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POLYNOMIAL-TIME IDENTIFICATION OF OPTIMAL ROBUST NETWORK FLOWS UNDER CERTAIN ARC FAILURES (PREPRINT)

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14. ABSTRACT

Network flow problems have a wide variety of important applications in many area. Although deterministic formulations of these problems are well-studied, in many practical situation one has to deal with uncertainties associated with possible failures of network components (e.g., each arc has a probability of failure). Formulations and optimal solutions of these problems need to be adjusted to take into account these uncertainty factors. The main difficulty arising in addressing these issues is the dramatic increase in the computational complexity of the resulting optimization problems. We propose LP-based solution methods for network flow problems under a set of failure scenarios, which allow for robust solutions to be found in polynomial time. We justify this fact by proving that for network flow problems under certainty with linear loss functions, the number of scenarios required to approximate the mean of the loss distribution for any fixed e>0 with probability (1-a), for a c (0,1), is polynomial with respect to the size of the network.
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POLYNOMIAL-TIME IDENTIFICATION OF OPTIMAL ROBUST NETWORK FLOWS UNDER UNCERTAIN ARC FAILURES

VLADIMIR BOGINSKI AND CLAYTON W. COMMANDER

ABSTRACT. Network flow problems have a wide variety of important applications in many areas. Although deterministic formulations of these problems are well-studied, in many practical situations one has to deal with uncertainties associated with possible failures of network components (e.g., each arc has a probability of failure). Formulations and optimal solutions of these problems need to be adjusted to take into account these uncertainty factors. The main difficulty arising in addressing these issues is the dramatic increase in the computational complexity of the resulting optimization problems. We propose LP-based solution methods for network flow problems under a set of failure scenarios, which allows for robust solutions to be found in polynomial time. We justify this fact by proving that for network flow problems under uncertainty with linear loss functions, the number of scenarios required to approximate the mean of the loss distribution for any fixed $\varepsilon > 0$ with probability $(1-\alpha)$, for $\alpha \in (0,1]$, is polynomial with respect to the size of the network.

1. INTRODUCTION

Network flow problems are among the most well-studied topics in operations research with numerous papers and textbooks devoted to their study [1, 3, 4, 9, 14, 15, 17, 20]. These problems have a wide variety of important applications in many areas. Although deterministic formulations of these problems are well-studied, in many practical situations one has to deal with uncertainties associated with possible failures of network components (e.g., some arcs have a probability of failure). Formulations and optimal solutions of these problems need to be adjusted to take into account these uncertainty factors.

The main difficulty arising in addressing these issues is the dramatic increase in the computational complexity of the resulting optimization problems. For example, Corea and Kulkarni [6] have considered the MINIMUM COST FLOW PROBLEM where the length (cost) of each edge is a random variable. They construct continuous time Markov chains to derive stable algorithms computing the distribution of the minimum cost. The stochasticity of the edge weights makes the problem particularly challenging in that the size of the state space grows exponentially with the size of the graph. Glockner and Nemhauser [12] have also considered dynamic network flow problems where the arc capacities are random variables. They describe novel decomposition techniques for the resulting multi-stage stochastic linear program. Doulliez and Rao [10] have studied the problem of determining the maximum flow in a graph subject to a single arc failure. Along this line, Aneja et al. [2] considered the MAXIMUM FLOW PROBLEM and present a strongly polynomial algorithm for maximizing the residual flow in a graph after a single arc is destroyed. Cormican et al. [7] have formulated stochastic programming formulations of the NETWORK INTERDICTION PROBLEM in which arcs are removed in order to minimize the expected maximum flow on the network. These problems are modeled as two-stage stochastic programming problems and are $\mathcal{NP}$-hard.

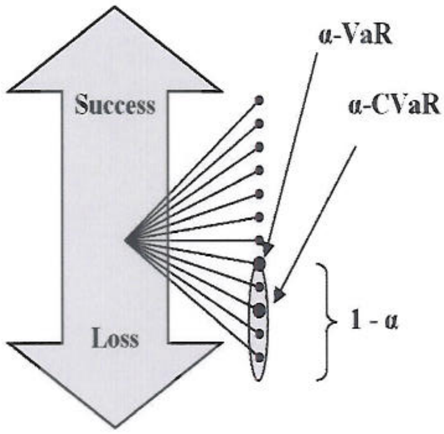


FIGURE 1. A graphical depiction of VaR and CVaR [5].

In this paper, we propose linear programming based solution methods for network flow problems under uncertainty. In particular, we provide a *polynomially solvable* formulation for the ROBUST MINIMUM COST FLOW PROBLEM (RMCF), in which each arc has a probability of failure. To our knowledge, this is the first formulation of its type. Moreover, we prove for linear programming problems under uncertainty with linear loss functions, the required number of scenarios needed in order to ensure that the sample mean is within $\varepsilon > 0$ of the true mean with probability $(1-\alpha)$, for $0 < \alpha \leq 1$, is polynomial in the size of the input. To our knowledge, this is the first time such questions have been considered for optimization problems under uncertainty.

2. SCENARIO-BASED APPROACHES FOR OPTIMIZATION PROBLEMS UNDER UNCERTAINTY

Optimization problems that involve uncertainties are typically formulated using random samples (or, *scenarios*) from loss distributions [8]. An important question that needs to be answered is, *how many scenarios are needed to accurately reflect the statistical properties of the loss distributions with a high confidence level*? More specifically, does the required number of scenarios grow *polynomially* or *exponentially* with respect to the size of the input? Clearly this is an important problem, and to the best of our knowledge has not been previously addressed in the literature on optimization under uncertainty. In this paper, we will formulate the ROBUST MINIMUM COST FLOW PROBLEM under uncertainty and prove that for network optimization problems with linear loss functions, for any fixed positive $\varepsilon > 0$ and $\alpha \in (0,1]$, $O(m^2)$ scenarios are sufficient to ensure that the sample mean of the total loss associated with possible edge failures is within ε from the true mean with probability $(1-\alpha)$ (where m is the number of edges in the network).

Next, we discuss the basic statistical concepts (sometimes referred to as *risk measures*), which we utilize in this study.

2.1. **Quantitative Loss Measures.** One of the most well-known risk measures used in robust optimization under uncertainty is known as *Value-at-Risk* (VaR) [13]. VaR provides an upper bound, or percentile on a given loss distribution. For example, consider an application in which a constraint must be satisfied within a specific confidence level $\alpha \in (0,1]$.

Then the corresponding α-VaR value is the lowest value ζ such that with probability α, the loss does not exceed ζ [16]. In economic terms, VaR is simply the maximum amount at risk to be lost from an investment. VaR is the most widely applied risk measure in probabilistic settings primarily because it is conceptually simple and easy to incorporate into a mathematical model [5]. However with this ease of use come several complicating factors. Some disadvantages are that the inclusion of VaR constraints increases the number of discrete variables in a problem. Thus a polynomially solvable problem is likely to become $\mathcal{NP}$-hard [11] after the VaR constraints are added to the model. Also, VaR is not a so-called *coherent* risk measure, implying among other things that it is non-convex and not sub-additive.

Another risk measure closely related to VaR is the so-called Conditional Value-at-Risk (CVaR). CVaR is a more conservative measure of risk, defined as the *conditional expectation* of the loss under the condition that VaR is exceeded. Rockafellar and Uryasev [18] proved several important results regarding optimization of CVaR, which make this risk measure rather attractive from the optimization viewpoint. In particular, CVaR has been shown to possess the properties that VaR lacks; in particular, it is *coherent* (which includes *convexity* among other properties). This makes this statistical measure much more convenient to handle in optimization models. A graphical representation of the relationship between CVaR and VaR is shown in Figure 1. In order to define CVaR and Var we need to determine the cumulative distribution function for a given decision vector subject to some uncertainties. Suppose $L(x,y)$ is a loss function associated with a decision vector $x \in X \subseteq \mathbb{R}^n$, and a random vector $y \in \mathbb{R}^m$ which is the uncertainties that may affect the performance. Assume that y is governed by a probability measure P on a Borel set, say Y [5]. Then the loss $L(x,y)$ for each $x \in X$ is a random variable having a distribution in $\mathbb{R}$ induced by that of y. Therefore the probability of $L(x,y)$ not exceeding some value ζ is defined as

$$\psi(x,\zeta) := P\{y|L(x,y) \leq \zeta\}. \tag{1}$$

By fixing x, the cumulative distribution function of the loss associated with the decision x is thus given by $\psi(x,\zeta)$ [21].

Given the loss random variable $L(x,y)$ and any $\alpha \in (0,1)$, we can use equation (1) to define α-VaR as

$$\zeta_\alpha(x) := \min\{\zeta \in \mathbb{R} : \psi(x,\zeta) \geq \alpha\}. \tag{2}$$

From this we see that the probability that the loss $L(x,y)$ exceeds $\zeta_\alpha(x)$ is $1-\alpha$. Using the definition above, CVaR is the conditional expectation that the loss according to the decision vector x dominates $\zeta_\alpha(x)$ [18]. Thus we have α-CVaR denoted as $\phi_\alpha(x)$ defined as

$$\phi_\alpha(x) := E\{L(x,y)|L(x,y) \geq \zeta_\alpha(x)\}. \tag{3}$$

In order to include CVaR and VaR constraints in optimization models we must characterize $\zeta_\alpha(x)$ and $\phi_\alpha(x)$ in terms of a function $F_\alpha : X \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$F_\alpha(x,\zeta) := \zeta + \frac{1}{(1-\alpha)} E\{\max\{L(x,y)-\zeta, 0\}\}. \tag{4}$$

In [18], Rockafellar and Uryasev prove that as a function of ζ, $F_\alpha(x,\zeta)$ is convex and continuously differentiable. Moreover, they show that α-CVaR of the loss associated with any $x \in X$, i.e., $\phi_\alpha(x)$, is equal to the global minimum of $F_\alpha(x,\zeta)$, over all $\zeta \in \mathbb{R}$. Further, if $A_\alpha(x) := \operatorname{argmin}_{\zeta\in\mathbb{R}} F_\alpha(x,\zeta)$ is the set consisting of the values of ζ for which F is minimized, then $A_\alpha(x)$ is a non-empty, closed and bounded interval and $\zeta_\alpha(x)$ is the left endpoint of $A_\alpha(x)$. In particular, it is always the case that $\zeta_\alpha(x) \in \text{argmin}_{\zeta \in \mathbb{R}} F_\alpha(x,\zeta)$ and $\psi_\alpha(x) = F_\alpha(x, \zeta_\alpha(x))$ [18].

This result gives a linear optimization algorithm for computing α-CVaR. It is a result of the convexity of $F_\alpha(x,\zeta)$, that we are able to minimize CVaR for $x \in X$ without having to numerically calculate $\phi_\alpha(x)$ for every x. This has been shown by Rockafellar and Uryasev in [19]. Further, it has been shown in [19] that for any probability threshold α and loss tolerance C, that constraining $\phi_\alpha(x) \leq C$ is equivalent to constraining $F_\alpha(x,\zeta) \leq C$.

3. Minimum Cost Flow Problem under Uncertainty with CVaR Constraints

An instance of the MINIMUM COST FLOW PROBLEM (MCF) consists of a directed graph $G = (V,E)$, where each edge (arc) $(i,j) \in E$ has an associated cost c_{ij} per unit of flow along this edge, as well as a capacity u_{ij} denoting the maximum amount of flow that can traverse edge (i,j). For each node $i \in V$, d_i denotes demand (supply) of node i. Then the MCF can be formulated as the following linear program

$$\textbf{(MCF)} \quad \min \quad \sum_{(i,j)\in E} c_{ij}x_{ij} \tag{5}$$

s.t.

$$\sum_{\{j:(i,j)\in E\}} x_{ij} - \sum_{\{j:(j,i)\in E\}} x_{ji} = d_i, \ \forall\, i \in V, \tag{6}$$

$$0 \leq x_{ij} \leq u_{ij}, \ \forall\, i,j \in V. \tag{7}$$

Suppose that $L(x,y)$ is a *loss function*, where $x : V \times V \mapsto \{0,1\}$ is a vector of decision variables. In the context of a network flow problem, x_{ij} represents the total amount of flow through arc (i,j). Further let y be a random vector representing the edge failures in the graph defined as follows:

$$y_{ij} := \begin{cases} 1, & \text{with probability } p_{ij}, \\ 0, & \text{with probability } 1 - p_{ij}, \end{cases}$$

where p_{ij} is the probability of failure for the arc (i,j).

Furthermore, for each scenario $s = 1,2,\ldots,S$,

$$y^s_{ij} := \begin{cases} 1, & \text{with probability } p_{ij}, \\ 0, & \text{with probability } 1 - p_{ij}. \end{cases} \tag{8}$$

Then, the random variable $L(x,y)$ representing the total loss is given as

$$L(x,y) = \sum_{i,j=1}^{n} x_{ij}y_{ij}. \tag{9}$$

Further, for each scenario $s = 1,\ldots,S$,

$$L^s(x,y) = \sum_{i,j=1}^{n} x_{ij}y^s_{ij}. \tag{10}$$

Then the characteristic function $F_\alpha(x,\zeta)$ for CVaR constraints is given as

$$F_\alpha(x,\zeta) = \zeta + \frac{1}{1-\alpha}\sum_{s=1}^{S} \pi_s \max\left\{\sum_{s=1}^{S}\sum_{i,j=1}^{n} x_{ij}y^s_{ij} - \zeta, 0\right\}, \tag{11}$$

where π_s is the probability of scenario y^s, for $s = 1,2,\ldots,S$. Since $L(x,y)$ is linear with respect to x, $F_\alpha(x,\zeta)$ is convex and piecewise linear [16]. If we assume that each scenario is equally likely, that is $\pi_s = \frac{1}{S}$, $\forall\, s = 1,2,\ldots,S$, then (11) reduces to

$$F_\alpha(x,\zeta) = \zeta + \frac{1}{S(1-\alpha)} \sum_{s=1}^{S} \max\left\{ \sum_{i,j=1}^{n} x_{ij} y_{ij}^s - \zeta, 0 \right\}. \tag{12}$$

Then we can model the the *robust* MCF problem with CVaR constraints as

$$\textbf{(RMCF-1)} \quad \text{Minimize} \quad \sum_{\{(i,j)\in E\}} c_{ij} x_{ij} + \sum_{i\in V} M_i p_i \tag{13}$$

s.t.

$$\sum_{\{j:(i,j)\in E\}} x_{ij} - \sum_{\{j:(j,i)\in E\}} x_{ji} = d_i - p_i, \forall\, i \in V, \tag{14}$$

$$0 \le x_{ij} \le u_{ij}\ \forall\, (i,j) \in E,$$

$$\zeta + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} \max\left\{ \sum_{i,j=1}^{n} x_{ij} y_{ij}^s - \zeta, 0 \right\} \le C, \tag{15}$$

$$\zeta \in \mathbb{R}, \tag{16}$$

$$p_i \ge 0,\ \forall\, i \in V. \tag{17}$$

Note that the mass-balance constraints in (6) from formulation **MCF** have been relaxed in formulation **RMCF-1**. This modification is made in order to ensure that arc failures do not render the instance infeasible. Notice that *penalty terms* representing the possible loss of flow (e.g., unsatisfied demand) at node i have been added to the objective function and the corresponding constraints. Clearly, it is desired that the penalty variables p_i are equal to zero in the optimal solution, which is modeled by multiplying these variables by sufficiently large coefficients M_i in the objective.

We can linearize $F_\alpha(x,\zeta)$ by using dummy variables t_s, $s = 1,2,\ldots,S$, and replacing $F_\alpha(x,\zeta)$ by the linear function $\zeta + \frac{1}{S(1-\alpha)} \sum_{s=1}^{S} t_s$ and adding the set of linear constraints

$$t_s \ \ge\ L(x,y^s) - \zeta,\ \forall\, s = 1,2,\ldots,S, \tag{18}$$

$$t_s \ \ge\ 0,\ \forall\, s = 1,2,\ldots,S. \tag{19}$$

Thus, the fully linearized mathematical programming formulation of the ROBUST MINIMUM COST FLOW PROBLEM with Conditional Value-at-Risk constraints is given as

$$\textbf{(RMCF-LP)} \quad \text{Minimize} \quad \sum_{(i,j)\in E} c_{ij}x_{ij} + \sum_{i\in V} M_i p_i \tag{20}$$

s.t.

$$\sum_{\{j:(i,j)\in E\}} x_{ij} - \sum_{\{j:(j,i)\in E\}} x_{ji} = d_i - p_i,\ \forall\, i \in V, \tag{21}$$

$$\zeta + \frac{1}{S(1-\alpha)} \sum_{i=1}^{S} t_s \leq C, \tag{22}$$

$$t_s \geq \sum_{i,j=1}^{n} x_{ij} y_{ij}^s - \zeta,\ \forall\, s = 1,2,\ldots,S, \tag{23}$$

$$t_s \geq 0,\ \forall\, s = 1,2,\ldots,S, \tag{24}$$

$$\zeta \in \mathbb{R}, \tag{25}$$

$$p_i \geq 0,\ \forall\, i \in V, \tag{26}$$

$$0 \leq x_{ij} \leq u_{ij},\ \forall\, i,j \in E. \tag{27}$$

Note that this LP formulation contains $O(|V|+|E|+S)$ variables and $O(|V|+|E|+S)$ constraints. Therefore, the subject of further investigation is how large the number of scenarios S needs to be to ensure sufficiently good statistical properties of the optimal robust solutions. In the next section, we address this issue and identify the conditions under which S is polynomial with respect to $|E|$ (and $|V|$), which makes the proposed problem formulation polynomially solvable.

4. On the Required Number of Scenarios

In this section, we discuss the issues of obtaining a "good" approximation of the true distribution of the losses in the above formulations. More specifically, we investigate the issue of the *required* number of scenarios that would ensure that the true mean of the loss distribution is close enough to the sample (scenario-based) mean with a high confidence level. In particular, the subject of special interest would be the dependency of the required number of scenarios on the input size, e.g., the size of the network. This is particularly important as the number of scenarios considered will directly affect the number of additional constraints in the model. In particular, if the required number of scenarios needed is exponential in the size of the input, then the polynomial solvability of the problem is undermined by the exponential number of constraints. In this section, we prove that the required number of samples needed to accurately approximate the mean of the distribution of a linear loss function is in fact polynomial in the size of the input (e.g., the number of nodes and edges in the network) under certain conditions.

Let the sample mean loss be given as

$$\bar{L} := \frac{L_1 + L_2 + \cdots + L_S}{S} = \frac{1}{S}\sum_{s=1}^{S} L_s = \frac{1}{S}\sum_{s=1}^{S}\sum_{i,j=1}^{n} x_{ij} y_{ij}^s. \tag{28}$$

Also, let sample mean of each component of the random vector y_{ij} be defined as $\bar{y}_{ij} := \frac{1}{S}\sum_{s=1}^{S} y_{ij}^s$. Let the true mean of the loss distribution be μ_L. With this, we have the following proposition.

Proposition 1. *Consider a linear loss function for a network flow problem, such as the one defined above. Then for any $\varepsilon, \alpha \in \mathbb{R}$ such that $\varepsilon > 0$, and $\alpha \in (0,1]$, the required number* *of scenarios S to guarantee that $P(|\bar{L}-\mu_L|<\varepsilon)>1-\alpha$ is $O(m^2/\varepsilon^2)$, where $m=|E|$ is the number of edges in the original network. Furthermore, for any fixed ε, $S=O(m^2)$.*

Proof.

$$P(|\bar{L}-\mu_L|<\varepsilon) = P\left(|\frac{1}{S}\sum_{s=1}^{S}\sum_{i,j=1}^{n} x_{ij}y_{ij}^s - \sum_{i,j=1}^{n} x_{ij}\mu_{ij}|<\varepsilon\right) \quad (29)$$

$$= P\left(|\sum_{i,j=1}^{n} x_{ij}\left(\frac{1}{S}\sum_{s=1}^{S} y_{ij}^s - \mu_{ij}\right)|<\varepsilon\right) \quad (30)$$

$$= P\left(|\sum_{i,j=1}^{n} x_{ij}(\bar{y}_{ij}-\mu_{ij})|<\varepsilon\right). \quad (31)$$

Now, let $U:=\max_{i,j}\{u_{ij}\}$ be the maximum capacity of any single arc in the network. Then continuing from (31) we have that

$$P\left(|\sum_{i,j=1}^{n} x_{ij}(\bar{y}_{ij}-\mu_{ij})|<\varepsilon\right) \geq P\left(mU\max_{i,j}\{|\bar{y}_{ij}-\mu_{ij}|\}<\varepsilon\right) \quad (32)$$

$$= P\left(\max_{i,j}\{|\bar{y}_{ij}-\mu_{ij}|\}<\frac{\varepsilon}{mU}\right)>1-\alpha. \quad (33)$$

Let i^* and j^* be the indices for which $|\bar{y}_{ij}-\mu_{ij}|$ is maximum, and let $\sigma^2_{i^*j^*}$ be the variance of $y_{i^*j^*}$. Then, using the notation c_α representing the critical value of the normal distribution corresponding to the confidence level $(1-\alpha)$ (e.g., for $1-\alpha=0.95$, $c_\alpha=1.96$) the minimum required number of scenarios S, needed to satisfy the condition above, can be found as follows.

$$\frac{c_\alpha\sigma^2_{i^*j^*}}{\sqrt{S}} = \frac{\varepsilon}{mU} \quad (34)$$

$$\Rightarrow S = \frac{c_\alpha^2\sigma^2_{i^*j^*}U^2}{\varepsilon^2}m^2 = O(m^2/\varepsilon^2). \quad (35)$$

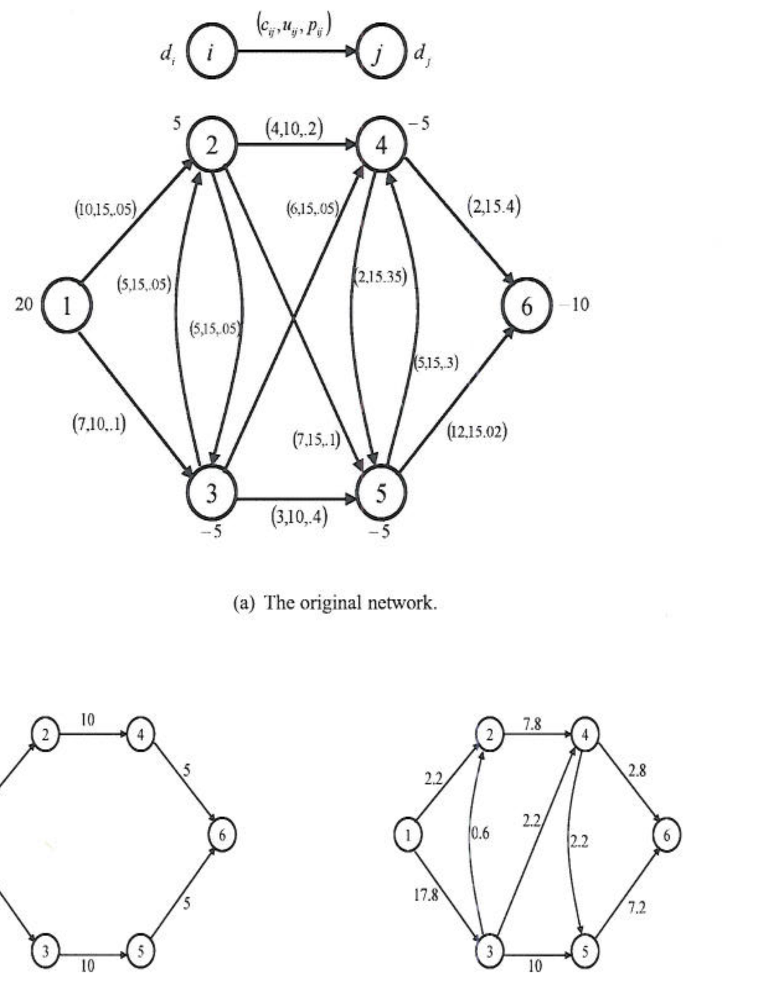
Furthermore, for any fixed ε, $S=O(m^2)$. Thus we have the desired result. □

Note that without the loss of generality, the result of Proposition 1 holds for any loss functions that are linear with respect to the uncertain parameters y, where these parameters can denote any uncertainties in the model, such as costs, capacities, or other application specific parameters.

The proof of Proposition 1 provides the justification for the use of Conditional Value-at-Risk constraints for network flow problems. That is, by incorporating CVaR constraints we are able to guarantee that the sample mean of the loss distribution can be made arbitrarily accurate while the size of the problem remains polynomial with respect to the input. Thus we can find accurate, robust solutions to these problems in polynomial time using standard linear programming techniques.

5. Numerical Example

In this section, we present a numerical example demonstrating the effectiveness of the proposed approach. Consider the three graphs shown in Figure 2. Suppose we are considering the MINIMUM COST FLOW PROBLEM. Subfigure 2(a) represents the original network and the triplets (c_{ij},u_{ij},p_{ij}), correspond to the cost, maximum capacity, and probability of failure for each arc (i,j).



(a) The original network.

(b) The ideal optimal solution. The objective value is 295.

(c) The optimal solution with CVaR. In this example, $S = 100$ and the objective value is 340.2.

FIGURE 2. The min-cost flow problem is solved for the graph in (a). c_{ij}, u_{ij}, and $p_{i,j}$ represent the cost, maximum capacity, and probability of failure for arc (i, j) respectively.

In Subfigure 2(b), the optimal *ideal* solution is shown. The ideal case represents the deterministic MCF problem, or equivalently, the probabilistic version when all edge failure probabilities are 0. The optimal solution for this instance is 295 units of flow. However by taking into account the edge failure probabilities, we see that this is not the preferred solution. In 2(c), the optimal solution is shown with CVaR constraints. In this example, the total loss in the worst 10% of scenarios is constrained to be less than 0.23 units of flow. We generated 100 scenarios uniformly at random for this example, with up to 20%

of the arcs failing in any scenario. These are arbitrary values and can be specified by the user as the situation calls. In this case, the objective function value is 340.2. We see that in fact, the solution tends to push more flow over arcs with lower probabilities of failure accepting the higher cost for the hedging. This is exactly the behavior we would predict. An interpretation is that by diversifying the flow across various arcs, we are able to ensure that the solution is robust in the event of (multiple) arc failures. Moreover, note that the optimal solutions for robust formulations of the MINIMUM COST FLOW PROBLEM can be *fractional*. Recall that for the deterministic version of the MCF problem, the optimal solution values are integral if all the parameters of the original model are integer. Depending on specific applications, one can either operate with fractional flows, or use certain rounding techniques to convert the flows to integer values.

6. CONCLUSIONS

In this paper we proposed polynomial-time linear programming based solution approaches for network flow problems subject to uncertain arc failures. Further, we have shown that for linear loss functions, a polynomial number of scenarios is sufficient in order to approximate the true mean of the loss distribution arbitrarily close. A numerical example was given demonstrating the effectiveness of the proposed procedure in the context of the MINIMUM COST FLOW PROBLEM.

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