given for VG-ICA (Madan and Yen, 2004)
Implied Risk Aversion Coefficient

![Graph showing the relationship between Risk Aversion Coefficient and Risk (SD). The graph includes several labeled lines for years 2010 to 2040 and a horizontal line indicating income.]
## Portfolios for Comparison

<table>
<thead>
<tr>
<th>Model</th>
<th>G Fund</th>
<th>F Fund</th>
<th>C Fund</th>
<th>S Fund</th>
<th>I Fund</th>
</tr>
</thead>
<tbody>
<tr>
<td>VG-ICA (Daily)</td>
<td>0%</td>
<td>1%</td>
<td>43%</td>
<td>30%</td>
<td>26%</td>
</tr>
<tr>
<td>Riskless</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>TSP “Market Portfolio”</td>
<td>39%</td>
<td>6%</td>
<td>39%</td>
<td>8%</td>
<td>8%</td>
</tr>
<tr>
<td>L 2030</td>
<td>16%</td>
<td>9%</td>
<td>38%</td>
<td>16%</td>
<td>21%</td>
</tr>
<tr>
<td>L 2040</td>
<td>5%</td>
<td>10%</td>
<td>42%</td>
<td>18%</td>
<td>25%</td>
</tr>
</tbody>
</table>

**NOTE:** These portfolios are created with returns assumed to be Normally distributed.
Stochastic Dominance

- Generalizes utility theory; don’t need a specific utility function

- First-Order Stochastic Dominance (FOSD)
  - Assumes only monotonicity; strongest result
  - A FOSD B IFF \( F_B(x) \geq F_A(x), \forall x \)

- Second-Order Stochastic Dominance (SOSD)
  - Also assumes risk aversion
  - A SOSD B IFF \( \int_{-\infty}^{x} [F_B(u) - F_A(u)] du \geq 0, \forall x \)

- Easy to test with empirical data
Traditional Risk Measures

• Dispersion Measures
  – Variance (or Standard Deviation)
    • Treats gains and losses equally
  – Semi-Variance
    • Only considers observations below mean
  – Mean Absolute Deviation (MAD)
    • Average absolute deviation from the mean

• “Safety Risk” Measures
  – Value-at-Risk (VaR)
  – Expected Tail Loss (ETL)
Value-at-Risk (VaR)

- “Expected maximum loss over a fixed horizon for a given confidence level”
  \[ P(X \geq \text{VaR}_\lambda(X)) = \lambda \]
- Standard risk measure for past 12 years
- Does not reward diversification
- Addresses size but not shape of tail
Coherent Measures of Risk
(Artzner, Delbaen, Eber, & Heath, 1999)

• Axioms for coherency:
  – Translation invariance \( \rho(X - \alpha) = \rho(X) - \alpha \)
  – Monotonicity \( X > Y \Rightarrow \rho(X) > \rho(Y) \)
  – Sub-additivity \( \rho(X + Y) \leq \rho(X) + \rho(Y) \)
  – Positive homogeneity \( \rho(\lambda X) = \lambda \rho(X) \)

• Variance: not monotonic or translation invariant
• VaR: not sub-additive in non-Gaussian world
• Other measures that are coherent exist.
Conditional VaR

- “Expected value of all losses greater than VaR for a specified $\lambda$.”

$$CVaR_{\lambda}(X) = E[X \mid X > VaR_{\lambda}(X)]$$

- Also known as Expected Shortfall (Rockafellar & Uryasev, 2001) and Tail VaR (Acerbi, Nordio, et al., 2001)

- Accounts for size and shape of left tail but ignores rest of distribution
Classes of Weighted VaR
*(Cherny, 2006; Cherny & Madan, 2007)*

\[
WVaR_\mu(X) = \int_{[0,1]} CVaR_{\lambda}(X) \mu(dx)
\]

- **Beta VaR(\(\alpha, \beta\))**
  \[
  \mu_{\alpha,\beta}(dx) = B(\beta + 1, \alpha - \beta)^{-1} x^\beta (1 - x)^{\alpha - \beta - 1} dx, \quad x \in [0,1]
  \]
  - Expectation of average of the \(\beta\) biggest of \(\alpha\) independent copies of portfolio loss
  - Faster to estimate than CVaR

- **Alpha VaR(\(\alpha\))**
  - Essentially Beta VaR with \(\beta=1\)
  - Expectation of biggest of \(\alpha\) copies of portfolio loss
Effect of Alpha and Beta

Can allow for more risk by decreasing $\alpha$ or increasing $\beta$
Performance (Reward-Risk) Measures

- Sharpe Ratio
  \[ SR = \frac{E(X)}{\sigma_X} \]

- STARR Ratio
  \[ STARR = \frac{E(X)}{CVaR_\lambda(X)} \]

- R-Ratio
  \[ R = \frac{CVaR_{\lambda_1}(-X)}{CVaR_{\lambda_2}(X)} \]

(Rachev)
New Portfolio Performance Measures
(Nestler, 2007b)

• Similar to R-Ratio but use Alpha-VaR and Beta-VaR in place of CVaR

• AVaR-Ratio: \[ AVR = \frac{AVaR_{\alpha_1}(-X)}{AVaR_{\alpha_2}(X)} \]

• BVaR-Ratio: \[ BVR = \frac{BVaR_{\alpha_1,\beta_1}(-X)}{BVaR_{\alpha_2,\beta_2}(X)} \]
Monthly Contribution

• Assumes saving 10% of base pay each month (median for TSP)

TOTAL CONTRIBUTIONS: $170K
Realistic Scenario: Portfolio Value  
(5000 sample paths)

Expected Value:
- VG-ICA: $418,381
- L2040: $318,840
- L2030: $313,936
- TSP MP: $310,247
- Riskless: $259,642

Upside Potential:
- VG-ICA: $1,992,133
- L2040: $1,127,069
- L2030: $1,093,643
- TSP MP: $1,064,902

Total Contrib. $170,505
PDF of Discounted Portfolio Value
Realistic Scenario: CDF Comparison

NOTE: No SD.
Realistic Scenario: Zoomed CDF Comparison

Portfolio Gain/Loss

F(Gain/Loss)

LEGEND
- VG-ICA
- TSP MP
- L 2030
- L2040

95% VaR

13%

28%

32%

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### Realistic Scenario: Risk & Performance Measures

<table>
<thead>
<tr>
<th>Risk Measure (↓ better)</th>
<th>VG-ICA</th>
<th>TSP MP</th>
<th>L 2030</th>
<th>L 2040</th>
</tr>
</thead>
<tbody>
<tr>
<td>Std Dev</td>
<td>$168,885</td>
<td>$80,890</td>
<td>$94,515</td>
<td>$105,525</td>
</tr>
<tr>
<td>95% VaR</td>
<td>$43,382</td>
<td>$44,146</td>
<td>$60,378</td>
<td>$66,910</td>
</tr>
<tr>
<td>95% CVaR</td>
<td>$68,056</td>
<td>$54,789</td>
<td>$74,684</td>
<td>$82,783</td>
</tr>
<tr>
<td>Alpha VaR(50)</td>
<td>$77,575</td>
<td>$59,250</td>
<td>$81,352</td>
<td>$87,754</td>
</tr>
<tr>
<td>Beta VaR(50,5)</td>
<td>$43,938</td>
<td>$44,010</td>
<td>$60,203</td>
<td>$67,757</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Performance Measure (↑ better)</th>
<th>VG-ICA</th>
<th>TSP MP</th>
<th>L 2030</th>
<th>L 2040</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sharpe Ratio</td>
<td>0.94</td>
<td>0.63</td>
<td>0.57</td>
<td>0.54</td>
</tr>
<tr>
<td>STARR Ratio</td>
<td>2.33</td>
<td>2.90</td>
<td>0.72</td>
<td>0.69</td>
</tr>
<tr>
<td>R-Ratio(.05,.05)</td>
<td>8.87</td>
<td>2.82</td>
<td>4.10</td>
<td>4.07</td>
</tr>
<tr>
<td>AVR</td>
<td>9.35</td>
<td>5.41</td>
<td>4.62</td>
<td>4.67</td>
</tr>
<tr>
<td>BVR</td>
<td>11.60</td>
<td>5.05</td>
<td>4.24</td>
<td>4.12</td>
</tr>
</tbody>
</table>
Traditional Reward-Risk Profile

![Graph showing the relationship between Reward ( Expected Return ) and Risk ( Standard Deviation ) for various funds and strategies. The graph includes points for G Fund, F Fund, C Fund, S Fund, TSP MP, and L 2010, L 2020, L 2030, and L 2040, indicating their risk and reward profiles. The graph also highlights a trend line for 20% each distribution.]

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New Reward-Risk Profile
Some Possible Answers

• Why might I be more risk tolerant than I currently believe?
  – Counting military or government pension as “pseudo-bonds” could change the target stock-bond asset mix.

• What are the L (Lifecycle) funds? How are they constructed? Why might they be of interest (or not) to me?
  – “Set it and forget it” funds built using mean-variance optimization with returns assumed to be distributed Normally.
  – Depends on an individual’s level of interest and involvement.

• What if stock and index fund returns are not normally distributed, as is commonly assumed?
  – Possible to take advantage of information contained in higher moments.

• How does choosing reward-risk measures affect optimal TSP portfolios?
  – Ability to capture information from entire distribution is useful.
  – Need to do further work on optimizing performance measures instead of using expected utility.
QUESTIONS?