Eavesdropping (or Jamming) of Communication Networks
Project# FA9550-05-1-0137

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December, 2007

REPORT DOCUMENTATION PAGE			*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
	Final	

4. TITLE AND SUBTITLE	
Eavesdropping (or Jamming) of Communication Networks	5a. CONTRACT NUMBER FA9550-05-1-0137
	5b. GRANT NUMBER AFOSR
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) Dr. Stanislav Uryasev	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) University of Florida PO Box 116595 303 Weil Hall, Gainesville, FL 32611-6595	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) AFOSR 875 Randolph Arlington VA 22203	10. SPONSOR/MONITOR'S ACRONYM(S) AFOSR
	11. SPONSOR/MONITOR'S REPORT NUMBER(S) AFRL-SR-AR-TR-08-0329

12. DISTRIBUTION/AVAILABILITY STATEMENT

Distribution A: Approved for Public Release

13. SUPPLEMENTARY NOTES

14. ABSTRACT

This report presents research on optimization formulations for jamming wireless communication networks. The first chapter addresses deterministic situations which were the focus of the first research year. In particular, several formulations of the deterministic WIRELESS NETWORK JAMMING PROBLEM were derived and theoretical proof of problem statements equivalences were obtained. The second year addressed situations in which a network is to be jammed, but no apriori information (i.e., topology, number of nodes, etc.) were assumed known. Proofs of upper and lower bounds on the required number of jamming devices as well as convergence results were derived. A heuristic for this setup was also proposed. During the third year the robust optimization formulations were researched. Information such as the number and placement of the communication nodes and other parameters were considered subject to some uncertainty.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			19b. TELEPHONE NUMBER *(Include area code)*

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Summary

This report presents research on optimization formulations for jamming wireless communication networks. The first chapter addresses deterministic situations which were the focus of the first research year. In particular, several formulations of the deterministic WIRELESS NETWORK JAMMING PROBLEM were derived and theoretical proof of problem statements equivalences were obtained. The second year addressed situations in which a network is to be jammed, but no apriori information (i.e., topology, number of nodes, etc.) were assumed known. Proofs of upper and lower bounds on the required number of jamming devices as well as convergence results were derived. A heuristic for this setup was also proposed. During the third year the robust optimization formulations were researched. Information such as the number and placement of the communication nodes and other parameters were considered subject to some uncertainty. We developed fast and robust percentile formulations for these cases. Equivalence of problem statements was proven for stochastic the robust formulations and computational experiments were performed

Introduction

In adversarial environments, disabling the communication capabilities of the enemy is a high priority task.

The first chapter of the report introduced the problem of determining the optimal number and locations for a set of jamming devices in order to neutralize a wireless communication network. This problem is known as the WIRELESS NETWORK JAMMING PROBLEM. We developed several mathematical programming formulations based on covering the communication nodes and limiting the connectivity index of the nodes. Two case studies are presented comparing the formulations with the addition of various percentile constraints.

The second chapter considers the case where there is no information about the network to be jammed. The problem is reduced to jamming all points in the area of interest. The optimal solution determines the locations of the minimum number of jamming devices required to suppress the network. We consider a subproblem which places jamming devices on the nodes of a uniform grid over the area of interest. The objective here is to determine the maximum grid step size. We derived upper and lower bounds for this problem and provided a convergence result. We proved that due to the cumulative effect of the jamming devices, the proposed method produces better solutions than the classical technique of covering the region with uniform circles.

The deterministic formulations of the wireless network jamming problem are extended in the third chapter to tackle the stochastic jamming problem formulations. Robust variants of previously developed deterministic formulations are introduced. These formulations consider the case when the exact topology of the network to be jammed is not known. Particularly, we considered instances with several likely topologies, and developed robust approaches for placing jamming devices to suppress the network regardless of which topology is realized. We derived several formulations and included percentile constraints to account for a variety of scenarios. Case studies are presented and the results are analyzed.

Chapter 1

The wireless network jamming problem

This chapter presents the results published in [7] Commander, C., Pardalos, P., Ryabchenko, V. , Uryasev, S. and G. Zrazhevsky. The wireless network jamming problem. Journal of Combinatorial Optimization, 14:4, pp. 481-498, 2007.

1.1 Introduction

Military strategists are constantly seeking ways to increase the effectiveness of their force while reducing the risk of casualties. In any adversarial environment, an important goal is always to neutralize the communication system of the enemy. In this work, we are interested in jamming a wireless communication network. Specifically, we introduce and study the problem of determining the optimal number and placement for a set of jamming devices in order to neutralize communication on the network. This is known as the WIRELESS NETWORK JAMMING PROBLEM (WNJP). Despite the enormous amount of research on optimization in telecommunications [23], this important problem for military analysts has received little attention by the research community.

The organization of the chapter is as follows. Section 3.2 contains several formulations based on covering the communication nodes with jamming devices. In Section 3.4, we use tools from graph theory to define an alternative formulation based on limiting the connectivity index of the network nodes. Next, we incorporate percentile constraints to develop formulations which provide solutions requiring less jamming devices, but whose solution quality favors the exact methods. In Section 3.5, we present two case studies comparing the solutions and computation time for all formulations. Finally, conclusions and future directions of research are addressed.

We will now briefly introduce some of the idiosyncrasies, symbols, and notations we will employ throughout this chapter. Denote a graph $G = (V, E)$ as a pair consisting of a set of vertices V, and a set of edges E. All graphs in this chapter are assumed to be undirected and unweighted. We use the symbol "$b := a$" to mean "the expression a defines the (new) symbol b" in the sense of [17]. Of course, this could be conveniently extended so that a statement like "$(1-\epsilon)/2 := 7$" means "define the symbol ϵ so that $(1-\epsilon)/2 = 7$ holds." Finally, we will use *italics* for emphasis and SMALL CAPS for problem names. Any other locally used terms and symbols will be defined in the sections in which they appear.

1.2 Coverage Formulations

Before formally defining the problem statement, we will state some basic assumptions about the jamming devices and the communication nodes being jammed. We assume that parameters such as the frequency range of the jamming devices are known. In addition, the jamming devices are assumed to have omnidirectional antennas. The communication nodes are also assumed to be outfitted with omnidirectional antennas and function as both receivers and transmitters. Given a graph $G = (V, E)$, we can represent the communication devices as the vertices of the graph. An undirected edge would connect two nodes if they are within a certain communication threshold.

Given a set $\mathcal{M} = \{1, 2, \dots, m\}$ of communication nodes to be jammed, the goal is to find a set of locations for placing jamming devices in order to suppress the functionality of the network. The *jamming effectiveness* of device j is calculated using $d : (V \times V) \mapsto \mathbb{R}$, where d is a decreasing function of the distance from the jamming device to the node being jammed. Here we are considering radio transmitting nodes, and correspondingly, jamming devices which emit electromagnetic waves. Thus the jamming effectiveness of a device depends on the power of its electromagnetic emission, which is inversely proportional to the squared distance from the jamming device to the node being jammed. Specifically,

$$d_{ij} := \frac{\lambda}{r^2(i,j)},$$

where $\lambda \in \mathbb{R}$ is a constant, and $r(i,j)$ represents the distance between node i and jamming device j. Without the loss of generality, we can set $\lambda = 1$.

The cumulative level of jamming energy received at node i is defined as

$$Q_i := \sum_{j=1}^{n} d_{ij} = \sum_{j=1}^{n} \frac{1}{r^2(i,j)},$$

where n is the number of jamming devices. Then, we can formulate the WIRELESS NETWORK JAMMING PROBLEM (WNJP) as the minimization of the number of jamming devices placed, subject to a set of *covering* constraints:

$$\textbf{(WNJP)} \quad \text{Minimize} \quad n \tag{1.1}$$

$$\text{s.t.} \quad Q_i \geq C_i, \quad i = 1, 2, \dots, m. \tag{1.2}$$

The solution to this problem provides the optimal number of jamming devices needed to ensure a certain jamming threshold C_i is met at every node $i \in \mathcal{M}$. A continuous optimization approach where one is seeking the optimal placement coordinates $(x_j, y_j), j = 1, 2, \dots, n$ for jamming devices given the coordinates $(X_i, Y_i), i = 1, 2, \dots, m$, of network nodes, leads to highly non-convex formulations. For example, consider the covering constraint for network node i, which is given as

$$\sum_{j=1}^{n} \frac{1}{(x_j - X_i)^2 + (y_j - Y_i)^2} \geq C_i.$$

It is easy to verify that this constraint is non-convex. Finding the optimal solution to the resulting nonlinear programming problem would require an extensive amount of computational effort.

To overcome the non-convexity of the above formulation, we propose several integer programming models for the problem. Suppose now that along with the set of communication nodes $\mathcal{M} = \{1, 2, \dots, m\}$, there is a fixed set $\mathcal{N} = \{1, 2, \dots, n\}$ of possible locations for the jamming devices. This assumption is reasonable because in real battlefield scenarios, the set of possible placement locations will likely be limited. Define the decision variable x_j as

$$x_j := \begin{cases} 1, & \text{if a jamming device is installed at location } j, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

If we redefine $r(i,j)$ to be the distance between communication node i and jamming location j, then we have the OPTIMAL NETWORK COVERING (ONC) formulation of the WNJP given as

$$\textbf{(ONC)} \quad \text{Minimize} \quad \sum_{j=1}^{n} c_j x_j \tag{1.4}$$

s.t.

$$\sum_{j=1}^{n} d_{ij} x_j \geq C_i, \quad i = 1, 2, \dots, m, \tag{1.5}$$

$$x_j \in \{0, 1\}, \quad j = 1, 2, \dots, n, \tag{1.6}$$

where C_i and d_{ij} are defined as above. Here the objective is to minimize the number of jamming devices used while achieving some minimum level of coverage at each node. The coefficients c_j in (4) represent the costs of installing a jamming device at location j. In a battlefield scenario, placing a jamming device in the direct proximity of a network node may be theoretically possible; however, such a placement might be undesirable due to security considerations. In this case, the location considered would have a higher placement cost than would a safer location. If there are no preferences for device locations, then without the loss of generality,

$$c_j = 1, \quad j = 1, 2, \ldots, n.$$

Though we have removed the non-convex covering constraints, this formulation remains computationally difficult. Notice that ONC is formulated as a MULTIDIMENSIONAL KNAPSACK PROBLEM which is known to be $\mathcal{NP}$-hard in general [10].

1.3 Connectivity Formulation

In the general WNJP, it is important that the distinction be made that the objective is not simply to jam all of the nodes, but to destroy the functionality of the underlying communication network. In this section, we use tools from graph theory to develop a method for suppressing the network by jamming those nodes with several communication links and derive an alternative formulation of the WNJP. Given a graph $G = (V, E)$, the *connectivity index* of a node is

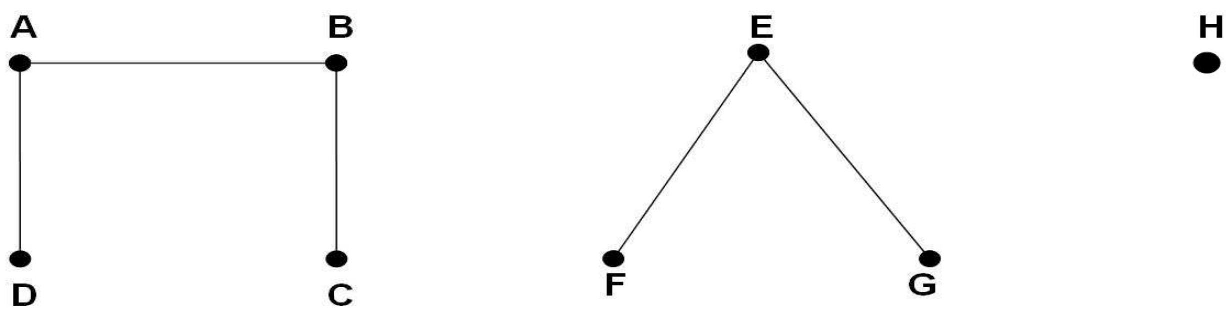


Figure 1.1: Connectivity Index of nodes A,B,C,D is 3. Connectivity Index of E,F,G is 2. Connectivity Index of H is 0.

defined as the number of nodes reachable from that vertex (see Figure 1.1 for examples). To constrain the network connectivity in optimization models, we can impose constraints on the connectivity indices instead of using covering constraints.

We can now develop a formulation for the WNJP based on the connectivity index of the communication graph. We assume that the set of communication nodes $\mathcal{M} = \{1, 2, \ldots, m\}$ to be jammed is known and a set of possible locations $\mathcal{N} = \{1, 2, \ldots, n\}$ for the jamming devices is given. Note than in the communication graph, $V \equiv \mathcal{M}$. Let $S_i := \sum_{j=1}^{n} d_{ij} x_j$ denote the cumulative level of jamming at node i. Then node i is said to be jammed if S_i exceeds some threshold value C_i. We say that communication is severed between nodes i and j if at least one of the nodes is jammed. Further, let $y : \mathcal{M} \times \mathcal{M} \mapsto \{0, 1\}$ be a surjection where $y_{ij} := 1$ if there exists a path from node i to node j in the jammed network. Lastly, let $z : \mathcal{M} \mapsto \{0, 1\}$ be a surjective function where z_i returns 1 if node i is not jammed.

The objective of the CONNECTIVITY INDEX PROBLEM (CIP) formulation of the WNJP is to minimize the total jamming cost subject to a constraint that the connectivity index of each node does not exceed some pre-described level

L. The corresponding optimization problem is given as:

$$
\begin{aligned}
\textbf{(CIP)} \quad \text{Minimize} \quad & \sum_{j=1}^{n} c_j x_j && (1.7)\\
\text{s.t.} \quad & && \\
& \sum_{\substack{j=1 \\ j\neq i}}^{m} y_{ij} \leq L, \; \forall\, i \in \mathcal{M}, && (1.8)\\
& M(1-z_i) > S_i - C_i \geq -Mz_i, \; \forall\, i \in \mathcal{M}, && (1.9)\\
& x_j \in \{0,1\}, \; \forall\, j \in \mathcal{N}, && (1.10)\\
& z_i \in \{0,1\} \,\forall\, i \in \mathcal{M}, && (1.11)\\
& \forall\, i,j \in \mathcal{M}, y_{ij} \in \{0,1\}, \; \forall\, i,j \in \mathcal{M}, && (1.12)
\end{aligned}
$$

where $M \in \mathbb{R}$ is some large constant.

Let $v : \mathcal{M} \times \mathcal{M} \mapsto \{0,1\}$ and $v' : \mathcal{M} \times \mathcal{M} \mapsto \{0,1\}$ be defined as follows:

$$
v_{ij} \quad := \quad \begin{cases} 1, \text{ if } (i,j) \in E, \\ 0, \text{ otherwise,} \end{cases} \tag{1.13}
$$

and

$$
v'_{ij} \quad := \quad \begin{cases} 1, \text{ if } (i,j) \text{ exists in the jammed network,} \\ 0, \text{ otherwise.} \end{cases} \tag{1.14}
$$

With this, we can formulate an equivalent integer program as

$$
\begin{aligned}
\textbf{(CIP-1)} \quad \text{Minimize} \quad & \sum_{j=1}^{n} c_j x_j, && (1.15)\\
\text{s.t.} \quad & && \\
& y_{ij} \geq v'_{ij}, \; \forall\, i,j \in \mathcal{M}, && (1.16)\\
& y_{ij} \geq y_{ik} y_{kj}, \; k \neq i,j; \; \forall\, i,j \in \mathcal{M}, && (1.17)\\
& v'_{ij} \geq v_{ij} z_j z_i, \; i \neq j; \; \forall\, i,j \in \mathcal{M}, && (1.18)\\
& \sum_{\substack{j=1 \\ j\neq i}}^{m} y_{ij} \leq L, \; \forall\, i \in \mathcal{M}, && (1.19)\\
M(1-z_i) \quad > \quad & S_i - C_i \geq -Mz_i, \; \forall\, i \in \mathcal{M}, && (1.20)\\
& z_i \in \{0,1\}, \; \forall\, i \in \mathcal{M}, && (1.21)\\
x_j \in \{0,1\}, \; \forall\, j \in \mathcal{N}, \quad & y_{ij} \in \{0,1\} \,\forall\, i,j \in \mathcal{M}, && (1.22)\\
v_{ij} \in \{0,1\}, \; \forall\, i,j \in \mathcal{M}, \quad & v'_{ij} \in \{0,1\}, \; \forall\, i,j \in \mathcal{M}. && (1.23)
\end{aligned}
$$

Lemma 1. *If* CIP *has an optimal solution then,* CIP-1 *has an optimal solution. Further, any optimal solution x^* of the optimization problem* CIP-1 *is an optimal solution of* CIP.

Proof. It is easy to establish that if i and j are reachable from each other in the jammed network then in CIP-1, $y_{ij} = 1$. Indeed, if i and j are adjacent then there exists a sequence of pairwise adjacent vertices:

$$
\{(i_0, i_1), ..., (i_{m-1}, i_m)\}, \tag{1.24}
$$

where $i_0 = i$, and $i_m = j$. Using induction it can be shown that $y_{i_0 i_k} = 1, \; \forall\, k = 1, 2, \ldots, m$. From (3.12), we have that $y_{i_k i_{k+1}} = 1$. If $y_{i_0 i_k} = 1$, then by (3.13), $y_{i_0 i_{k+1}} \geq y_{i_0 i_k} y_{i_k i_{k+1}} = 1$, which proves the induction step.

The proven property implies that in CIP-1:

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} y_{ij} \geq \text{connectivity index of } i. \tag{1.25}$$

Therefore, if (x^*, y^*) and (x^{**}, y^{**}) are optimal solutions of CIP-1 and CIP correspondingly, then:

$$V(x^*) \geq V(x^{**}), \tag{1.26}$$

where V is the objective in CIP-1 and CIP.
As (x^{**}, y^{**}) is feasible in CIP, it can be easily checked that y^{**} satisfies all feasibility constraints in CIP-1 (it follows from the definition of y_{ij} in CIP). So, (x^{**}, y^{**}) is feasible in CIP-1; thus proving the first statement of the lemma. Hence from CIP-1,

$$V(x^{**}) \geq V(x^*). \tag{1.27}$$

From (1.26) and (1.27):

$$V(x^{**}) = V(x^*). \tag{1.28}$$

Let us define y such that

$$y_{ij} := 1 \Leftrightarrow j \text{ is reachable from } i \text{ in the network jammed by } x^*.$$

Using (1.25), (x^*, y) is feasible in CIP-1, and hence optimal. From the construction of y it follows that (x^*, y) is feasible in CIP. Relying on (1.28) we can claim that x^* is an optimal solution of CIP. The lemma is proved. □

We have therefore established a one-to-one correspondence between formulations CIP and CIP-1. Now, we can linearize the integer program CIP-1 by applying some standard transformations. The resulting linear 0-1 program, CIP-2 is given as

$$\textbf{(CIP-2)} \quad \text{Minimize} \quad \sum_{j=1}^{n} c_j x_j \tag{1.29}$$

s.t.

$$y_{ij} \geq v'_{ij}, \ \forall\, i, j = 1, \ldots, \mathcal{M}, \tag{1.30}$$

$$y_{ij} \geq y_{ik} + y_{kj} - 1, \ k \neq i, j; \ \forall\, i, j \in \mathcal{M}, \tag{1.31}$$

$$v'_{ij} \geq v_{ij} + z_j + z_i - 2, \ i \neq j; \ \forall\, i, j \in \mathcal{M}, \tag{1.32}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} y_{ij} \leq L, \ \forall\, i \in \mathcal{M}, \tag{1.33}$$

$$M(1 - z_i) > S_i - C_i \geq -M z_i, \ \forall\, i \in \mathcal{M}, \tag{1.34}$$

$$z_i \in \{0, 1\}, \ \forall\, i \in \mathcal{M}, \tag{1.35}$$

$$x_j \in \{0, 1\}, \ \forall\, j \in \mathcal{N}, \qquad y_{ij} \in \{0, 1\} \ \forall\, i, j \in \mathcal{M}, \tag{1.36}$$

$$v_{ij} \in \{0, 1\}, \ \forall\, i, j \in \mathcal{M}, \qquad v'_{ij} \in \{0, 1\}, \ \forall\, i, j \in \mathcal{M}. \tag{1.37}$$

In the following lemma, we provide a proof of equivalence between CIP-1 and CIP-2.

Lemma 2. *If* CIP-1 *has an optimal solution then* CIP-2 *has an optimal solution. Furthermore, any optimal solution* x^* *of* CIP-2 *is an optimal solution of* CIP-1.

Proof. For 0-1 variables the following equivalence holds:

$$y_{ij} \geq y_{ik} y_{kj} \Leftrightarrow y_{ij} \geq y_{ik} + y_{kj} - 1$$

The only differences between CIP-1 and CIP-2 are the constraints:

$$v'_{ij} = v_{ij} z_j z_i \tag{1.38}$$
$$v'_{ij} \geq v_{ij} + z_i + z_j - 2 \tag{1.39}$$

Note that (3.40) implies (3.41) ($v_{ij} z_j z_i \geq v_{ij} + z_i + z_j - 2$). Therefore, the feasibility region of CIP-2 includes the feasibility region of CIP-1. This proves the first statement of the lemma.

From the last property we can also deduce that for all $x_1,\ x_2$ such that x_1 is an optimal solution of CIP-1, and x_2 is optimal for CIP-2, that

$$V(x_1) \geq V(x_2), \tag{1.40}$$

where $V(x)$ is the objective of CIP-1 and CIP-2.

Let $(x^*, y^*, v^{'*}, z^*)$ be an optimal solution of CIP-2. Construct $v^{''*}$ using the following rules:

$$v^{''*}_{ij} \quad := \quad \begin{cases} 1, \text{ if } v_{ij} + z^*_i + z^*_j - 2 = 1, \\ 0, \text{ otherwise.} \end{cases} \tag{1.41}$$

$v^{'*}_{ij} \geq v^{''*}_{ij} \Rightarrow (x^*, y^*, v^{''*}, z^*)$ is feasible in CIP-2 ($y_{ij} \geq v^{''*}_{ij}$), hence optimal (the objective value is $V(x^*)$, which is optimal). Using (3.43), $(v^{''*}, z^*)$ satisfies:

$$v^{''*}_{ij} = v_{ij} z^*_j z^*_i.$$

Using this we have that $(x^*, y^*, v^{''*}, z^*)$ is feasible for CIP-1. If x_1 is an optimal solution of CIP-1 then:

$$V(x_1) \leq V(x^*) \tag{1.42}$$

On the other hand, using (3.42):

$$V(x^*) \leq V(x_1). \tag{1.43}$$

(3.45) and (3.46) together imply $V(x_1) = V(x^*)$. The last equality proves that x^* is an optimal solution of CIP-1. Thus, the lemma is proved. □

We have as a result of the above lemmata the following theorem which states that the optimal solution to the linearized integer program CIP-2 is an optimal solution to the original connectivity index problem CIP.

Theorem 1. *If* CIP *has an optimal solution then* CIP-2 *has an optimal solution. Furthermore, any optimal solution of* CIP-2 *is an optimal solution of* CIP.

Proof. The theorem is an immediate corollary of **Lemma 1** and **Lemma 2**. □

1.4 Deterministic Setup with Percentile Constraints

As we have seen, to suppress communication on a wireless network may not necessarily imply that all nodes must be jammed. We might instead choose to constrain the connectivity index of the nodes as in the CIP formulations. Alternatively, it may be sufficient to jam some percentage of the total number of nodes in order to acquire an effective control over the network. The latter can be accomplished by adding *percentile risk constraints* to the mathematical formulation. Used extensively in financial engineering applications and optimization of stochastic systems, risk measures have also proven effective when applied to deterministic problems [18]. In this section, we review two risk measures, namely Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) and provide formulations of the WNJP with the incorporation of these risk measures.

1.4.1 Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR)

The Value-at-Risk (VaR) percentile measure is perhaps the most widely used in all applications of risk management [14]. Stated simply, VaR is an upper percentile of a given loss distribution. In other words, given a specified confidence level α, the corresponding α-VaR is the lowest amount ζ such that, with probability α, the loss is less or equal to ζ [19]. VaR type risk measures are popular for several reasons including their simple definition and ease of implementation.

An alternative risk measure is Conditional Value-at-Risk (CVaR). Developed by Rockafellar and Uryasev, CVaR is a percentile risk measure constructed for estimation and control of risks in stochastic and uncertain environments. However, CVaR-based optimization techniques can also be applied in a deterministic percentile framework. CVaR is defined as the conditional expected loss under the condition that it exceeds VaR [26]. Figure 1.2 provides a graphical representation of the VaR and CVaR concepts. As we will see, CVaR has many properties that offer nice alternatives to VaR.

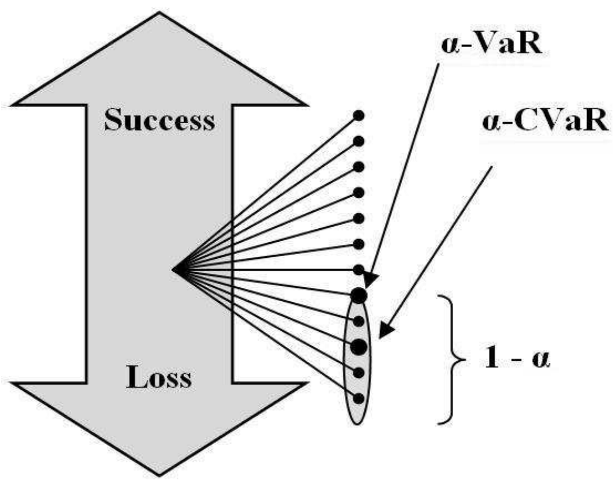


Figure 1.2: Graphical representation of Var and CVaR.

Let $f(x, y)$ be a performance or loss function associated with the decision vector $x \subseteq X \subseteq \mathbb{R}^n$, and a random vector in $y \in \mathbb{R}^m$. The y vector can be interpreted as the uncertainties that may affect the loss. Then, for each $x \in X$, the corresponding loss $f(x, y)$ is a random variable having a distribution in $\mathbb{R}$ which is induced by y. We assume that y is governed by a probability measure P on a Borel set, say Y. Therefore, the probability of $f(x, y)$ not exceeding some threshold value ζ is given by

$$\psi(x, \zeta) := P\{y | f(x, y) \leq \zeta\}. \tag{1.44}$$

For a fixed decision vector x, $\psi(x, \zeta)$ is the cumulative distribution function of the loss associated with x. This function is fundamental for defining VaR and CVaR [19].

With this, the α-VaR and α-CVaR values for the loss random variable $f(x, y)$ for any specified $\alpha \in (0, 1)$ are denoted by $\zeta_\alpha(x)$ and $\phi_\alpha(x)$ respectively. From the aforementioned definitions, they are given by

$$\zeta_\alpha(x) := \min\{\zeta \in \mathbb{R} : \psi(x, \zeta) \geq \alpha\}, \tag{1.45}$$

and

$$\phi_\alpha(x) := E\{f(x, y) | f(x, y) \geq \zeta_\alpha(x)\}. \tag{1.46}$$

Notice that the probability that $f(x, y) \geq \zeta_\alpha(x)$ is equal to $1 - \alpha$. Finally by definition, we have that $\phi_\alpha(x)$ is the conditional expectation that the loss corresponding to x is greater than or equal to $\zeta_a(x)$ [24].

The key to including VaR and CVaR constraints into a model are the characterizations of $\zeta_\alpha(x)$ and $\phi_\alpha(x)$ in terms of a function $F_\alpha : X \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$F_\alpha(x, \zeta) := \zeta + \frac{1}{(1-\alpha)} E\{\max\{f(x,y) - \zeta, 0\}\}. \tag{1.47}$$

The following theorem, which provides the crucial properties of the function F_α follow directly from the chapter by [24].

Theorem 2. *As a function of ζ, $F_\alpha(x,\zeta)$ is convex and continuously differentiable. The α-CVaR of the loss associated with any $x \in X$ can be determined from the formula*

$$\phi_\alpha(x) = \min_{\zeta \in \mathbb{R}} F_\alpha(x, \zeta). \tag{1.48}$$

In this formula, the set consisting of the values of ζ for with the minimum is attained, namely

$$A_\alpha(x) = \operatorname*{argmin}_{\zeta \in \mathbb{R}} F_\alpha(x, \zeta), \tag{1.49}$$

is a nonempty, closed, bounded interval, and the α-VaR of the loss is given by

$$\zeta_\alpha(x) = \textit{ left endpoint of } A_\alpha(x). \tag{1.50}$$

In particular, it is always the case that

$$\zeta_\alpha(x) \in \operatorname*{argmin}_{\zeta \in \mathbb{R}} F_\alpha(x, \zeta) \quad \textit{and} \quad \psi_\alpha(x) = F_\alpha(x, \zeta_\alpha(x)). \tag{1.51}$$

This result provides an efficient linear optimization algorithm for CVaR. However, from a numerical perspective, the convexity of $F_\alpha(x,\zeta)$ with respect to x and ζ as provided by Theorem 2 is more valuable than the convexity of $\phi_\alpha(x)$ with respect to x. As we will see in the following theorem due to [25], this allows us to minimize CVaR without having to proceed numerically through repeated calculations of $\phi_\alpha(x)$ for various decisions x.

Theorem 3. *Minimizing $\phi_\alpha(x)$ with respect to $x \in X$ is equivalent to minimizing $F_\alpha(x,\zeta)$ over all $(x,\zeta) \in X \times \mathbb{R}$, in the sense that*

$$\min_{x \in X} \phi_\alpha(x) \quad = \quad \min_{(x,\zeta) \in X \times \mathbb{R}} F_\alpha(x, \zeta), \tag{1.52}$$

where moreover

$$(x^*, \zeta^*) \in \operatorname*{argmin}_{(x,\zeta) \in X \times \mathbb{R}} F_\alpha(x,\zeta) \Leftrightarrow x^* \in \operatorname*{argmin}_{x \in X} \phi_\alpha(x),\ \zeta^* \in \operatorname*{argmin}_{\zeta \in \mathbb{R}} F_\alpha(x^*, \zeta). \tag{1.53}$$

In the deterministic setting of the WNJP, we are not particularly interested in minimizing VaR or CVaR as it pertains to the loss. Rather, we would like to impose percentile constraints on the optimization model in order to handle a desired probability threshold. The following theorem from [25] provides this capability.

Theorem 4. *For any selection of probability thresholds α_i and loss tolerances $\omega_i, i = 1, \ldots, m$, the problem*

$$\min_{x \in X} \quad g(x) \tag{1.54}$$

s.t.

$$\phi_{\alpha_i}(x) \le \omega_i, \textit{ for } i = 1, \ldots, m, \tag{1.55}$$

where g is any objective function defined on X, is equivalent to the problem

$$\min_{(x,\zeta_1,\ldots,\zeta_m) \in X \times \mathbb{R}^m} \quad g(x) \tag{1.56}$$

s.t.

$$F_{\alpha_i}(x, \zeta_i) \le \omega_i, \textit{ for } i = 1, \ldots, m. \tag{1.57}$$

Indeed, $(x^*, \zeta_1^*, \ldots, \zeta_m^*)$ solves the second problem if and only if x^* solves the first problem and the inequality $F_{\alpha_i}(x, \zeta_i) \le \omega_i$ holds for $i = 1, \ldots, m$.

Furthermore, $\phi_{\alpha_i}(x^*) \le \omega_i$ holds for all $i = 1, \ldots, m$. In particular, for each i such that $F_{\alpha_i}(x^*, \zeta^*) = \omega_i$, one has that $\phi_{\alpha_i}(x^*) = \omega_i$.

1.4.2 Percentile Constraints and the WNJP

In this section, we investigate the use of VaR and CVaR constraints when applied to the formulations of the WNJP derived in Sections 3.2 and 3.4 above. As we have seen, risk measures are generally designed for optimization under uncertainty. Since we are considering deterministic formulations of the WNJP, we can interpret each communication node $i \in \mathcal{M}$ as a random scenario, and apply the desired risk measures in this context.

We begin with the OPTIMAL NETWORK COVERING formulation of the WNJP. Suppose it is determined that jamming some fraction $\alpha \in (0, 1)$ of the nodes is sufficient for effectively dismantling the network. This can be accomplished by the inclusion of α-VaR constraints in the original model. Let $y : \mathcal{M} \mapsto \{0, 1\}$ be a surjection defined by

$$y_i \quad := \quad \begin{cases} 1, & \text{if node } i \text{ is jammed,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.58}$$

Recall from Section 3.2 that $\mathcal{N} = \{1, \ldots, n\}$ is the set of locations for the jamming devices, and x is a binary vector of length n where $x_j = 1$ if a jamming device is placed at location j. Then to find the minimum number of jamming devices that will allow for covering $\alpha \cdot 100\%$ of the network nodes with prescribed levels of jamming C_i, we must solve the following integer program

$$\textbf{(ONC-VaR)} \quad \text{Minimize} \quad \sum_{j=1}^{n} c_j x_j \tag{1.59}$$

s.t.

$$\sum_{i=1}^{m} y_i \geq \alpha m, \tag{1.60}$$

$$\sum_{j=1}^{n} d_{ij} x_j \geq C_i y_i, \quad i = 1, 2, \ldots, m, \tag{1.61}$$

$$x_j \in \{0, 1\}, \quad j = 1, 2, \ldots, n, \tag{1.62}$$

$$y_i \in \{0, 1\}, \quad i = 1, 2, \ldots, m. \tag{1.63}$$

Notice that this formulation differs from the ONC formulation with the addition of the α-VaR constraint (1.60). According to (1.61), if $y_i = 1$ then node i is jammed. Lastly, we have from (1.60) that at least $100 \cdot \alpha\%$ of the y variables are equal to 1.

The optimal solution to the ONC-VaR formulation will provide the minimum number of jamming devices required to suppress communication on at least $\alpha \cdot 100\%$ of the network nodes. The resulting solution may provide coverage levels comparable to those provided by the ONC model, while potentially reducing the number of jamming devices used. However, notice that the remaining $(1 - \alpha) \cdot 100\%$ of the nodes for which y_i is potentially 0, there is no guarantee that they will receive any amount of coverage. Furthermore, the addition of the m binary variables adds a computational burden to a problem which is already $\mathcal{NP}$-hard.

We can also reformulate the CONNECTIVITY INDEX PROBLEM to include Value-at-Risk constraints. Let $\rho : \mathcal{M} \mapsto \mathbb{Z}^+$ be a surjection where ρ_i returns the connectivity index of node i. That is, $\rho_i := \sum_{j=1, j \neq i}^{m} y_{ij}$. Further let $w : \mathcal{M} \mapsto \{0, 1\}$ be a decision variable having the property that if $w_i = 1$, then $\rho_i \leq L$. With this, the connectivity index formulation of WNJP with VaR percentile constraints is given as

$$\textbf{(CIP-VaR)} \quad \text{Minimize} \quad \sum_{j=1}^{n} c_j x_j \tag{1.64}$$

s.t.

$$\rho_i \le L w_i + (1 - w_i) M, \; i = 1, 2, \dots, m, \tag{1.65}$$

$$\sum_{i=1}^{m} w_i \ge \alpha m, \tag{1.66}$$

$$x_j \in \{0, 1\}, \; j = 1, 2, \dots, n, \tag{1.67}$$

$$w_i \in \{0, 1\}, \; i = 1, 2, \dots, m, \tag{1.68}$$

$$\rho_i \in \{0, 1\}, \; i = 1, 2, \dots, m, \tag{1.69}$$

where $M \in \mathbb{R}$ is some large constant.

Analogous to constraints (1.60)-(1.61), constraints (1.65)-(1.66) guarantee that at least $\alpha \cdot 100\%$ of the nodes will have connectivity index less than L. As with the ONC-VaR formulation, there are two drawbacks of CIP-VaR. First, there is no control guarantee at all on any of the remaining $(1 - \alpha) \cdot 100\%$ nodes for which $w_i = 0$. Secondly, the addition of m binary variables adds a tremendous computational burden to the problem. As an alternative to VaR, we now examine formulations of the WNJP using Conditional Value-at-Risk constraints [24].

We first consider the OPTIMAL NETWORK COVERING problem. In order to put this into our derived framework, we need to define the loss function associated with an instance of the ONC. We introduce the function $f : \{0, 1\}^n \times \mathcal{M} \mapsto \mathbb{R}$ defined by

$$f(x, i) := C_i - \sum_{j=1}^{n} x_j d_{ij}. \tag{1.70}$$

That is, given a decision vector x representing the placement of the jamming devices, the loss function is defined as the difference between the energy required to jam the network node i and the cumulative amount of energy received at node i due to x. With this, we can formulate the ONC with the addition of CVaR constraints as the following integer linear program:

$$\textbf{(ONC-CVaR)} \quad \text{Minimize} \sum_{j=1}^{n} c_j x_j \tag{1.71}$$

s.t.

$$\zeta + \frac{1}{(1 - \alpha) m} \sum_{i=1}^{m} \max \left\{ C_{\min} - \sum_{j=1}^{n} x_j d_{ij} - \zeta, \; 0 \right\} \le 0, \tag{1.72}$$

$$\zeta \in \mathbb{R}, \tag{1.73}$$

$$x_j \in \{0, 1\}, \tag{1.74}$$

where $C_{\min}$ is the minimal prescribed jamming level and d_{ij} is defined as above. The expression on the left hand side of (3.70) is $F_\alpha(x, \zeta)$. Further, from Theorem 4 we see that constraint (3.70) corresponds to having $\phi_\alpha(x) \le \omega = 0$ [25]. Said differently, the CVaR constraint (3.70) implies that in the $(1 - \alpha) \cdot 100\%$ of the worst (least) covered nodes, the average value of $f(x) \le 0$. For the case when $C_i \equiv C$ for all i, it follows that the average level of jamming energy received by the worst $(1 - \alpha) \cdot 100\%$ of nodes exceeds C.

The important point about this formulation is that we have not introduced additional integer variables to the problem in order to add the percentile constraints. Recall, that in ONC-VaR we introduced m discrete variables. Since we have to add only m real variables to replace max-expressions under the summation and a real variable ζ, this formulation is much easier to solve than ONC-VaR.

In a similar manner, we can formulate the CONNECTIVITY INDEX PROBLEM with the addition of CVaR constraints. As before, we need to first define an appropriate loss function. Recall that the definition of ρ_i, the connectivity index of node i, is given as the number of nodes reachable from i. Then can define the loss function f' for a network node i

as the difference between the connectivity index of i and the maximum allowable connectivity index L which occurs as a result of the placement of the jamming devices according to x. That is, let $f' : \{0,1\}^n \times \mathcal{M} \mapsto \mathbb{Z}$ be defined by

$$f'(x,i) := \rho_i - L. \tag{1.75}$$

With this, the CIP-CVaR formulation is given as follows.

$$\textbf{(CIP-CVaR)} \quad \text{Minimize} \sum_{j=1}^{n} c_j x_j \tag{1.76}$$

s.t.

$$\zeta + \frac{1}{(1-\alpha)m} \sum_{i=1}^{m} \max\{\rho_i - L - \zeta,\ 0\} \leq 0, \tag{1.77}$$

$$\rho_i \in \mathbb{Z}, \tag{1.78}$$

$$\zeta \in \mathbb{R}, \tag{1.79}$$

where ρ_i is defined as above. As with the previous formulation, the expression on the left-hand side of (1.77) is $F_\alpha(x,\zeta)$ from (3.53). Furthermore, we have from from Theorem 4 that (1.77) corresponds to having $\phi_\alpha(x) \leq \omega = 0$. This constraint on CVaR provides that for the $(1-\alpha)\cdot 100\%$ of the worst cases, the average connectivity index will not exceed L. Again, we see that in order to include the CVaR constraint, we only need to add $(m+1)$ real variables to the problem. Computationally, CVaR provides a more conservative solution and will be much easier to solve than the CIP-VaR formulation as we will see in the next section.

1.5 Case Studies

In order to demonstrate the advantages and disadvantages of the proposed formulations for the WNJP, we will present two case studies. The experiments were performed on a PC equipped with a 1.4MHz Intel Pentium® 4 processor with 1GB of RAM, working under the Microsoft Windows® XP SP1 operating system. In the first study, an example network is given and the problem is modeled using the proposed coverage formulation. The problem is then solved exactly using the commercial integer programming software package, CPLEX®. Next, we modify the problem to include VaR and CVaR constraints and again use CPLEX® to solve the resulting problems. Numerical results are presented and the three formulations are compared. In the second case study, we model and solve the problem using the connectivity index formulation. We then include percentile constraints re-optimize. Finally, we analyze the results.

1.5.1 Coverage Formulation

Optimal Solutions	Regular Constraints	VaR Constraints
Number of Jammers	6	4
Level of Jamming	100% ∀ nodes	100% for 96% of nodes, 85% (of reqd.) for 4% of nodes
CPLEX® Time	0.81 sec	0.98 sec

Table 1.1: Optimal solutions using the coverage formulation with regular and VaR constraints.

Here we present two networks and solve the WNJP using the network covering (ONC) formulation. The first network has 100 communication nodes and the number of available jamming devices is 36. The cost of placing a jamming device at location j, c_j is equal to 1 for all locations. This problem was solved using the regular constraints and the VaR type constraints. Recall that there is a set of possible locations at which jamming devices can be placed. In these examples, this set of points constitutes a uniform grid over the battlespace. The placement of the jamming devices from each solution can be seen in Figure 1.3. The numerical results detailing the level of jamming for the network nodes is given in Table 1.1. Notice that the VaR solution called for 33% less jamming devices than the original problem while providing almost the same jamming quality.

Opt Solns	Reg (all)	VaR (.9 conf)	CVaR (.7 conf)
# Jammers	9	8	7
Jamming Level	100% ∀ nodes	100% for 90% of nodes, 72% for 10% of nodes	100% for 57% of nodes, 90% for 20% of nodes, 76% for 23% of nodes
CPLEX® Time	15 sec	15h 55min 11sec	41 sec

Table 1.2: Optimal solutions using the coverage formulation with regular and VaR, and CVaR constraints.

In the second example, the network has 100 communication nodes and 72 available jammers. This problem was solved using the regular constraints as well as both types of percentile constraints. The resulting graph is shown in Figure 1.4. The corresponding numerical results are given in Table 1.2.

In this example, the VaR formulation requires 11% less jamming devices with almost the same quality as the formulation with the standard constraints. However, this formulation requires nearly 16 hours of computation time. The CVaR formulation gives a solution with a very good jamming quality and requires 22% less jamming devices than the standard formulation and 11% less devices than the VaR formulation. Furthermore, the CVaR formulation requires an order of magnitude less computing time than the formulation with VaR constraints.

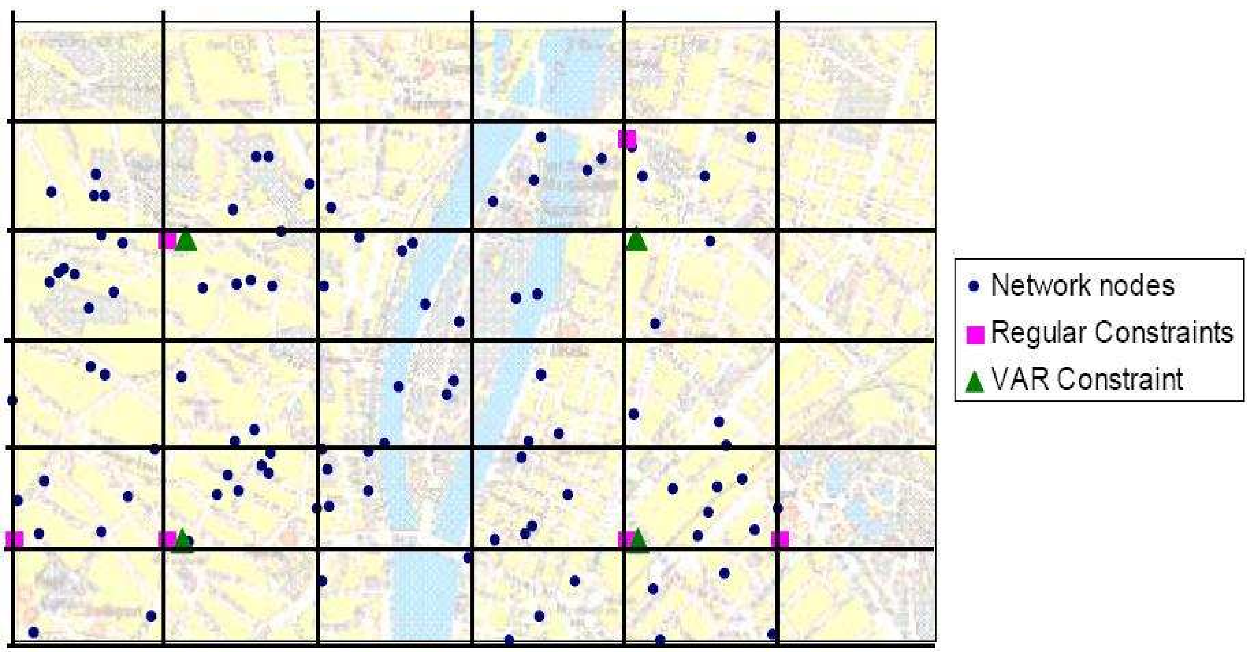


Figure 1.3: Case study 1. The placement of jammers is shown when the problem is solved using the original and VaR constraints.

1.5.2 Connectivity Formulation

We now present a case study where the WNJP was solved using the connectivity index formulation (CIP). The communication graph consists of 30 nodes and 60 edges. The maximal number of jamming devices available is 36. We set the maximal allowed connectivity index of any node to be 3. In Figure 1.5 we can see the original graph with the communication links prior to jamming. The result of the VaR and CVaR solutions is seen in Figure 1.6. The confidence level for both the VaR and CVaR formulations was 0.9. Both formulations provide optimal solutions for the given instance. The resulting computation time for the VaR formulation was 15 minutes 34 seconds, while the CVaR formulation required only 7 minutes 33 seconds.

1.6 Extensions and Conclusions

[6]

In this chapter we introduced the deterministic WIRELESS NETWORK JAMMING PROBLEM and provided several formulations using node covering constraints as well as constraints on the connectivity indices of the network nodes. We also incorporated percentile constraints into the derived formulations. Further, we provided two case studies comparing the two formulations with and without the risk constraints.

With the introduction of this problem, we also recognize that several extensions can be made. For example, all of the formulations presented in this chapter assume that the network topology of the enemy network is known. It is reasonable to assume that this is not always the case. In fact, there may be little or no a priori information about the network to be jammed. In this case, stochastic formulations should be considered and analyzed.

A generalization of the node coverage formulation including uncertainties in the number of communication nodes and their coordinates might be considered. For the connectivity index problem, there might exist uncertainties in the number of network nodes, their locations, and the probability that a node will recover a jammed link. Also, efficient heuristics such as Greedy Randomized Adaptive Search Procedure (GRASP) [22], Genetic Algorithms [12], and Tabu

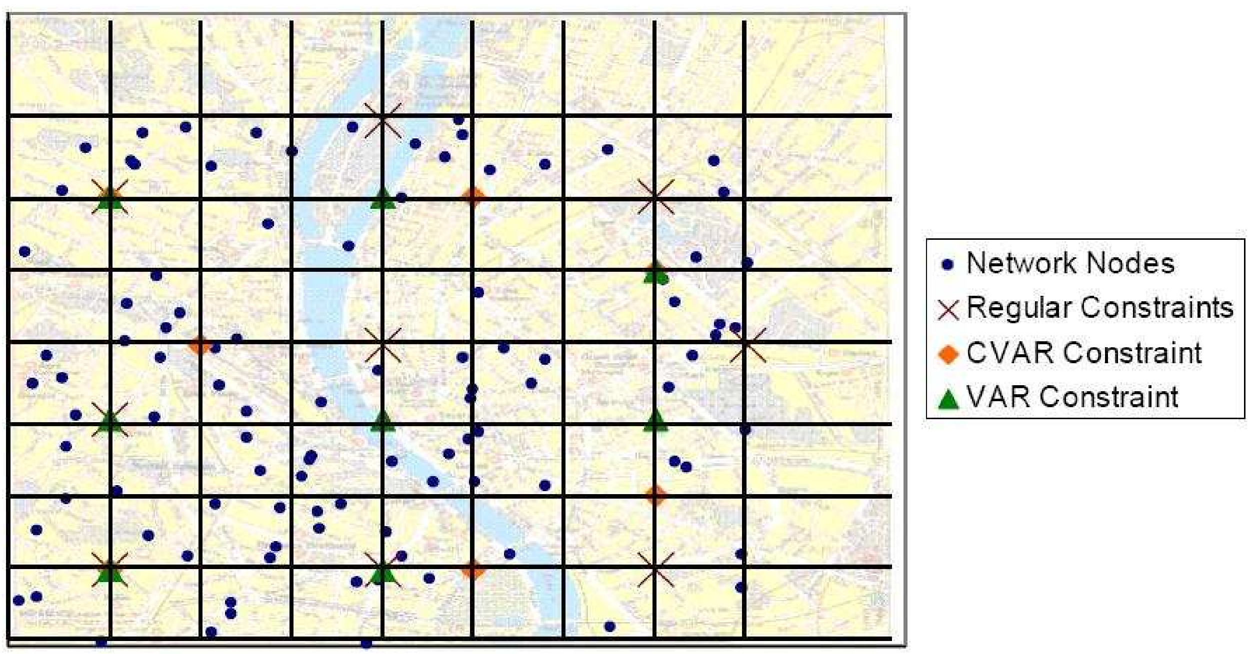


Figure 1.4: Case study 1 continued. The placement of jammers is shown when the problem is solved using VaR and CVaR constraints.

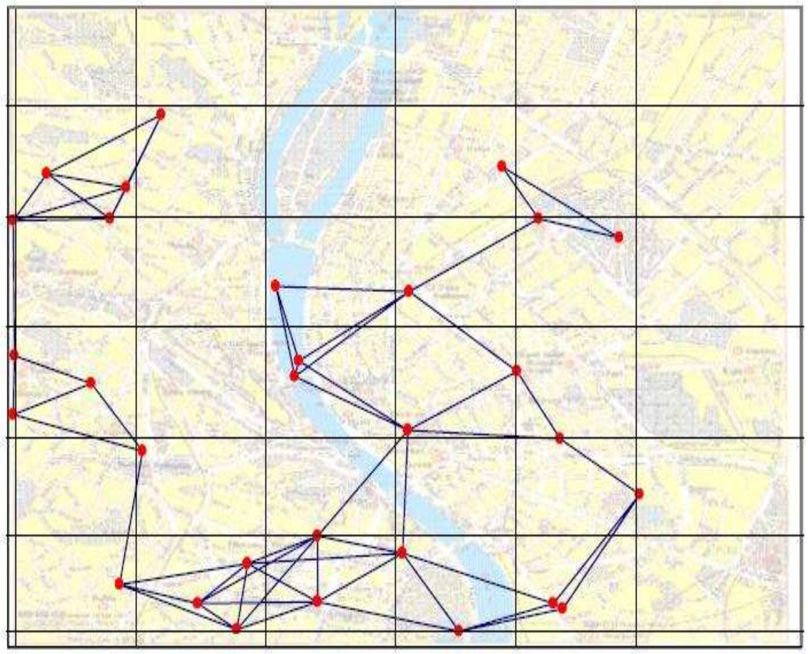
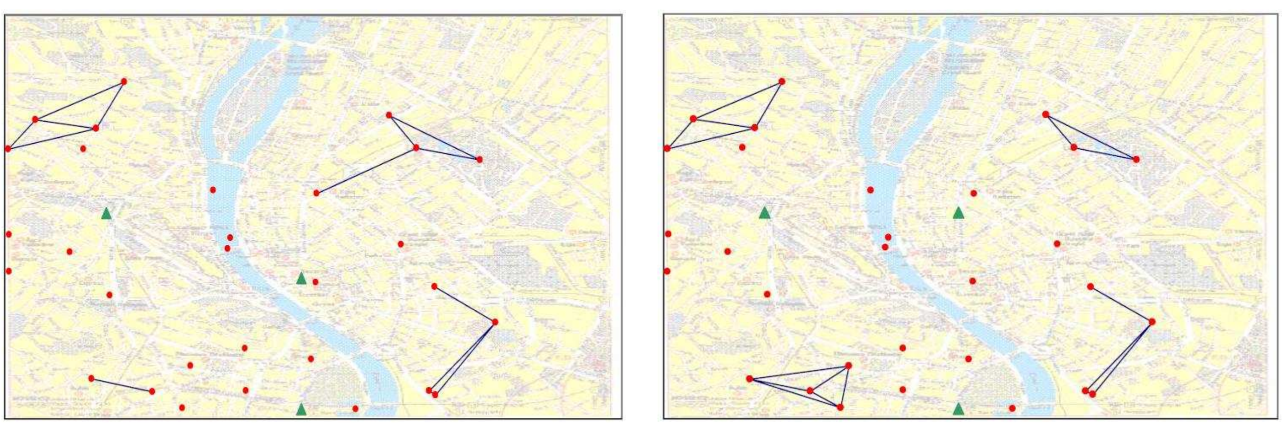


Figure 1.5: Case Study 2: Original graph.

Search [11], should be designed so that larger real-world instances can be solved. These are only a few ideas and extensions that can be derived from this new and interesting combinatorial optimization problem.



(a)

(b)

Figure 1.6: (a) VaR Solution. (b) CVaR Solution. In both cases, the triangles represent the jammer locations.

Chapter 2

Jamming communication networks under complete uncertainty

This chapter presents the results published in [6] Commander, C. , Pardalos, P. , Ryabchenko, V. , Shylo, O. , Uryasev, S. and G. Zrazhevsky. Jamming communication networks under complete uncertainty. Optimization Letters, 2:1, pp. 53-70, 2007.

2.1 Introduction

This chapter describes a problem of interdicting/jamming communication networks in uncertain environments. Jamming communication networks is an important problem but has not been intensively researched despite the vast amount of work on optimizing telecommunication systems [23]. Most papers on network interdiction are about preventing jamming and analyzing network vulnerability [20, 9]. To our knowledge, the only literature on network interdiction involving optimal placement of jamming devices is the work of Commander et al. [7] in which several mathematical programming formulations were given for the deterministic WIRELESS NETWORK JAMMING PROBLEM. The only other thoroughly studied cases are problems of minimizing the maximal network flow and maximizing the shortest path between given nodes via arc interdiction using limited resources. Wood [27], Israeli et al. [15], and Cormican et al. [8] studied stochastic and deterministic cases and suggested efficient heuristics. A similar setup but with a different objective was recently studied by Held in 2005 [13].

Since most situations arise in military battlefield scenarios, exact information about the topology of the adversary's network is unknown. Thus, deterministic network interdiction approaches have limited applicability. In this case, a stochastic approach involving some risk measure for evaluating the efficiency of the jamming device placement may be helpful. However, choosing an appropriate risk measure is a challenging problem in its own right. In this chapter, we consider an extreme case where there is no a priori information about the topology of the network to be jammed. The only information used in our approach is a bounding area, containing the communication network.

The organization of the chapter is as follows. Section 2 gives a formal description of the problem and the jamming model. We derive bounds and prove a convergence result for the case of complete uncertainty in Section 3. Here we also demonstrate the advantage of the proposed method compared to the simplified case which does not account for the cumulative effect of the jamming devices. Section 4 provides some concluding remarks.

2.2 Descriptions, Assumptions, and Definitions

In general, the problem of jamming a communication network is to determine the minimum number of jamming devices required to interdict or suppress functionality of the network. Starting with this general statement, more specific ones can be obtained by considering various types of jamming devices and interdiction criteria. Depending on the given information about the communication nodes and the network topology, stochastic or deterministic setups can be constructed [7]. Below we provide assumptions and basic definitions of the considered framework.

We consider radio-transmitting communication networks and jamming devices operating with electromagnetic waves. We assume that the jamming devices have omnidirectional antennas and emit electromagnetic waves in all directions with the same intensity. We also assume that jamming power decreases reciprocally to the squared distance from a device.

Definition 1. *A point (communication node) X is said to be jammed or covered if the cumulative energy received from all jamming devices exceeds some threshold value E:*

$$\sum_i \frac{\lambda}{\mathcal{R}^2(X,i)} \geq E, \tag{2.1}$$

where $\lambda \in \mathbb{R}$ and $\mathcal{R}(X,i)$ represents the distance from X to jamming device i. This condition can be rewritten as:

$$\sum_i \frac{1}{\mathcal{R}^2(X,i)} \geq \frac{1}{L^2}, \tag{2.2}$$

where $L = \sqrt{\frac{\lambda}{E}}$.

The latter inequality implies that a jamming device covers any point inside a circle of radius L.

Definition 2. *A connection (arc) between two communication nodes is considered blocked if any of the two nodes is covered.*

Usually, interdiction efficiency is determined by a fraction of covered nodes and/or arcs. More complicated criteria used are based on the amount of information transmitted through the network or the length of the shortest path between pairs of nodes. We do not consider a specific criterium because we are interested in the case of complete uncertainty. Thus, we are assuming that we have no knowledge of the network topology, including information about the node coordinates.

2.3 Jamming Under Complete Uncertainty

If we ignore the cumulative effect of the jamming devices, then the problem reduces to determining the optimal covering of an area on a plane by circles. This covering problem was solved in 1936 by Kershner [16]. The current capter shows that accounting for the cumulative effect of all the devices can lead to significant losses in costs, i.e. required number of jamming devices.

Since we assume no information is known about the network to be jammed, the only reasonable approach is to cover all points in some area known to contain the network. This approach would also be appropriate when some information about the network is available, but is potentially inaccurate.

We consider a case when a communication network is located inside a square. However, all of the following theorems can be formulated for a more general case. For example, to obtain results when the network is contained inside a rectangular region in the plane, the only modification required to the calculations is an appropriate updating of the summation bounds.

An optimal covering is one which contains the minimum number of jamming devices that jam all points in the particular area of interest. However, finding a globally optimal solution for the general problem is difficult [7]. Therefore, we consider a subproblem of covering a square with jamming devices located at the nodes of a uniform grid. The solution to this problem will provide a feasible solution (optimal in certain cases) to the general problem. Suppose the grid step size is R. If the length of a square side a is not a multiple of R, then we cover a bigger square with a side of length $R([\frac{a}{R}]+1)$. See Figure 2.1 for an example. The optimal solution in the considered problem is a uniform grid with the largest possible step size which covers the square. The problem remains non-trivial, even for this simplified setup.

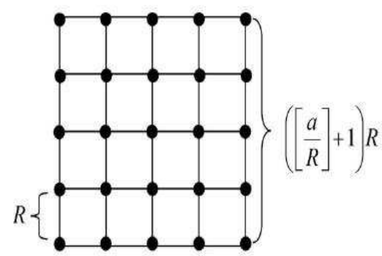


Figure 2.1: Uniform grid with jamming devices

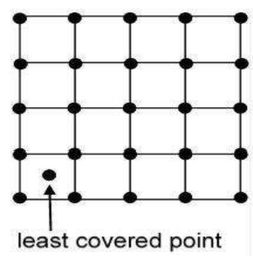


Figure 2.2: The least covered point is shown in the lower left grid cell.

Lemma 1. *For any covering of a square with a uniform grid, a point which receives the least amount of jamming energy lies inside a corner grid cell (see Figure 2.2).*

Proof. Consider a corner cell S_0 and an arbitrary non-corner cell S_i. We prove that for any point $P \in S_i$, there is a corresponding point $P' \in S_0$ such that $E(P) > E(P')$, where $E(X)$ is the cumulative jamming energy from all devices received at point X.

Let P' be a symmetric correspondence of point P inside S_0. Here, symmetry implies that P and P' are equidistant from the sides of their respective cells. We split the square into the four rectangles A, B, C, and D, where A is the

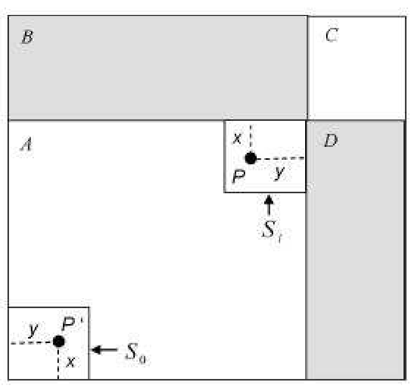


Figure 2.3: Square Decomposition

rectangle containing cells S_0 and S_i (see Figure 2.3). Denote the other two corner cells of rectangle A by C_1 and C_2. Let also T_1 and T_2 be points inside C_1 and C_2 respectively, such that T_1PT_2P' is a rectangle with sides parallel to the

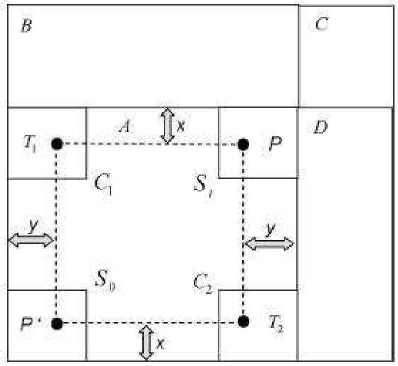


Figure 2.4: Equivalent Points

sides of the square as in Figure 2.4. Using symmetry we get the following relations:

$$E(P', A) = E(P, A), \quad (2.3)$$
$$E(P', B) < E(T_1, B) = E(P, B), \quad (2.4)$$
$$E(P', D) < E(T_2, D) = E(P, D), \quad (2.5)$$
$$E(P', C) < E(P, C), \quad (2.6)$$

where $E(X, I)$ is the cumulative jamming energy from all devices inside rectangle I received by point X. Relations (2.3) - (2.6) imply

$$\begin{aligned} E(P') &= E(P', A) + E(P', B) + E(P', C) + E(P', D) \\ &< E(P, A) + E(P, B) + E(P, C) + E(P, D) \\ &= E(P), \end{aligned} \quad (2.7)$$

and the lemma is proved. □

Below we formulate theorems for upper $\overline{R}$ and lower $\underline{R}$ bounds for the optimal grid step size $R^* : \underline{R} < R^* < \overline{R}$. In all formulated theorems, we consider covering a square with side length a.

Theorem 1. *The unique solution of the equation*

$$\frac{1}{2R^2}\left(\pi \ln(\frac{a}{R} + 1) + \pi - 3\right) = \frac{1}{L^2} \quad (2.8)$$

is a lower bound $\underline{R}$ for the optimal grid step size R^*.

Proof. In Lemma 1, we proved that the least covered point lies inside a corner cell. Consider now a grid with step size R. Without the loss of generality, let $P(x_0, y_0)$ be a point inside the bottom left corner cell as shown in Figure 2.5. I_1, I_2, and I_3 are cumulative jamming energy received at P by jamming devices located in regions C, A, and B correspondingly. Similarly, I_4 is the jamming energy from the jamming device located at the bottom left node O. With

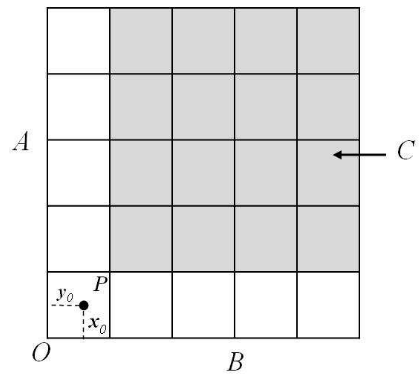


Figure 2.5: Cumulative emanation of jamming devices.

this, the jamming energy received at point P is calculated through the expression

$$E(P) = I_1 + I_2 + I_3 + I_4, \text{ where} \tag{2.9}$$

$$I_1 = \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} \frac{1}{(R - x_0 + i \cdot R)^2 + (R - y_0 + j \cdot R)^2}, \tag{2.10}$$

$$I_2 = \sum_{i=0}^{T-1} \frac{1}{(R - x_0 + i \cdot R)^2 + y_0^2}, \tag{2.11}$$

$$I_3 = \sum_{j=0}^{T-1} \frac{1}{x_0^2 + (R - y_0 + j \cdot R)^2}, \tag{2.12}$$

$$I_4 = \frac{1}{x_0^2 + y_0^2}, \tag{2.13}$$

$$T = \left[\frac{a}{R}\right] + 1. \tag{2.14}$$

Notice that we can estimate $I_2 + I_3$ as

$$I_2 + I_3 \geq 2 \cdot \sum_{i=0}^{T-1} \frac{1}{R^2(1+i)^2 + R^2} \geq \frac{2}{R^2} \int_0^T \frac{1}{1 + (1+x)^2} dx. \tag{2.15}$$

This follows from the fact that

$$\sum_{i=0}^{N} f(i) \geq \int_0^{N+1} f(x) dx, \tag{2.16}$$

where $f(x)$ is a decreasing function. This property can be easily established geometrically. Notice in Figure 2.6 that the left side of inequality (2.16) represents the shaded region in the figure, while the right side represents the area under $f(x)$. Continuing from (2.15) above we have

$$\begin{aligned} \int_0^T \frac{1}{1 + (1+x)^2} dx &= \arctan(T+1) - \frac{\pi}{4} \\ &= \frac{\pi}{2} - \arctan\left(\frac{1}{T+1}\right) - \frac{\pi}{4} \\ &\geq \frac{\pi}{4} - \frac{1}{T+1}. \end{aligned} \tag{2.17}$$

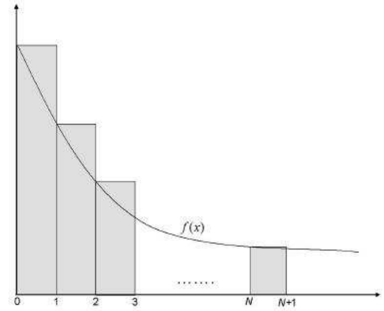


Figure 2.6: Integral Lower Bound.

Here and further, we use the inequalities given below:

$$\arctan(x) \leq x,\ 0 \leq x \leq 1, \tag{2.18}$$

$$\arctan(x) \geq x - \frac{x^3}{3}, 0 \leq x \leq 1. \tag{2.19}$$

Now combining (2.15) and (2.17), we obtain

$$I_2 + I_3 \geq \frac{2}{R^2}\left(\frac{\pi}{4} - \frac{1}{T+1}\right). \tag{2.20}$$

We also have the following approximation for I_4 which follows clearly

$$I_4 \geq \frac{1}{2R^2}. \tag{2.21}$$

For estimating I_1 we use a property similar to (2.16), but in a higher dimension. Namely,

$$\sum_{i=0}^{N}\sum_{j=0}^{N} f(i,j) \geq \int_0^{N+1}\int_0^{N+1} f(x,y)dxdy, \tag{2.22}$$

where as above, $f(x, y)$ is a decreasing function of x and y. Using this inequality, we derive the following approximation for I_1.

$$\begin{aligned} I_1 &\geq \int_0^T\int_0^T \frac{dxdy}{(R - x_0 + x\cdot R)^2 + (R - y_0 + y\cdot R)^2} \\ &\geq \int_0^T\int_0^T \frac{dxdy}{(R + x\cdot R)^2 + (R + y\cdot R)^2} \\ &= \frac{1}{R^2}\int_1^{T+1}\int_1^{T+1} \frac{dxdy}{x^2+y^2}. \end{aligned} \tag{2.23}$$

Furthermore,

$$\begin{aligned}
\int_1^{T+1}\int_1^{T+1}\frac{dxdy}{x^2+y^2} &= \int_1^{T+1}\frac{1}{x}\arctan\left(\frac{T+1}{x}\right)dx-\int_1^{T+1}\frac{1}{x}\arctan\left(\frac{1}{x}\right)dx\\
&\geq \int_1^{T+1}\frac{1}{x}\arctan\left(\frac{T+1}{x}\right)dx-\int_1^{T+1}\frac{dx}{x^2}\\
&= \int_1^{T+1}\frac{1}{x}\left(\frac{\pi}{x}-\arctan\left(\frac{x}{T+1}\right)\right)dx-1+\frac{1}{T+1} \qquad (2.24)\\
&= \frac{\pi}{2}\ln(T+1)-1+\frac{1}{T+1}-\int_0^{T+1}\frac{1}{x}\arctan\left(\frac{x}{T+1}\right)dx\\
&\geq \frac{\pi}{2}\ln(T+1)-1+\frac{1}{T+1}-\int_0^{T+1}\frac{1}{x}\left(\frac{x}{T+1}\right)dx\\
&= \frac{\pi}{2}\ln(T+1)-2\left(1-\frac{1}{T+1}\right).
\end{aligned}$$

Combining this result with (2.23) we have

$$I_1 \geq \frac{1}{R^2}\left(\frac{\pi}{2}\ln(T+1)-2\left(1-\frac{1}{T+1}\right)\right). \qquad (2.25)$$

Summing (2.20), (2.21), and (2.25) we obtain an overestimate of the total coverage at point P. That is

$$\begin{aligned}
E(P) &\geq \frac{1}{R^2}\cdot\left(\frac{\pi}{2}\ln(T+1)-2+\frac{2}{T+1}+\frac{\pi}{2}-\frac{2}{T+1}+\frac{1}{2}\right)\\
&= \frac{1}{R^2}\left(\frac{\pi}{2}\ln(T+1)+\frac{\pi}{2}-\frac{3}{2}\right) \qquad (2.26)\\
&\geq \frac{1}{2R^2}\left(\pi\cdot\ln\left(\frac{a}{R}+1\right)+\pi-3\right).
\end{aligned}$$

To guarantee coverage of point P, it is sufficient to claim that

$$f(R)=\frac{1}{2R^2}\left(\pi\cdot\ln\left(\frac{a}{R}+1\right)+\pi-3\right)\geq\frac{1}{L^2}. \qquad (2.27)$$

Since $f(R)$ is monotonically decreasing on $(0,+\infty)$, the largest R satisfying the above inequality is the unique solution $\underline{R}$ of the equation

$$f(R)=\frac{1}{L^2}. \qquad (2.28)$$

Thus, a uniform grid with step size $\underline{R}$ jams any point P inside a corner cell. According to Lemma 1, the grid jams the least covered point in the square implying that the whole square is jammed. Thus we have the desired result. □

Since the function $f(R)=\frac{1}{2R^2}(\pi\ln(\frac{a}{R}+1)+\pi-3)$ is monotonic, equation (2.8) can be easily solved using a numerical procedure such as a binary search. Therefore, using (2.8), we can obtain a step size $\underline{R}$ such that the corresponding uniform grid covers the entire square. Further, the number of jamming devices in the grid does not exceed

$$N_1=\left(\frac{a}{\underline{R}}+2\right)^2. \qquad (2.29)$$

A more straightforward solution of the initial problem could be based on the property that a jamming device covers all the points inside a circle of radius L as mentioned in Definition 1. Using that, we could reduce the problem to finding the optimal covering of a square with circles of radius L. A direct result from [16] (that was mentioned in [20]) is that in the limit, the minimum number of circles to cover an area a^2 is

$$N_2=\frac{2a^2}{3\sqrt{3L^2}}. \qquad (2.30)$$

To compare the approaches, we consider the ratio

$$\begin{aligned}\frac{N_2}{N_1} &= \left(\frac{R}{L^2}\right)\frac{2}{3\sqrt{3}}\frac{1}{(1+2\frac{R}{a})^2} \\ &= \frac{2x^2}{3\sqrt{3}}\frac{1}{(1+\frac{2x}{k})^2},\end{aligned} \tag{2.31}$$

where $x = \frac{R}{L}$ and $k = \frac{a}{L}$. Using these substitutions, equation (2.8) can be rewritten in terms of variables x and k as follows

$$\frac{1}{x^2}\left(\pi \ln\left(\frac{k}{x}+1\right)+\pi-3\right) = 2. \tag{2.32}$$

By solving (2.32) for different values of k, one can find corresponding values of x and $\frac{N_2}{N_1}$. To evaluate the advantage of the uniform grid approach over the naive one, we provide some computational results in the Table 2.1. From the

k	x	$\frac{N_2}{N_1}$
10^2	2.44	2.3
10^4	3.54	4.8
10^6	4.40	7.5
10^8	5.14	10.2

Table 2.1: Comparing $\frac{N_2}{N_1}$ for various values of k.

table, we see that as k increases, the advantage of using our approach becomes more significant. In fact, it can be proved that $\lim_{a\to\infty}\frac{N_2}{N_1} = \infty$. This will follow as a corollary of Theorem 3.

To establish the quality of the lower bound rigorously, we need to first establish a similar result for an upper bound. This follows in the next theorem.

Theorem 2. *The unique solution of the equation*

$$\frac{1}{R^2}\left(\frac{\pi}{2}\ln\left(\frac{2a}{R}+1\right)-\frac{1}{6(\frac{a}{R}+1)}+\frac{\pi}{2}+\frac{19}{3}\right) = \frac{1}{L^2} \tag{2.33}$$

is an upper bound $\overline{R}$ of the optimal grid step size R^*.

Proof. Let $P(x_0, y_0)$ be the least jammed point, that lies inside a corner cell according to Lemma 1. Without the loss of generality, as in the proof of Theorem 1, we assume that P is inside the bottom left corner cell. The jamming energy received at point P is calculated through the expressions (2.9) - (2.14). Since P is the least covered point, the following inequality holds.

$$E(P) \le E\left(P'\left(x=\frac{R}{2}, y=0\right)\right) = I_1' + I_2' + I_3' + I_4', \text{ where} \tag{2.34}$$

$$I_1' = \sum_{i=0}^{T-1}\sum_{j=0}^{T-1}\frac{1}{(\frac{R}{2}+i\cdot R)^2+(R+j\cdot R)^2}, \tag{2.35}$$

$$I_2' = \sum_{i=0}^{T-1}\frac{1}{(\frac{R}{2}+i\cdot R)^2}, \tag{2.36}$$

$$I_3' = \sum_{j=0}^{T-1}\frac{1}{(\frac{R}{2})^2+(R+j\cdot R)^2}, \tag{2.37}$$

$$I_4' = \frac{1}{(\frac{R}{2})^2}. \tag{2.38}$$

I_2' and I_3' can be estimated through integrals similarly to the techniques used in the proof of Theorem 1. The following inequality holds

$$\sum_{i=1}^{N} f(i) \leq \int_0^N f(x)dx, \tag{2.39}$$

where $f(x)$ is a decreasing function. This property can also be proven geometrically. Figure 2.7 represents a graphical interpretation of this relation. The left side of the inequality is represented by the shaded area. The right side of (2.39)

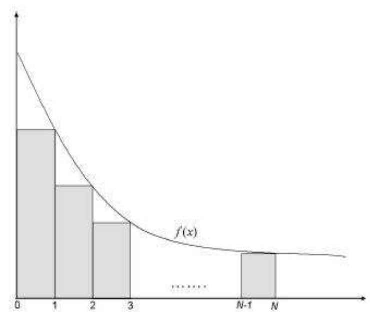


Figure 2.7: Integral Upper Bound.

is the area under $f(x)$. With this property we have from (2.36) that

$$\begin{aligned} I_2' &\leq \frac{1}{(\frac{R}{2})^2} + \int_0^{T-1} \frac{dx}{(\frac{R}{2} + x \cdot R)^2} \\ &= \frac{1}{R^2}\left(6 - \frac{1}{T - \frac{1}{2}}\right). \end{aligned} \tag{2.40}$$

Furthermore, using inequalities (2.18) and (2.19), we see that (2.37) is estimated by

$$\begin{aligned} I_3' &\leq \frac{1}{(\frac{R}{2})^2 + (R + x \cdot R)^2} \\ &= \frac{2}{3R^2} + \frac{2}{R^2}\left(\arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{2T}\right)\right) \\ &\leq \frac{2}{3R^2} + \frac{2}{R^2}\left(\frac{1}{2} - \frac{1}{2T} + \frac{1}{24T^3}\right) \\ &= \frac{1}{R^2}\left(\frac{5}{3} - \frac{1}{T} + \frac{1}{12T^3}\right). \end{aligned} \tag{2.41}$$

To estimate I_1' a property similar to (2.39) can be used. This inequality is given by

$$\sum_{i=1}^{N}\sum_{j=1}^{N} f(i,j) \leq \int_0^N \int_0^N f(x,y)dxdy + \int_0^N f(x,0)dx + \int_0^N f(0,y)dy, \tag{2.42}$$

where $f(x, y)$ is a decreasing function of x and y. With the above inequality,

$$\begin{aligned} I_1' &\leq \frac{1}{(\frac{R}{2}^2) + R^2} + \int_0^{T-1} \frac{dx}{(\frac{R}{2})^2 + (R + x \cdot R)^2} + \int_0^{T-1} \frac{dx}{(\frac{R}{2} + x \cdot R)^2 + R^2} + \\ &+ \int_0^{T-1} \int_0^{T-1} \frac{dxdy}{(\frac{R}{2} + x \cdot R)^2 + ((R + y \cdot R)^2} \\ &= \frac{4}{5R^2} + \frac{C}{R^2} + \frac{1}{R^2} \int_0^{T-1} \int_0^{T-1} \frac{d(x + \frac{1}{2})dy}{(\frac{1}{2} + x)^2 + (y+1)^2}, \text{ where} \end{aligned} \tag{2.43}$$

$$
\begin{aligned}
C &= 2\arctan(2T) - \arctan(2) + \arctan\left(T - \frac{1}{2}\right) - \frac{\pi}{2} \\
&= \frac{\pi}{2} - 2\arctan\left(\frac{1}{2T}\right) + \arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{2}{2T-1}\right) \\
&\leq \frac{\pi}{2} - 2\left(\frac{1}{2T} - \frac{1}{24T^3}\right) + \frac{1}{2} - \left(\frac{2}{2T-1} - \frac{8}{3(2T-1)^3}\right) \\
&\leq \frac{\pi+1}{2}.
\end{aligned}
\tag{2.44}
$$

The double integral in (2.43) is bounded as follows

$$
\begin{aligned}
&\int_0^{T-1}\int_0^{T-1} \frac{d(x+\frac{1}{2})dy}{(\frac{1}{2}+x)^2+(y+1)^2} = \int_{\frac{1}{2}}^{T-\frac{1}{2}}\int_1^T \frac{dtdy}{t^2+y^2} \\
&= \int_{\frac{1}{2}}^{T-\frac{1}{2}} \frac{1}{t}\left(\arctan\left(\frac{T}{t}\right) - \arctan\left(\frac{1}{t}\right)\right) dt \\
&\leq \int_{\frac{1}{2}}^{T-\frac{1}{2}} \frac{1}{t}\left(\frac{\pi}{2} - \arctan\left(\frac{t}{T}\right)\right) dt - \int_{\frac{1}{2}}^{T-\frac{1}{2}} \frac{1}{t}\left(\frac{1}{t} - \frac{1}{3t^3}\right) dt \\
&\leq \frac{\pi}{2}\left(\ln\left(T-\frac{1}{2}\right) - \ln\left(\frac{1}{2}\right)\right) - \int_{\frac{1}{2}}^{T-\frac{1}{2}} \frac{1}{t}\left(\frac{t}{T} - \frac{t^3}{3T^3}\right) dt - \\
&- \left(\frac{4}{3} - \frac{1}{T-\frac{1}{2}} + \frac{1}{6(T-\frac{1}{2})^2}\right) \\
&= \frac{\pi}{2}\ln(2T-1) - \frac{20}{3} + \frac{5}{6T} + \frac{1}{12T^2} - \frac{1}{36T^3} + \frac{1}{T-\frac{1}{2}} - \frac{1}{6(T-\frac{1}{2})^2} \\
&< \frac{\pi}{2}\ln(2T-1) - \frac{20}{3} + \frac{5}{6T} + \frac{1}{T-\frac{1}{2}} - \frac{1}{12(T-\frac{1}{2})^2}.
\end{aligned}
\tag{2.45}
$$

Combining the results from (2.43), (2.44), and (2.45) gives the overestimate for I_1' as

$$
I_1' < \frac{1}{R^2}\left(\frac{\pi}{2}\ln(2T-1) + \frac{\pi}{2} - \frac{16}{3} + \frac{5}{6T} + \frac{1}{T-\frac{1}{2}} - \frac{1}{12(T-\frac{1}{2})^2}\right). \tag{2.46}
$$

Recall equation (2.34) stated $E(P) \leq I_1' + I_2' + I_3' + I_4$. So using the expression for I_4' given in (2.38) and the overestimates for I_1', I_2', and I_3' derived in equations (2.46), (2.40), and (2.41) respectively, we obtain

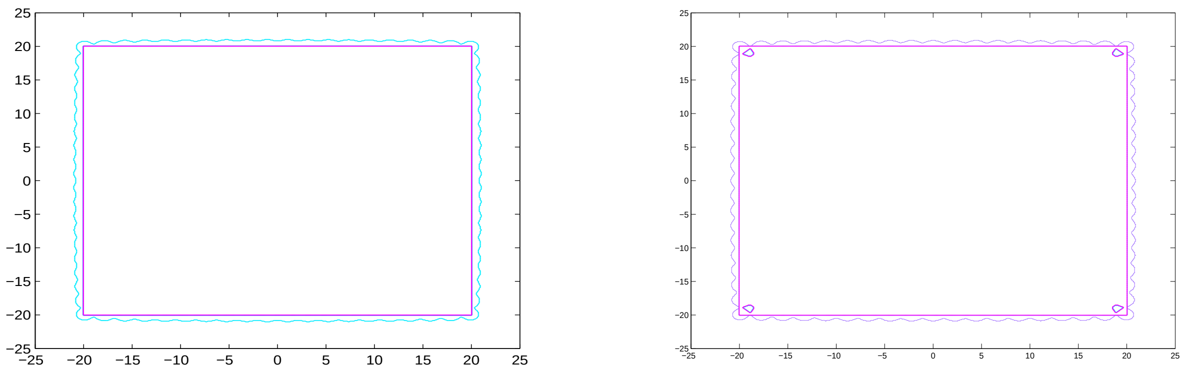
$$
E(P) \quad \leq \quad \frac{1}{R^2}\left(\frac{\pi}{2}\ln(2T-1) - \frac{1}{6T} + \frac{\pi}{2} + \frac{19}{3}\right). \tag{2.47}
$$

Finally, if we let $T = [\frac{a}{R}] + 1 \leq \frac{a}{R} + 1$, we get

$$
E(P) < \frac{1}{R^2}\left(\frac{\pi}{2}\ln\left(\frac{2a}{R}+1\right) - \frac{1}{6(\frac{a}{R}+1)} + \frac{\pi}{2} + \frac{19}{3}\right) \tag{2.48}
$$

The function $f(R) = \frac{1}{R^2}\left(\frac{\pi}{2}\ln\left(\frac{2a}{R}+1\right) - \frac{1}{6(\frac{a}{R}+1)} + \frac{\pi}{2} + \frac{19}{3}\right)$ is monotone, hence the equation $f(R) = \frac{1}{L^2}$ has a unique solution $\overline{R}$. Equation (2.48) implies that a grid with step size $\overline{R}$ does not cover the entire square. That is, there exists at least one point P that remains uncovered. Thus $\overline{R}$ is an upper bound for the optimal grid covering problem. Since the optimal grid step size $R^* < \overline{R}$, the theorem is proved. □

In Figure 2.8, we see an example in which we are covering at 40×40 square and the required jamming level at each point is 3.0 units. In part (a), we see the coverage associated with the required number of devices from the lower bound of Theorem 2. In this case, $20^2 = 400$ jamming devices are used to cover the area. Notice that there are no holes in the region. This, together with the scallop shell outside the bounding box indicates that all points within the region are covered. In part (b), we see the coverage corresponding to the placement of the jamming devices on a uniform grid according to the upper bound of Theorem 3. Here, the required number of devices is $19^2 = 361$. Notice the holes located at the four corners of the region indicating that these points are uncovered. This validates the theoretical results obtained in Theorem 2 and Theorem 3.



(a) (b)

Figure 2.8: (a) The coverage when jamming devices are placed according to the lower bound from Theorem 2. The total number of jamming devices required is $20^2 = 400$. (b) We see the coverage associated with the result obtained from Theorem 3. In this case, $19^2 = 361$ devices are placed. Notice the corner points are not jammed.

Now that we have established both upper and lower bounds for an optimal grid step size, we can determine the quality of the bounds. The result is obtained in the following theorem.

Theorem 3.

$$\lim_{a\to\infty} \frac{\overline{R}}{\underline{R}} = 1, \tag{2.49}$$

where $\overline{R}$ *and* $\underline{R}$ *are bounds obtained from equations (2.8) and (2.33), correspondingly. Moreover, the following inequality holds:*

$$1 \leq \frac{\overline{R}}{\underline{R}} \leq \sqrt{1 + \frac{c}{\ln(a)}}, \tag{2.50}$$

for constants $M \in \mathbb{R}, c \in \mathbb{R}$*, such that* $\overline{R} > M$.

Proof. By letting $x = \frac{\underline{R}}{L}$ and $y = \frac{\overline{R}}{L}$, equations (2.8) and (2.33) can be respectively rewritten as

$$a = L \cdot x \left(e^{\frac{2}{\pi}(x^2+\frac{3}{2})-1} - 1 \right), \text{ and} \tag{2.51}$$

$$\frac{\pi}{2} \ln\left(\frac{2a}{L \cdot y} + 1\right) = y^2 - \frac{19}{3} - \frac{\pi}{2} + \frac{L \cdot y}{6(a + L \cdot y)}. \tag{2.52}$$

To prove the theorem, we need to show that

$$\lim_{a\to\infty} \frac{y}{x} = 1, \tag{2.53}$$

where $x > 0$ and $y > 0$ are solutions of (2.51) and (2.52), correspondingly. From (2.52), we obtain

$$\frac{\pi}{2}\ln\left(\frac{2a}{L\cdot y}+1\right) > y^2 - C_1, \text{ where} \tag{2.54}$$

$$C_1 = \frac{19}{3}+\frac{\pi}{2}, \text{ and} \tag{2.55}$$

$$a > \frac{L\cdot y}{2}\left(e^{\frac{2}{\pi}(y^2-C_1)}-1\right). \tag{2.56}$$

From (2.51) and (2.56) we see that

$$x\left(e^{\frac{2}{\pi}(x^2+C_2)}\cdot C_3 - 1\right) > \frac{y}{2}\left(e^{\frac{2}{\pi}(y^2-C_1)}-1\right), \text{ where} \tag{2.57}$$

$$C_2 = \frac{3}{2}, \text{and} \tag{2.58}$$

$$C_3 = e^{-1}. \tag{2.59}$$

Since $y\cdot L$ and $x\cdot L$ are upper and lower bounds, correspondingly, the following relation holds

$$\frac{y}{x} > 1. \tag{2.60}$$

With (2.51) and (2.60) above, we can also conclude that

$$\lim_{a\to\infty} x = \infty \quad \text{and} \quad \lim_{a\to\infty} y = \infty. \tag{2.61}$$

For all $M \in I\!R$, where $M > \sqrt{C_1}$, there exists $Q \in I\!R$ such that (2.57) can be reduced to

$$\frac{y}{x} \quad < \quad Q\cdot e^{\frac{2}{\pi}(x^2-y^2)}, \text{ and } y > M. \tag{2.62}$$

Moreover, for $c = \frac{\pi}{2}\ln(Q)$ the following inequality holds

$$\left(\frac{y}{x}\right)^2 - 1 \leq \frac{c}{x^2}, \text{ and } y > M. \tag{2.63}$$

Assume for the sake of contradiction that the inequality in (2.63) does not hold for some (x^*, y^*). That is assume that $\left(\frac{y^*}{x^*}\right)^2 - 1 > \frac{c}{x^{*2}}$. Using (2.62) we have

$$\frac{y^*}{x^*} < Q\cdot e^{-\frac{2}{\pi}x^{*2}\left((\frac{y^*}{x^*})^2-1\right)} < Q\cdot e^{-\frac{2}{\pi}x^{*2}\cdot\frac{c}{x^{*2}}} = 1, \tag{2.64}$$

which contradicts (2.60).

Applying (2.60) and (2.63) we get

$$1 < \frac{y}{x} \leq \sqrt{1+\frac{c}{x^2}}, \text{ and } y > M. \tag{2.65}$$

Letting a tend to ∞ and taking (2.61) into account, we see that in fact

$$\lim_{a\to\infty}\frac{y}{x} = 1. \tag{2.66}$$

Finally, by using (2.65) and (2.51), the following relation can be obtained

$$1 < \frac{y}{x} \leq \sqrt{1+\frac{k}{\ln(a)}}, \tag{2.67}$$

for some constant $k \in I\!R$, when $y > M$. Thus, the theorem is proved. □

2.4 Conclusion

In this chapter, we introduced the problem of jamming a communication network under complete uncertainty. We examined the case when the network is known to lie in a square with area a^2. We derived upper and lower bounds for the optimal number of jamming devices required when they are located at the vertices of a uniform grid. We also provided a convergence result indicating that the proposed bounds are tight. Furthermore, we proved that our approach is more efficient than the solution provided by optimally covering the square with circles of radius L.

Chapter 3

Robust Wireless Network Jamming Problems

This chapter presents the results published in [5] C.W. Commander, P.M. Pardalos, V. Ryabchenko, S. Sarykalin, T. Turko, and S. Uryasev. Robust wireless network jamming problems. C.W. Commander, M.J. Hirsch, R.A. Murphey, and P.M. Pardalos (editors), Optimization and Cooperative Control Strategies, Lecture Notes in Control and Information Sciences, Springer, 2008.

3.1 Introduction

Research on suppressing and eavesdropping communication networks has seen a surge recently in the optimization community. Two recent papers by Commander et al. [6, 7] represent the current state-of-the-art. These problems have several important military applications and represent a critical area of research as optimization of telecommunication systems improve technological capabilities [23]. In [6], the authors develop lower and upper bounds for the optimal number of wireless jamming devices required to suppress a network contained in a given area such as a map grid. In this work, there were no a priori assumptions made about the topology of the network to be jammed other than the geographical region in which it was contained. This problem is particularly important in the global war on terrorism as improvised explosive devices (IEDs) continue to plague the coalition forces. In fact, IEDs account for approximately 65% of all combat injuries in Iraq [21]. These homemade bombs are almost always detonated by some form of radio frequency device such as cellular telephones, pagers, and garage door openers. The ability to suppress radio waves in a given region will help prevent casualties resulting from IEDs [4].

In [7] the WIRELESS NETWORK JAMMING PROBLEM was introduced and several formulations derived. In the WNJP the topology of the network is assumed and various objectives can be considered from jamming all the communication nodes to constraining the connectivity index of the nodes. In this chapter, we introduce robust variants of those formulations which account for the fact that the exact topology of the network to be jammed may not be known entirely. Particularly, we consider instances in which several topologies are considered likely, and develop robust scenarios for placing jamming devices which are able to suppress the network regardless of which candidate topology is realized. The overarching goal is to develop robust formulations with respect to the uncertainties in the information about the network. These models will provide a more realistic interpretation of combat scenarios in urban and dynamic environments.

The organization of the chapter is as follows. In Section 3.2, we derive several formulations of the ROBUST WIRELESS NETWORK JAMMING PROBLEM (R-WNJP). In Section 3.3, we review several percentile measures and incorporate percentile constraints into the models in Section 3.4. The results of several case studies are presented in Section 3.5 and the results are analyzed. We conclude with directions of future research.

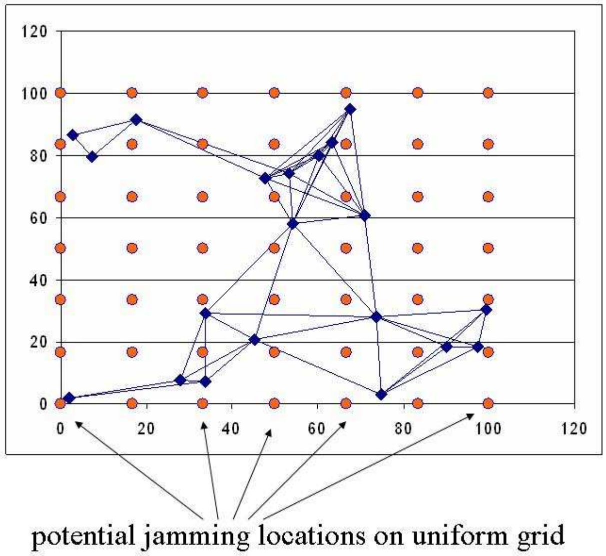


Figure 3.1: Example network shown with potential jamming device locations.

3.2 Problem Formulations

Denote a graph $G = (V, E)$ as a pair consisting of a set of vertices V, and a set of edges E. All graphs in this chapter are assumed to be undirected and unweighted. We use the symbol "$b := a$" to mean "the expression a defines the (new) symbol b" in the sense of King [17]. Finally, we will use *italics* for emphasis and SMALL CAPS for problem names.

We assume that the communication network to be jammed comprises a set $\mathcal{M} = \{1, 2, \ldots, m\}$ of radio devices which are outfitted with omnidirectional antennas and function as both transmitters and receivers as in [7]. Further we assume that the coordinates of the nodes and various parameters such as the frequency range are given by probability distributions. For example, we can assume that a Kalman filter provides some estimates of the locations of the nodes. In a deterministic setup, the topology which represents the communication pattern could be represented by a graph in which an edge connects two nodes if they are within a certain communication threshold.

As for the set of jamming devices, we assume that they too are outfitted with omnidirectional antennas that the effectiveness of a jamming device on a communication node is inversely proportional to their distance squared. Suppose that the set of jamming devices is giving by $\mathcal{N} = \{1, 2, \ldots, n\}$, and we are given a set potential locations in which to place them. Figure 3.1 provides an example of the communication network and the potential jamming device locations. Moreover, each potential location j has an associated cost $c_j, j = 1, 2, \ldots, n$. We can describe the jamming power received by network node i located at a point $(\xi_i, \eta_i) \in \mathbb{R} \times \mathbb{R}$, from jamming device $j \in \mathcal{N}$ located at (x, y) is given by

$$d_j(\xi_i, \eta_i) \equiv d_{ij} := \frac{\lambda}{(x_j - \xi_i)^2 + (y_j - \eta_i)^2}, \tag{3.1}$$

where $\lambda \in \mathbb{R}$ is a constant. Without the loss of generality, we can let $\lambda = 1$. We say that node $i \in \mathcal{M}$ located at (ξ_i, η_i) is jammed if the total energy received at this point from all jamming devices exceeds some threshold value C_i for all $i \in \mathcal{M}$. That is, node $i \in \mathcal{M}$ is jammed if

$$\sum_{j=1}^{n} d_j(\xi_i, \eta_i) \geq C_i. \tag{3.2}$$

As mentioned above, we are considering robust formulations of the WNJP. Since the exact locations of the network nodes are unknown, we assume that a set of intelligence data has been collected and from that a set $\mathcal{S}$

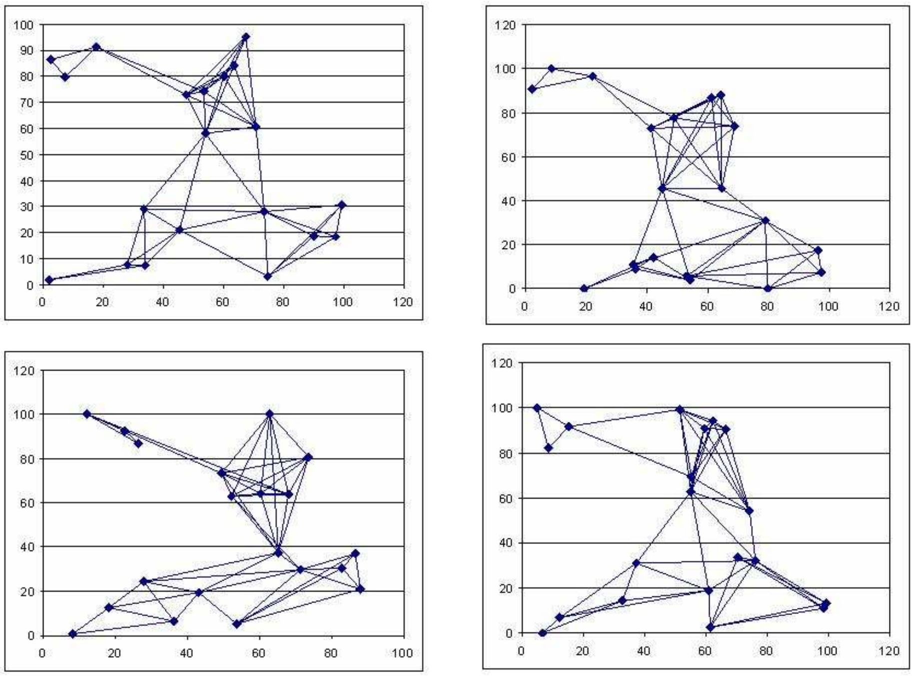


Figure 3.2: Four example scenarios are shown.

of the most likely scenarios have been compiled. We assume that for scenario $s \in \mathcal{S}$ both the node locations $\{(\xi_1^s, \eta_1^s), (\xi_2^s, \eta_2^s), \dots, (\xi_m^s, \eta_m^s)\}$ and the set of jamming thresholds $\{C_1^s, C_2^s, \dots, C_m^s\}$ are modeled in these scenarios. For each scenario $s \in \mathcal{S}$, we do not assume that the number of communication devices to jammed are equal. Therefore, we define for each scenario $s \in \mathcal{S}$, the set $\mathcal{M}_s = \{1, 2, \dots, m_s\}$ which represents the set of nodes to be jammed. For example, the networks shown in Figure 3.2 represent a set of possible topologies for the network to be jammed.

3.2.1 The Robust Connectivity Index Problem

Given a graph $G = (V, E)$, the connectivity index of a node is defined as the number of nodes reachable from that vertex. The first formulation of the WNJP we consider imposes constraints on the connectivity indices of the network nodes. The connectivity index of a network vertex is defined as the number of nodes reachable from that vertex. The degree to which the connectivity index of a given node is constrained may be determined by its relative importance or how crucial it is for maintaining connectivity among many components. It is at the discretion of the analyst whether to assign arbitrary values to each node or use some heuristic for determining a relative importance. One way to determine the connectivity indices is to identify the so-called *critical nodes* of the graph and impose relatively tighter constraints on these nodes. Critical nodes are those vertices whose removal from the graph induces a set of disconnected components whose sizes are minimally variant [1]. Critical node detection has been recently applied to interdicting wired communication networks [4], to network security applications [2], and most recently to the analysis of protein-protein interaction networks in the context of computational drug design [3].

Suppose for example that for the scenarios shown in Figure 3.2 the maximum allowable connectivity index is set to 3 for each node. Then the objective of the ROBUST CONNECTIVITY INDEX PROBLEM (RCIP) is to determine the minimum number and locations for the jamming devices so that each node has no more than 3 neighbors in each of the four scenarios presented. Figure 3.3 provides an example solution for this case.

Suppose that communication between two nodes in the communication graph is said to be severed if at least one of the nodes is jammed. Then the objective of the ROBUST CONNECTIVITY INDEX PROBLEM (RCIP) is to determine the minimum required jamming devices such that the connectivity index of each node i in each scenario s does not

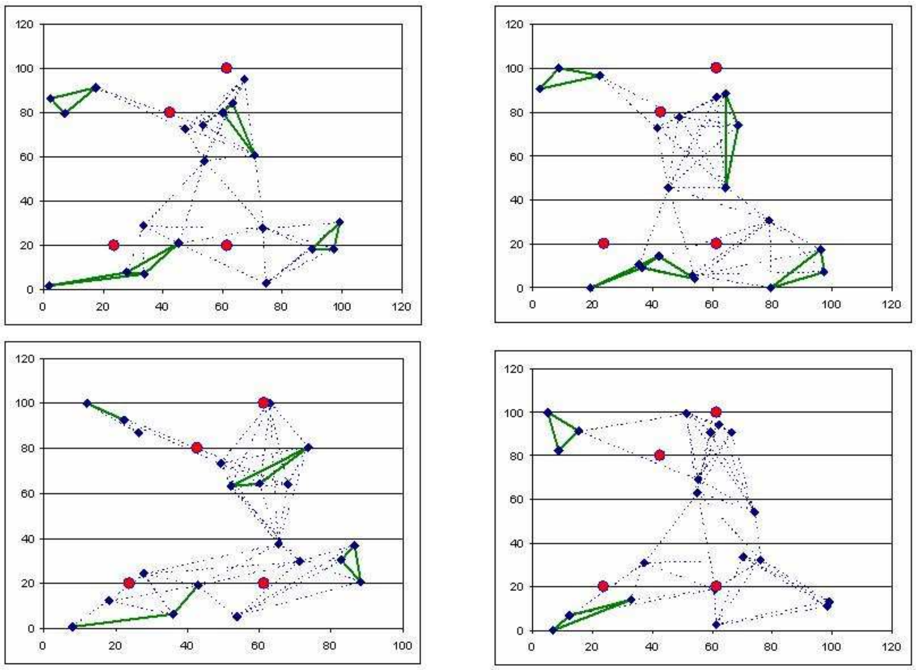


Figure 3.3: The optimal solution for the networks in Figure 3.2 is given for the case when the maximum connectivity index is 3 for all nodes.

exceed some predefined values L_i^s. In order to define the corresponding mathematical formulation we must define the following functions. First let $y^s : \mathcal{M}_s \times \mathcal{M}_s \mapsto \{0,1\}$ be a surjection where $y_{ij}^s := 1$ if there exists a path from node i to node j in the jammed network according to scenario $s \in \mathcal{S}$. Next let the function $z^s : \mathcal{M}_s \mapsto \{0,1\}$ be a surjective function where z_i^s returns 1 if node i is not jammed in scenario $s \in \mathcal{S}$. Finally, let $x_i, i = 1, \ldots, n$ be a set of decision variables where $x_i := 1$ if a jamming device location i is utilized. If c_k and d_{ij} are defined as above, then we can formulate the S-CIP as the following optimization problem.

$$\textbf{(RCIP)} \quad \min \quad \sum_{k=1}^{n} c_k x_k \tag{3.3}$$

s.t.

$$\sum_{\substack{j=1 \\ j \neq i}}^{m_s} y_{ij}^s \leq L_i^s, \ \forall\, i \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{3.4}$$

$$M(1 - z_i^s) > \sum_{k=1}^{n} d_{ik}^s x_k - C_i^s \geq -M z_i^s, \ \forall\, i \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{3.5}$$

$$x_j \in \{0,1\}, \ \forall\, j \in \mathcal{N}, \tag{3.6}$$

$$z_i^s \in \{0,1\}, \ \forall\, i \in \mathcal{M}_s, \tag{3.7}$$

$$y_{ik}^s \in \{0,1\}, \ \forall\, i, k \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{3.8}$$

where $M \in \mathbb{R}$ is some large constant.

In a manner similar to that shown in [7], we can formulate an equivalent integer programming formulation as follows. First let $v^s : \mathcal{M}_s \times \mathcal{M}_s \mapsto \{0,1\}$ and $v^{s'} : \mathcal{M}_s \times \mathcal{M}_s \mapsto \{0,1\}$ be respectively defined as

$$v_{ij}^s := \begin{cases} 1, \ \text{if } (i,j) \in E^s, \\ 0, \ \text{otherwise}, \end{cases} \tag{3.9}$$

and

$$v_{ij}^{s'} := \begin{cases} 1, & \text{if } (i,j) \text{ exists in the jammed network of scenario } s, \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

An equivalent integer program is then given by

$$\textbf{(RCIP-1)} \quad \min \quad \sum_{k=1}^{n} c_k x_k, \tag{3.11}$$

s.t.

$$y_{ij}^s \geq v_{ij}^{s'}, \; \forall\, i,j \in \mathcal{M}_s, \; \forall\, s \in \mathcal{S}, \tag{3.12}$$

$$y_{ij}^s \geq y_{ik}^s y_{kj}^s, \; k \neq i,j; \; \forall\, i,j \in \mathcal{M}_s, \; \forall\, s \in \mathcal{S}, \tag{3.13}$$

$$v_{ij}^{s'} \geq v_{ij}^s z_j^s z_i^s, \; i \neq j; \; \forall\, i,j \in \mathcal{M}_s, \; \forall\, s \in \mathcal{S}, \tag{3.14}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} y_{ij}^s \leq L_i^s, \; \forall\, i \in \mathcal{M}_s, \; \forall\, s \in \mathcal{S}, \tag{3.15}$$

$$M(1 - z_i^s) > \sum_{k=1}^{n} d_{ik}^s x_k - C_i^s \geq -M z_i^s, \; \forall\, i \in \mathcal{M}_s, \; \forall\, s \in \mathcal{S}, \tag{3.16}$$

$$x_j \in \{0,1\}, \; \forall\, j \in \mathcal{N}, \tag{3.17}$$

$$z_i^s \in \{0,1\}, \; \forall\, i \in \mathcal{M}_s, \; \forall\, s \in \mathcal{S}, \tag{3.18}$$

$$y_{ij}^s \in \{0,1\} \; \forall\, i,j \in \mathcal{M}_s, \; \forall\, s \in \mathcal{S}, \tag{3.19}$$

$$v_{ij}^s \in \{0,1\}, \; \forall\, i,j \in \mathcal{M}_s, \; \forall\, s \in \mathcal{S}, \tag{3.20}$$

$$v_{ij}^{s'} \in \{0,1\}, \; \forall\, i,j \in \mathcal{M}_s, \; \forall\, s \in \mathcal{S}. \tag{3.21}$$

We establish the equivalence of formulations RCIP and RCIP-1 in the following theorem. The proof follows similarly to a result for the single scenario problem in [7].

Theorem 5. *If* RCIP *has an optimal solution, then* RCIP-1 *has an optimal solution. Furthermore, any optimal solution* x^* *of the integer programming problem* RCIP-1 *is an optimal solution of the optimization problem* RCIP.

Proof. It is easy to see that if communication nodes i and j are reachable in the jammed network of a given scenario $s \in \mathcal{S}$, then $y_{ij}^s = 1$ in RCIP-1. Indeed if i and j are reachable, then there exists a sequence of pairwise adjacent vertices

$$\{(i_0, i_1), \ldots, (i_{m-1}, i_m)\}, \tag{3.22}$$

where $i_0 = i$ and $i_m = j$. By inducting along the vertices, we can establish the fact that $y_{i_0,i_{k+1}}^s = 1$ for all $k = 1, \ldots, m$. To do this, first note that from (3.12) we have that $y_{i_k,i_{k+1}}^s = 1$. Then if $y_{i_0,i_k}^s = 1$, then by (3.13) we have that

$$y_{i_0,i_{k+1}}^s \geq y_{i_0,i_k}^s y_{i_k,i_{k+1}}^s = 1. \tag{3.23}$$

This completes the induction step. Thus far we have shown that

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} y_{ij}^s \geq \text{ connectivity index of node } i.$$

Let F be the objective function in RCIP-1 and RCIP. Furthermore, suppose (x^*, y^*) and $(\hat{x}^*, \hat{y}^*)$ represent optimal solutions for each formulation respectively. Then so far, we have confirmed that

$$F(x^*) \geq F(\hat{x}^*). \tag{3.24}$$

It is easy to verify that $(\hat{x}^*, \hat{y}^*)$ is feasible in RCIP-1. This follows from the definition of y^s_{ij} in RCIPand the fact that $(\hat{x}^*, \hat{y}^*)$ satisfies the feasibility constraints in RCIP. This proves the first statement of the theorem. Hence from RCIP-1, we have that

$$F(x^*) \le F(\hat{x}^*). \tag{3.25}$$

Therefore using (3.24) and (3.25), we have

$$F(x^*) = F(\hat{x}^*). \tag{3.26}$$

Now define y^s such that

$$y^s_{ij} := 1 \Leftrightarrow j \text{ is reachable from } i \text{ when scenario } s \text{ is jammed by } x^*. \tag{3.27}$$

Using the above results, we know that (x^*, y^s) is feasible in RCIP-1, and hence optimal. Also from the construction of y^s it follows that (x^*, y^s) is feasible in RCIP. According to (3.26), we can conclude that x^* is also optimal for RCIP. Thus the theorem is proved. □

With the previous theorem, we have established a one-to-one correspondence between the two formulations. By using some standard techniques, we can now reformulate RCIP-1 into the following integer linear program

(RCIP-2) min
$$\sum_{k=1}^{n} c_k x_k \tag{3.28}$$
s.t.
$$y^s_{ij} \ge v^{s'}_{ij},\ \forall\, i,j = 1,\ldots,\mathcal{M}_s,\ \forall\, s \in \mathcal{S}, \tag{3.29}$$
$$y^s_{ij} \ge y^s_{ik} + y^s_{kj} - 1,\ k \ne i,j;\ \forall\, i,j \in \mathcal{M}_s,\ \forall\, s \in \mathcal{S}, \tag{3.30}$$
$$v^{s'}_{ij} \ge v^s_{ij} + z^s_j + z^s_i - 2,\ i \ne j;\ \forall\, i,j \in \mathcal{M}_s,\ \forall\, s \in \mathcal{S}, \tag{3.31}$$
$$\sum_{\substack{j=1\\ j\ne i}}^{m} y^s_{ij} \le L^s_i,\ \forall\, i \in \mathcal{M}_s,\ \forall\, s \in \mathcal{S}, \tag{3.32}$$
$$M(1 - z^s_i) > \sum_{k=1}^{n} d^s_{ik} x_k - C^s_i \ge -M z^s_i,\ \forall\, i \in \mathcal{M}_s,\ \forall\, s \in \mathcal{S}, \tag{3.33}$$
$$x_j \in \{0,1\},\ \forall\, j \in \mathcal{N}, \tag{3.34}$$
$$z^s_i \in \{0,1\},\ \forall\, i \in \mathcal{M}_s,\ \forall\, s \in \mathcal{S}, \tag{3.35}$$
$$y^s_{ij} \in \{0,1\}\ \forall\, i,j \in \mathcal{M}_s,\ \forall\, s \in \mathcal{S}, \tag{3.36}$$
$$v^s_{ij} \in \{0,1\},\ \forall\, i,j \in \mathcal{M}_s,\ \forall\, s \in \mathcal{S}, \tag{3.37}$$
$$v^{s'}_{ij} \in \{0,1\},\ \forall\, i,j \in \mathcal{M}_s,\ \forall\, s \in \mathcal{S}. \tag{3.38}$$

Theorem 6. *If* RCIP-1 *has an optimal solution, then* RCIP-2 *has an optimal solution. Further, any optimal solution* x^* *of* RCIP-2 *is an optimal solution of* RCIP-1.

Proof. For binary variables, notice that the following equivalence holds

$$y^s_{ij} \ge y^s_{ik} y^s_{kj} \Leftrightarrow y^s_{ij} \ge y^s_{ik} + y^s_{kj} - 1. \tag{3.39}$$

Then the only other difference between the two formulations are the two constraints:

$$v^{s'}_{ij} \quad = \quad v^s_{ij} z^s_j z^s_i \tag{3.40}$$
$$v^{s'}_{ij} \quad \ge \quad v^s_{ij} + z^s_i + z^s_j - 2 \tag{3.41}$$

In the similar manner as in (3.39) above, it is easy to verify that (3.40) implies (3.41). Therefore the feasible region of RCIP-2 includes the feasible region of RCIP-1. With this we have the first statement of the theorem.

As in the previous proof, let F represent the objective function in RCIP-1 and RCIP-2 and let x^* and $\hat{x}^*$ represent respective optimal solutions. Then it follows that

$$F(x^*) \geq F(\hat{x}^*). \tag{3.42}$$

Let $(x^*, y^{s*}, v^{'s*}, z^{s*})$ be an optimal solution for formulation RCIP-2. Now, define $v^{''s*}$ as follows:

$$v_{ij}^{''s*} \quad := \quad \begin{cases} 1, \text{ if } v_{ij}^s + z_i^{s*} + z_j^{s*} - 2 = 1, \\ 0, \text{ otherwise.} \end{cases} \tag{3.43}$$

Notice that if $v_{ij}^{'s*} \geq v_{ij}^{''s*}$ then $(x^*, y^{s*}, v^{''s*}, z^{s*})$ is feasible in RCIP-2 according to constraint (3.29) (i.e., $y_{ij}^s \geq v_{ij}^{''s*}$). Furthermore, $(x^*, y^{s*}, v^{''s*}, z^{s*})$ is optimal in RCIP-2 as the the objective value is $F(x^*)$, which is optimal by definition. Using (3.43), $(v^{''s*}, z^{s*})$ satisfies:

$$v_{ij}^{''s*} = v_{ij}^s z_j^{s*} z_i^{s*}. \tag{3.44}$$

Using this we have that $(x^*, y^{s*}, v^{''s*}, z^{s*})$ is feasible for RCIP-1. If $\hat{x}^*$ is an optimal solution of RCIP-1 then it follows that

$$F(\hat{x}^*) \leq F(x^*) \tag{3.45}$$

On the other hand, we have shown in (3.42) above, that

$$F(x^*) \leq F(\hat{x}^*). \tag{3.46}$$

(3.45) and (3.46) together imply $F(x_1) = F(x^*)$. The last equality proves that x^* is an optimal solution of RCIP-1. Thus, the theorem is proved. □

Finally, we have the following theorem which establishes the equivalence between the optimization problem RCIP and the integer linear programming formulation RCIP-2 [7].

Theorem 7. *If* RCIP *has an optimal solution, then* RCIP-2 *has an optimal solution. Moreover, any optimal solution of* RCIP-2 *is an optimal solution of* RCIP.

Proof. The proof follows directly from Theorem 5 and Theorem 6. □

3.2.2 Robust Node Covering Problem

What follows is a robust formulation of the OPTIMAL NODE COVERING problem presented in WNJP. As before, we are given $\mathcal{M}_s$, the set of nodes to be jammed. We are also given the set of potential locations for the jamming devices, $\mathcal{N}$. For the ROBUST NETWORK COVERING PROBLEM (RNCP) is to minimize the number of jamming devices required to suppress communication on all of the nodes for each of the scenarios. Recall from Equation (3.2) that a node in a given scenario is said to be suppressed if the cumulative amount of energy received by that node from all jamming devices exceeds some threshold level. Let c_k, d_{ik}^s, and C_i^s be as defined in the previous subsection. Also, recall that the decision variable $x_k := 1$ if a jamming device is installed at location $k \in \mathcal{N}$. With this, we can formulate the RNCP as follows.

$$\textbf{(RNCP)} \quad \min \quad \sum_{k=1}^{n} c_k x_k, \tag{3.47}$$

$$\text{s.t.}$$

$$\sum_{k=1}^{n} d_{ik}^s x_k \geq C_i^s, \; i = 1, 2, \ldots, m_s, s = 1, 2, \ldots, S, \tag{3.48}$$

$$x_k \in \{0, 1\}, \; k = 1, 2, \ldots, n, \tag{3.49}$$

The RNCP is $\mathcal{NP}$-hard which can be easily shown by a reduction from the MULTIDIMENSIONAL KNAPSACK PROBLEM [10]. With this in mind, we recognize that solving large-scale instances is unreasonable, thus we must seek alternative solution methods. One possible way of doing this is accepting the fact that jamming a sufficient percentage of the network nodes will suffice given the intractability of the problem. We move along now and examine the R-WNJP with the inclusion of percentile constraints.

3.3 Percentile Constraints

As demonstrated in the seminal chapter on deterministic network jamming problems [7], it is often the case that a network can be sufficiently neutralized by suppressing communication on a fraction of the total nodes. This can be accomplished by the inclusion of percentile constraints into a mathematical model. In this section, we review two commonly used risk measures and derive formulations of the RCIP and a robust node covering problem.

3.3.1 Review of Value at Risk (VaR) and Conditional Value at Risk (CVaR)

The first percentile constraint we examine is the simplest risk measure used in optimization of robust systems and is known as the Value at Risk (VaR) measure [14]. VaR provides an upper bound, or percentile on a given loss distribution. For example, consider an application in which a constraint must be satisfied within a specific confidence level $\alpha \in [0, 1]$. Then the corresponding α-VaR value is the lowest value ζ such that with probability α, the loss does not exceed ζ [19]. In economic terms, VaR is simply the maximum amount at risk to be lost from an investment. VaR is the most widely applied risk measure in stochastic settings primarily because it is conceptually simple and easy to incorporate into a mathematical model [7]. However with this ease of use come several complicating factors as we will soon see. Some disadvantages of VaR are that the inclusion of VaR constraints adds to the number of discrete variables in a problem. Also, VaR is not a so-called *coherent* risk measure, implying among other things that it is not sub-additive.

Another commonly applied risk measure is the so-called Conditional Value-at-Risk (CVaR) developed by Rockafellar and Uryasev [24]. CVaR is a more conservative measure of risk, defined as the expected loss under the condition that VaR is exceeded. A graphical representation of the relationship between CVaR and VaR is shown in Figure 1.2. In order to define CVaR and Var we need to determine the cumulative distribution function for a given decision vector subject to some uncertainties. Suppose $f(x, y)$ is a performance (or loss) function associated with a decision vector $x \in X \subseteq \mathbb{R}^n$, and a random vector $y \in \mathbb{R}^m$ which is the uncertainties that may affect the performance. Assume that y is governed by a probability measure P on a Borel set, say Y [7]. Then, the loss $f(x, y)$ for each $x \in X$ is a random variable having a distribution in $\mathbb{R}$ induced by y. Then the probability of $f(x, y)$ not exceeding some value ζ is defined as

$$\psi(x, \zeta) := P\{y|f(x, y) \leq \zeta\}. \tag{3.50}$$

By fixing x, the cumulative distribution function of the loss associated with the decision x is thus given by $\psi(x, \zeta)$ [26].

Given the loss random variable $f(x, y)$ and any $\alpha \in (0, 1)$, we can use equation (3.50) to define α-VaR as

$$\zeta_\alpha(x) := \min\{\zeta \in \mathbb{R} : \psi(x, \zeta) \geq \alpha\}. \tag{3.51}$$

From this we see that the probability that the loss $f(x, y)$ exceeds $\zeta_\alpha(x)$ is $1 - \alpha$. Using the definition above, CVaR is the conditional expectation that the loss according to the decision vector x dominates $\zeta_\alpha(x)$ [24]. Thus we have α-CVaR denoted as $\phi_\alpha(x)$ defined as

$$\phi_\alpha(x) := E\{f(x, y)|f(x, y) \geq \zeta_\alpha(x)\}. \tag{3.52}$$

In order to include CVaR and VaR constraints in optimization models we must characterize $\zeta_\alpha(x)$ and $\phi_\alpha(x)$ in terms of a function $F_\alpha : X \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$F_\alpha(x, \zeta) := \zeta + \frac{1}{(1 - \alpha)} E\{\max\{f(x, y) - \zeta, 0\}\}. \tag{3.53}$$

In the seminal paper on CVaR [24], Rockafellar and Uryasev prove that as a function of ζ, $F_\alpha(x,\zeta)$ is convex and continuously differentiable. Moreover, they show that α-CVaR of the loss associated with any $x \in X$, i.e. $\phi_\alpha(x)$, is equal to the global minimum of $F_\alpha(x,\zeta)$, over all $\zeta \in \mathbb{R}$. Further, if $A_\alpha(x) = \operatorname{argmin}_{\zeta\in\mathbb{R}} F_\alpha(x,\zeta)$ is the set consisting of the values of ζ for which F is minimized, then $A_\alpha(x)$ is a non-empty, closed and bounded interval and $\zeta_\alpha(x)$ is the left endpoint of $A_\alpha(x)$. In particular, it is always the case that $\zeta_\alpha(x) \in \operatorname{argmin}_{\zeta\in\mathbb{R}} F_\alpha(x,\zeta)$ and $\psi_\alpha(x) = F_\alpha(x,\zeta_\alpha(x))$ [24].

This result gives a linear optimization algorithm for computing α-CVaR. It is a result of the convexity of $F_\alpha(x,\zeta)$, that we are able to minimize CVaR for $x \in X$ without having to numerically calculate $\phi_\alpha(x)$ for every x. This has been shown by Rockafellar and Uryasev in [25]. Further, it has been shown in [25] that for any probability threshold α and loss tolerance ω, that constraining $\phi_\alpha(x) \le \omega$ is equivalent to constraining $F_\alpha(x,\zeta) \le \omega$.

3.3.2 Robust Jamming with Percentile Constraints

Now that we have theoretical groundwork for the VaR and CVaR percentile measures, we develop formulations of the robust jamming problems incorporating these risk constraints. We begin with the ROBUST NODE COVERING PROBLEM. Since we are given a set of possible network scenarios, we would like to a develop formulation for the RNCP in which the optimal solution will be guaranteed to cover some predetermined fraction $\alpha \in (0,1)$ of the network nodes, regardless of the scenario realized. To do this, we can include α-VaR constraints in the RNCP as follows. First we define the surjection $\rho^s : \mathcal{M}_s \mapsto \{0,1\}$ by

$$\rho_i^s := \begin{cases} 1, & \text{if node } i \text{ is jammed in scenario } s, \\ 0, & \text{otherwise.} \end{cases} \tag{3.54}$$

Then we can formulate the ROBUST NODE COVERING PROBLEM with Value-at-Risk constraints as

$$\textbf{(RNCP-VaR)} \quad \min \quad \sum_{k=1}^{n} c_k x_k, \tag{3.55}$$

s.t.

$$\sum_{k=1}^{n} d_{ik}^s x_k \ge C_i^s \rho_i^s, \ \forall\, s \in \mathcal{S}, \forall\, i \in \mathcal{M}_s, \tag{3.56}$$

$$\sum_{i=1}^{m_s} \rho_i^s \ge \alpha m_s, \ \forall\, s \in \mathcal{S}, \tag{3.57}$$

$$x_k \in \{0,1\}, \ \forall\, k \in \mathcal{N}, \tag{3.58}$$

$$\rho_i^s \in \{0,1\}, \ \forall\, s \in \mathcal{S}, \forall\, i \in \mathcal{M}_s, \tag{3.59}$$

Notice that to include the VaR constraints an additional m_s binary variables are required for each scenario $s \in \mathcal{S}$.

In a similar manner, we can incorporate VaR constraints into the RCIP by introducing $\omega^s : \mathcal{M}_s \mapsto \{0,1\}$ as

$$\omega_i^s := \begin{cases} 1, & \text{connectivity index of node } i \text{ is constrained on scenario } s, \\ 0, & \text{no requirement on connectivity index of node } i. \end{cases} \tag{3.60}$$

Using this we can have

$$\textbf{(RCIP-VaR)} \quad \min \quad \sum_{k=1}^{n} c_k x_k \tag{3.61}$$

s.t.

$$M(1 - z_i^s) > \sum_{k=1}^{n} d_{ik}^s x_k - C_i^s \geq -M z_i^s, \; \forall\, s \in \mathcal{S}, \; \forall\, i \in \mathcal{M}_s, \tag{3.62}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{m_s} y_{ij}^s \leq L_i^s \omega_i^s + M(1 - \omega_i^s), \; \forall\, s \in \mathcal{S}, \; \forall\, i \in \mathcal{M}_s, \tag{3.63}$$

$$\sum_{j=1}^{m_s} \omega_j^s \geq \alpha m_s, \; \forall\, s \in \mathcal{S}, \tag{3.64}$$

$$x_k \in \{0, 1\}, \; \forall\, k \in \mathcal{N}, \tag{3.65}$$

$$z_i^s, \omega_i^s \in \{0, 1\}, \; \forall\, s \in \mathcal{S}, \; \forall\, i \in \mathcal{M}_s, \tag{3.66}$$

$$y_{ik}^s \in \{0, 1\}, \; \forall\, s \in \mathcal{S}, \; \forall\, i, k \in \mathcal{M}_s, \tag{3.67}$$

where $M \in \mathbb{R}$ is some large constant. As with the **RNCP-VaR** formulation, an additional m_s binary variables are required for each scenario $s \in \mathcal{S}$. We will see in the following section the dramatic effect that the inclusion of VaR constraints has on the computability of optimal solutions.

In order to develop formulations incorporating CVaR constraints, we must derive an appropriate loss function for each problem. We will begin with the RNCP. As in [7], we introduce the function $f^s : \{0, 1\}^n \times \mathcal{M}_s \mapsto \mathbb{R}$ defined by

$$f^s(x, i) := C_i^s - \sum_{j=1}^{n} x_j d_{ij}^s. \tag{3.68}$$

Given a decision vector x representing the placement of the jamming devices, the loss function is defined as the difference between the energy required to jam network node i, namely C_i^s, and the cumulative amount of energy received at node i due to x over each scenario [7]. With this we can formulate the RNCP with CVaR constraints as follows.

$$\textbf{(RNCP-CVaR)} \quad \min \sum_{k=1}^{n} c_k x_k, \tag{3.69}$$

s.t.

$$\zeta^s + \frac{1}{(1 - \alpha) m_s} \sum_{i=1}^{m_s} \max \left\{ C_{min}^s - \sum_{k=1}^{n} d_{ik}^s x_k - \zeta^s, 0 \right\} \leq 0, \; \forall\, s \in \mathcal{S}, \tag{3.70}$$

$$x_k \in \{0, 1\}, \; \forall\, k \in \mathcal{N}, \tag{3.71}$$

$$\zeta^s \in \mathbb{R}, \; \forall\, s \in \mathcal{S}. \tag{3.72}$$

The CVaR constraint (3.70) implies that for the $(1 - \alpha) \cdot 100\%$ of the worst (least) covered nodes, the average value of $f(x)$ is less than or equal to 0.

In a similar manner, we can formulate the ROBUST CONNECTIVITY INDEX PROBLEM with the addition of CVaR constraints. As before, we need to define an appropriate loss function. We define the loss function f_s' for a network node i in scenario s as the difference between the connectivity index of i and the maximum allowable connectivity index L_i^s which occurs as a result of the placement of the jamming devices according to x. That is, let $f' : \{0, 1\}^n \times \mathcal{M}_s \mapsto \mathbb{Z}$ be defined by

$$f_s'(x, i) := \sum_{j=1, j \neq i}^{m_s} y_{ij}^s - L_i^s. \tag{3.73}$$

With this, the CIP-CVaR formulation is given as follows.

$$\textbf{(RCIP-CVaR)} \quad \min \sum_{k=1}^{n} c_k x_k, \tag{3.74}$$

s.t.

$$\zeta^s + \frac{1}{(1-\alpha)m_s} \sum_{i=1}^{m_s} \max \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{m_s} y_{ij}^s - L_{max}^s - \zeta^s, 0 \right\} \leq 0, \; \forall\, s \in \mathcal{S}, \tag{3.75}$$

$$x_k \in \{0,1\}, \; \forall\, k \in \mathcal{N}, \tag{3.76}$$

$$y_{ik}^s \in \{0,1\}, \; \forall\, s \in \mathcal{S}, \; \forall\, i,k \in \mathcal{M}_s, \tag{3.77}$$

$$\zeta^s \in \mathbb{R}, \; \forall\, s \in \mathcal{S}, \tag{3.78}$$

where L_{max}^s is a maximum allowable connectivity index under scenario s which occurs as a result of the placement of the jamming devices. The constraint on CVaR provides that for the $(1-\alpha) \cdot 100\%$ of the worst cases, the average connectivity index will not exceed $L_{\max}$. Notice that to include the CVaR constraint, we only add real variables to the problem. The continuous nature of CVaR variables versus the discrete nature of the VaR variables will explain the vast difference in the computation times in the case studies presented in the following section.

3.4 Case Studies

In this section, we present some preliminary numerical results comparing the performance and solution quality of the proposed formulations. The experiments were performed on a PC equipped with a 1.4MHz Intel Pentium R 4 processor with 1GB of RAM, working under the Microsoft Windows R XP SP2 operating system. The optimal solutions for the case studies were calculated using CPLEX 9.0.

The problem considered is relatively small, but provides some insights into the solutions obtained using VaR and CVaR constraints. The network consists of 20 nodes which must be jammed. For this problem, we consider five network scenarios. We note here that the jamming thresholds of the nodes do not depend upon the scenarios. As for the placement of the jamming devices, we use the same approach as in [7], which consists of a 36 potential locations located on the vertices of a uniform grid placed over the region containing the network. One network scenario showing the potential locations of the jamming devices is shown in Figure 3.1. The remaining scenarios are depicted in Figure 3.2.

3.4.1 Node Covering Problems

We begin by examining the ROBUST NODE COVERING PROBLEM. For the first case study, we consider the RNCP with Value-at-Risk constraints. For this case, the loss threshold for this case is .9 which implies that the covering constraints must be satisfied for at least 90% of the network nodes. The optimal solution for this case requires 9 jamming devices. CPLEX computed the solution for this problem in 18 seconds. The results of this instance are provided in Table 3.1. The table shows the total jamming level as a percentage of the jamming threshold received by each node in each scenario. Notice that all but 7 (over all scenarios) were totally jammed; however, for those nodes not totally jammed VaR constraints provide no guarantee that they will receive any jamming energy whatsoever. Though not an important factor in this case, this fact could potentially lead to problems in large-scale instances of the problem.

Next, we examine the same problem only replacing the VaR constraints with Conditional Value-at-Risk constraints. As before, the loss threshold is .90, implying that the maximum allowable losses (uncovered nodes) exceeding VaR must no greater than 10%. Interestingly, the optimal solution in this case also requires 9 jamming devices. However this solution was computed in 0.922 seconds. The results from this study are shown in Table 3.2. Notice in this case that with the same number of jamming devices all but 2 nodes (across all scenarios) were totally jammed. We see that not only is this solution better in terms of the total number of jammed nodes, but it was also computed in an order of magnitude less time.

Table 3.1: Network coverage with VaR constraints. The total jamming level (%) for each scenario is shown.

Node	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
1	100	100	100	100	100
2	100	100	100	100	100
3	100	100	100	100	100
4	100	100	100	100	100
5	100	100	100	100	100
6	100	100	100	100	100
7	100	100	100	100	100
8	100	100	100	100	100
9	100	100	100	100	100
10	100	100	100	100	100
11	100	100	100	100	100
12	100	100	100	100	100
13	100	100	100	100	79.31
14	100	100	100	100	100
15	100	75.74	100	100	100
16	86.45	81.86	100	57.84	100
17	100	100	100	100	100
18	100	100	100	100	100
19	100	100	100	100	100
20	86.47	100	100	65.24	100

3.4.2 Connectivity Index Problems

Now we discuss the results of the case study for the RSCIP with VaR and CVaR constraints. In this case, both maximum connectivity index is $L = 3$. Again, the VaR threshold is $.90$. The optimal solution for this problem (without percentile constraints) is shown in Figure 3.3. This solution requires 4 jamming devices and was computed in $3 : 58$. The solution using VaR constraints is shown in Figure 3.4. This solution also required 4 jamming devices, but took $8 : 49 : 43$ to compute. The same solution was also obtained using CVaR constraints in a time comparable to the original formulation. Even for this small example, we see that including VaR constraints in an optimization model often leads to drastic increases in computation times. This provides more evidence that using CVaR constraints instead is usually more efficient and provides solutions.

3.5 Conclusion

In this chapter, we develop models for jamming communication networks under uncertainty. This work extends prior work by the authors in which deterministic cases of the problems were considered [4, 7]. In particular, we have developed formulations for jamming wireless networks when the exact topology of the underlying network is unknown. We have used scenario based techniques which provide robust solutions to the problems considered. Future areas of research include investigating the required number of scenarios to accurately model the statistical properties of the data. Due to the complexity of the problems considered, heuristics and advanced cutting plane techniques should also be investigated.

Table 3.2: Network coverage with CVaR constraints. The total jamming level (%) for each scenario is shown.

Node	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
1	100	100	100	100	100
2	100	100	100	100	100
3	100	100	100	100	100
4	100	100	100	100	100
5	100	100	100	100	100
6	100	100	100	100	100
7	100	100	100	100	100
8	100	100	100	100	100
9	100	100	100	100	100
10	100	100	100	100	100
11	100	100	100	100	100
12	100	100	100	100	100
13	100	100	97.25	100	100
14	100	100	100	100	100
15	100	100	100	100	100
16	100	100	100	93.93	100
17	100	100	100	100	100
18	100	100	100	100	100
19	100	100	100	100	100
20	100	100	100	100	100

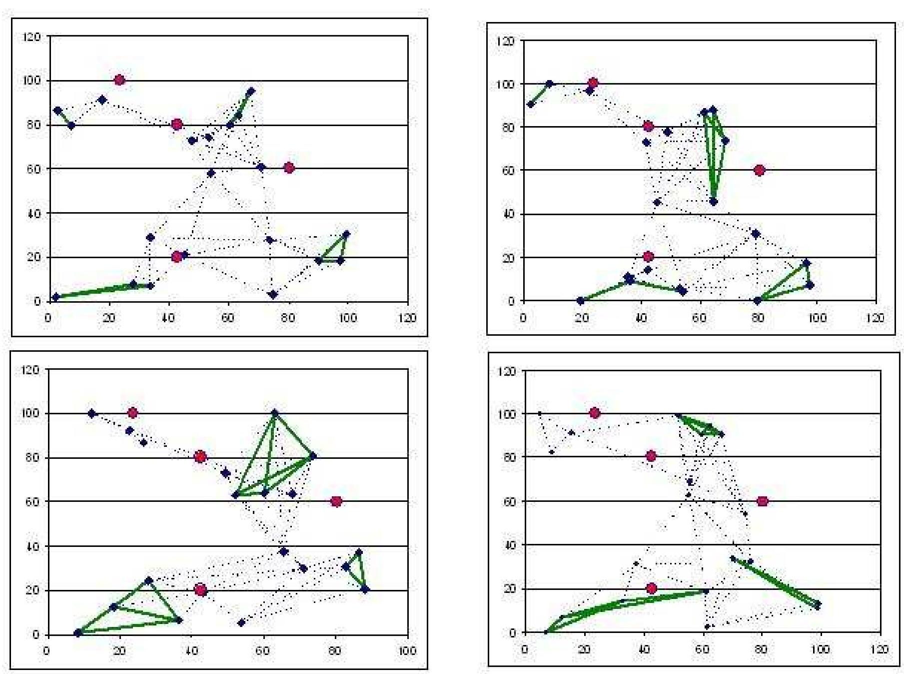


Figure 3.4: The optimal solution with VaR constraints for the networks in Figure 3.2 is given for the case when the maximum connectivity index is 3 for all nodes.

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