# ANGLE OF ARRIVAL OF A FOCUSED GAUSSIAN BEAM IN ATMOSPHERIC TURBULENCE

R. W. McMillan
U. S. Army Space and Missile Defense Command
Huntsville, Alabama 35807-3801

## ABSTRACT

The angles of arrival (AOA) of plane waves, spherical waves, and focused beams perturbed by atmospheric turbulence in the geometrical optics limit have been calculated by others [1]. Since geometrical optics theory allows beams to be focused to an infinitesimal point, the AOA of a focused beam in this limit becomes infinite because of a $D^{-1/3}$ dependence of this parameter, where $D$ is the beam diameter. If we apply Gaussian optics theory to this problem, the AOA is no longer infinite at the focal point because the beam size at focus is not infinitesimally small. In this paper we use Gaussian optics theory to show that the AOA varies according to the diameter and focal length of the focusing element and the wavelength of the radiation being focused, as well as the range. Calculated results are given.

## 1. INTRODUCTION

The atmosphere has significant effects on the propagation of electromagnetic waves. Among these effects are signal fading, blooming, and phase fluctuations. An effect that may be significant for laser weapon systems in particular is that of angle-of arrival fluctuations, which may cause a focused laser beam to wander off its target, resulting in the target not being destroyed. Although these effects are generally small, they can be significant for long range scenarios and tightly focused beams. In this paper we develop a simple theory of AOA for a focused beam and show that AOA varies with beam spot size and range.

## 2. THEORY

AOA has been derived for both plane and spherical waves propagating through atmospheric turbulence [1]. More recently, Churnside and Lataitis [2] have derived the AOA of a focused beam in the geometrical optics limit and have shown that their result reduces to those obtained for plane and spherical waves in the limits of infinite and zero focal lengths, respectively. They also show that the AOA approaches infinity in a predictable way as range approaches the focal length of the focusing optic. They used the simple model shown in Figure 1 as the basis of their calculations. The tilt angle $d\alpha$ is given by

$$d\alpha = \Delta n(z)dz / w(L), \tag{1}$$

so that the total tilt angle over path L is

$$\alpha = \frac{1}{w(L)}\int_0^L \Delta n(z)dz. \tag{2}$$

Using this result, and assuming that there is no average gradient of refractive index, Churnside and Lataitis derive an expression for the one-way AOA based on geometrical optics.

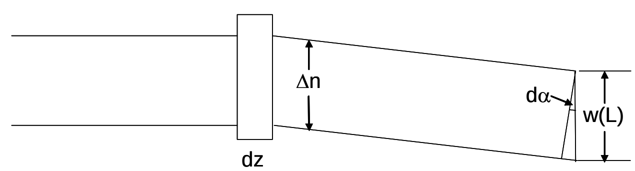


**Figure 1.** A thin atmospheric layer with varying index causes beam steering

To apply Gaussian optics theory, we rewrite the geometrical optics expression obtained in [2] for the AOA variance $\sigma_t^2$ as

$$\sigma_t^2 = 2.92\frac{C_n^2}{w^2(L)}\int_0^L [w(z)]^{5/3}dz, \tag{3}$$

where $C_n^2$ is the atmospheric turbulence structure parameter, $w(z)$ is the $1/e$ half-beam size at range $z$, and $w(L)$ is the same parameter at range $L$. Consider the Gaussian beam geometry shown in Figure 2. To calculate the AOA variance for a Gaussian beam, we substitute the expression for the Gaussian beam size [3,4]

# Report Documentation Page

*Form Approved*
*OMB No. 0704-0188*



| | | |
|---|---|---|
| 1. REPORT DATE<br>**01 NOV 2006** | 2. REPORT TYPE<br>**N/A** | 3. DATES COVERED<br>**-** |
| 4. TITLE AND SUBTITLE<br>**Angle Of Arrival Of A Focused Gaussian Beam In Atmospheric Turbulence** | | 5a. CONTRACT NUMBER<br>5b. GRANT NUMBER<br>5c. PROGRAM ELEMENT NUMBER |
| 6. AUTHOR(S) | | 5d. PROJECT NUMBER<br>5e. TASK NUMBER<br>5f. WORK UNIT NUMBER |
| 7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)<br>**U. S. Army Space and Missile Defense Command Huntsville, Alabama 35807-3801** | | 8. PERFORMING ORGANIZATION REPORT NUMBER |
| 9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) | | 10. SPONSOR/MONITOR'S ACRONYM(S)<br>11. SPONSOR/MONITOR'S REPORT NUMBER(S) |

12. DISTRIBUTION/AVAILABILITY STATEMENT
**Approved for public release, distribution unlimited**

13. SUPPLEMENTARY NOTES
**See also ADM002075.**

14. ABSTRACT

15. SUBJECT TERMS

| 16. SECURITY CLASSIFICATION OF: | | | 17. LIMITATION OF ABSTRACT | 18. NUMBER OF PAGES | 19a. NAME OF RESPONSIBLE PERSON |
|---|---|---|---|---|---|
| a. REPORT<br>**unclassified** | b. ABSTRACT<br>**unclassified** | c. THIS PAGE<br>**unclassified** | **UU** | **5** | |

**Standard Form 298 (Rev. 8-98)**
Prescribed by ANSI Std Z39-18

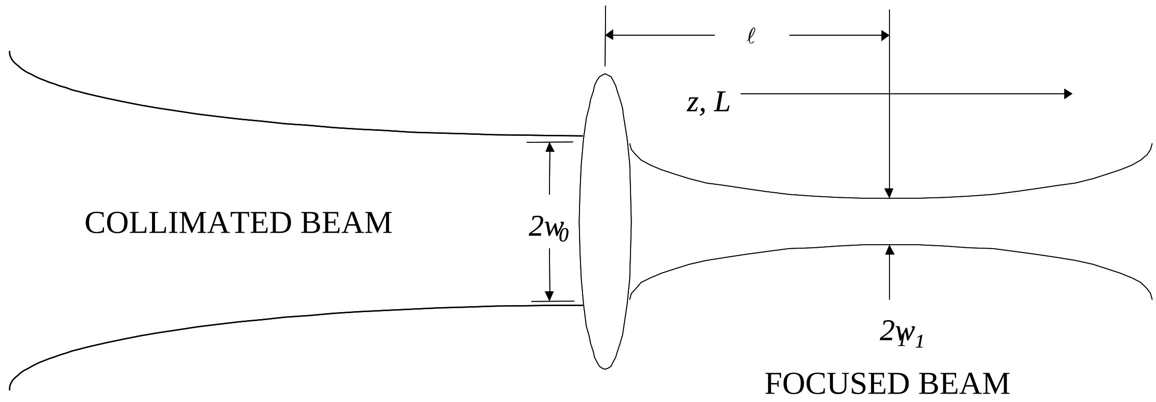


**Figure 2.** Gaussian beam geometry used in the calculation.

$$w(z) = 2w_1\left[1+\left(\lambda z/\pi w_1^2\right)^2\right]^{1/2}, \tag{4}$$

where $\lambda$ is wavelength and $2w_1$ is the *2(1/e)* beam diameter at the focal point $\ell$ as shown. We also transform the origin to $\ell$ with the result

$$\sigma_t^2 = 2.92\frac{(2w_1)^{5/3}}{4w^2(L-\ell)}\int_0^{L-\ell}\left\{1+\left[\frac{\lambda(z-\ell)}{\pi w_1^2}\right]^2\right\}^{5/6} dz. \tag{5}$$

From Yariv [4], we obtain expressions for the distance to the focal length $\ell$ and the half-beam waist dimension at $\ell$ in terms of the geometrical optics focal length $f$ as

$\ell = f/\left[1+(f/z_0)^2\right]$, where $z_0 = \pi w_0^2/\lambda$, and (6)

$$w_1^2 = \left(\frac{w_0 f}{z_0}\right)^2 / \left[1+\left(\frac{f}{z_0}\right)^2\right], \tag{7}$$

So that $\lambda\ell/\pi w_3^2 = z_0/f$. (8)

Substituting these expressions and $z/\ell = x$ into (3) above, we obtain the following equation for the AOA variance:

$$\sigma_t^2 = \frac{2.92\ell C_n^2\left[1+\left(f/z_0\right)\right]^{1/6}}{(2w_0)^{1/3}(f/z_0)^{1/3}\left\{1+\left[\frac{z_0}{f}\left(\frac{L}{\ell}-1\right)\right]^2\right\}} \bullet \int_0^{L/\ell-1}\left\{1+\left[\frac{z_0}{f}(x-1)\right]^2\right\}^{5/6} dx. \tag{9}$$

Normalizing this expression to the plane-wave variance at range $L - \ell$ for a beam size $2w_0$, which is $\sigma_p^2 = 2.92C_n^2(L-\ell)(2w_0)^{-1/3}$, we get the following expression for $\sigma_t^2/\sigma_p^2$:

$$\frac{\sigma_t^2}{\sigma_p^2} = \frac{\left[1+\left(f/z_0\right)^2\right]^{1/6}}{(f/z_0)^{1/3}(L/\ell-1)\left\{1+\left[(z_0/f)(L/\ell-1)\right]^2\right\}} \bullet \int_0^{L/\ell-1}\left\{1+\left[\frac{z_0}{f}(x-1)\right]^2\right\}^{5/6} dx. \tag{10}$$

We now show that this expression reduces to the plane-wave and spherical wave cases, respectively, for $f \to \infty$ and $f \to 0$. As $f \to \infty$, the second term in the integrand approaches zero, and integration gives simply $L/\ell -1$, which cancels the identical term in the denominator of the coefficient. The $1+(f/z_0)^2$ terms cancel as $f$ becomes very large and the bracketed term goes to zero. The result is that $\sigma_t^2/\sigma_p^2 \to 1$, showing that the AOA variances for beam and plane waves are identical for very long focal lengths.

The spherical wave case $f \to 0, L \to \infty$ is more complicated. Consider Equation (9). As $f \to 0, L \to \infty$,

$$\sigma_t^2 = \frac{2.92\,\ell C_n^2}{(2w_0)^{1/3}(f/z_0)^{1/3}(z_0/f)^2(L/\ell)^2}$$

$$\bullet \int_0^{L/\ell} \left\{1+\left[\frac{z_0}{f}(x-1)\right]^2\right\}^{5/6} dx. \tag{11}$$

As $L \to \infty$, most of the contribution to the integral occurs for $x = z/\ell >> 1$, so that

$$\int_0^{L/\ell} x^{5/3} dx = 3/8 (L/\ell)^{8/3}. \tag{12}$$

Making the indicated substitutions, we obtain

$$\sigma_t^2 = \frac{2.92(3/8)LC_n^2}{(2w_0 L/\ell)^{1/3}}. \tag{13}$$

The *2(1/e)* beam dimension at $z = L - \ell$ is

$$w(L-\ell) = \frac{(w_0 f/z_0)}{1+(f/z_0)}\left\{1+\left[\frac{z_0}{f}(L/\ell-1)\right]^2\right\}^{1/2}. \tag{14}$$

As $f \to 0$. $L \to \infty$,

$w_0(L-\ell) \to w_0(L/\ell)$ so that

$w^{1/3}(L-\ell) \to w_0^{1/3}(L/\ell)^{1/3}$.

Substituting in Equation (11) for this quantity in the denominator, we obtain

$$\sigma_t^2 = 2.92(3/8)LC_n^2\left[2w(L-\ell)\right]^{-1/3}, \tag{15}$$

which is the spherical wave result for the geometrical optics case.

There is another issue associated with the analysis detailed above. We assume that all of the radiation emanating from the optical system that focuses the beam is collected by whatever means is used for such collection. For example, if the radiation is incident on a target, all of the radiation is collected by the target. In other words, the beam is never larger than the target or the collection aperture. We must impose this condition because of the aperture averaging effect that occurs over large apertures. If the collection means is smaller than the beam, the AOA will simply be larger than that predicted above because the collection aperture is smaller. This reservation is consistent with the expression for plane-wave AOA variance preceding Equation (8) above.

## 3. RESULTS

We have integrated Equation (8) numerically for several different values of the parameter $\lambda f/\pi w_0^2 = f/z_0$, and the results are shown in Figure 3. Note that for small values of this parameter, corresponding to short wavelengths and focal lengths and large $w_0$, the angle of arrival variance is large near the beam waist, while for large values the variance is small. In other words, the smaller the spot size, the greater the AOA variance. Our results reduce to those obtained by other workers for plane and spherical waves in the limit as the focal length approaches infinity or zero, respectively, which is the geometrical optics limit. We have also calculated the AOA variance in the geometrical optics limit using the equation developed by Churnside and Lataitis [2], and the results are also shown in Figure 4. This result should be compared to that shown in Figure 3 *for* $f/z_0 = 0.1$. The result of Figure 4 is for the special case $f/z_0 = 0$. The results are similar for these two cases. These calculations aid in understanding the effects of the atmosphere on high-power laser weapon beams. Future work involves the inclusion of multiple laser cavity modes to account for less than ideal laser beam quality.

Figure 3 shows an interesting result that is apparent for large values of $f/z_0$ corresponding to long wavelengths, small focusing apertures, and long focal lengths. For these values of $f/z_0$, the AOA peak relative to that of a plane wave occurs before the beam waist, which means that the AOA peaks before that point where the beam is narrowest. There are at least three possible explanations for this behavior: (1) the narrowest point of the beam occurs at long ranges *L* because of the long focal lengths and wavelengths and the small focusing apertures. It is possible that the range becomes the dominant term in determining the AOA instead of the beam size; (2) the very simple model shown graphically in Figure 1 used to formulate the AOA variance given by Equation (1) is just too simple and does not account for this complicated phenomenon; (3) the approximations made by Churnside and Lataitis [2] lead to small errors in determining the AOA.

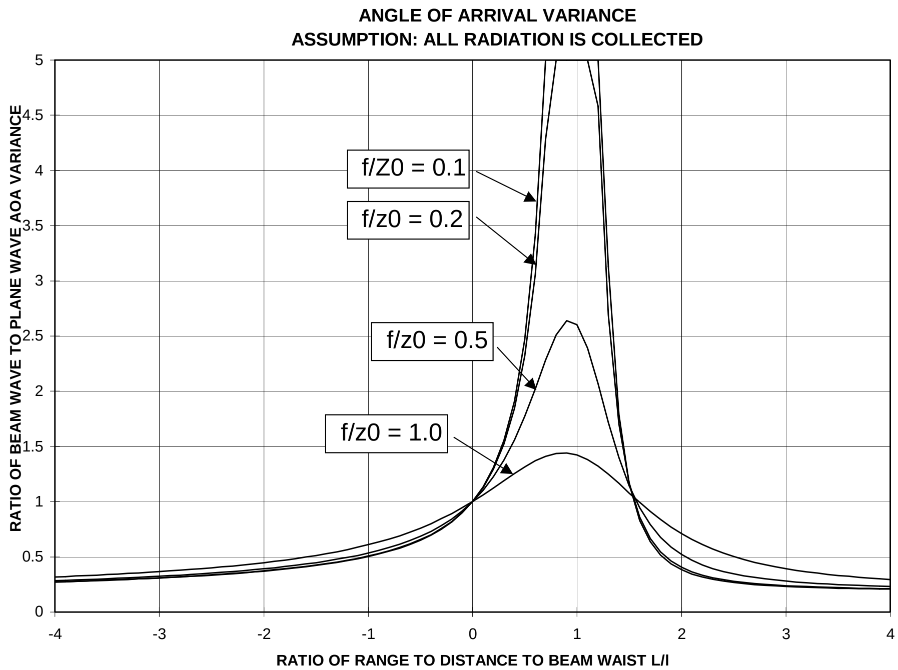


**Figure 3.** Angle of arrival variance for a focused Gaussian beam as a function of distance to beam waist.

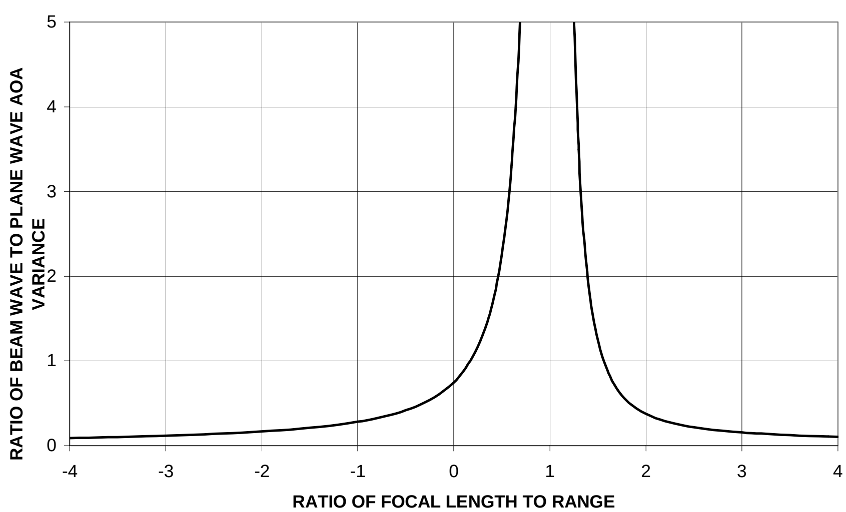


**Figure 4.** Angle of arrival variance for a geometrical optics focused beam calculated using the approach developed in Reference [2].

Regardless of the cause of this slight inconsistency, it is possible to show that it is inherent in the model used for this calculation. Consider Equation (10) and allow $\lambda f / \pi w_0^2 = f / z_0$ to become large, corresponding to the cases shown in Figure 3 for which the peak AOA occurs before the beam waist. We can then approximate the integrand of Equation (8) by the first term in its Taylor series expansion, resulting in

$$\frac{\sigma_t^2}{\sigma_p^2}=\frac{\left[1+\left(f/z_0\right)^2\right]^{1/6}}{\left(f/z_0\right)^{1/3}\left(L/\ell-1\right)\left\{1+\left[\left(z_0/f\right)\left(L/\ell-1\right)\right]^2\right\}}$$
$$\bullet\int_0^{L/\ell-1}\left\{1+\frac{5}{6}\left[\frac{z_0}{f}\left(x-1\right)\right]^2\right\}dx \qquad (14)$$

Note also that the numerator cancels the first term in the denominator for large $f/z_0$ and that the last term in the denominator becomes

$\frac{1}{1+\left[(z_0/f)(L/\ell-1\right]^2}\approx 1-\left[(z_0/f)(L/\ell-1\right]2$, using the approximation $1/(1+\varepsilon)\sim 1-\varepsilon$, where ε is a small number. The integration is now simple. Allowing $(f/z_0)$ to become large and performing the integration gives

$$\frac{\sigma_t^2}{\sigma_p^2}=\left\{1-\left[\left(z_0/f\right)\left(L/\ell\right)-1\right]^2\right\}$$
$$\bullet\left\{1+5/6\left(z_0/f\right)^2\left[1/3\left(L/\ell-1\right)^2-\left(L/\ell-1\right)+1\right]\right\} \qquad (15)$$

Differentiating this result with respect to $L/\ell$ and setting the result to zero, ignoring the terms in $(z_0/f)^2$, since this quantity is small, we obtain a quadratic equation in $L/\ell$ that has roots 0.76 and 1.70. If we solve Equation (8) for large values of $(z_0/f)$, the result appears to approach a limit of about 0.81, which is close to the smaller root obtained by the process described above. We do not understand the significance, if any, of the larger root at this time.

There are two possible explanations for this result. The first is that the simple model devised by Churnside and Lataitis [2], although it gives very reasonable results, is deficient for large values of $(z_0/f)$. The second is that for long ranges, long focal lengths, and small focusing apertures, the L term dominates, resulting in a maximum for the AOA short of the actual beam waist. In either case, this small anomaly does not appear to be significant.

## 4. CONCLUSIONS

We have determined the AOA variance for a focused Gaussian beam wave in atmospheric turbulence. We show that the AOA approaches infinity for small values of the parameter $f/z_0$ corresponding to short wavelengths, short focal lengths and large focusing apertures, which is the geometrical optics limit. This result agrees with classical AOA theory which predicts that this parameter approaches infinity for infinitesimal beam sizes. Furthermore, the theory developed in this paper shows that the AOA for a focused Gaussian beam is well behaved for larger beam sizes and approaches the classical result in an orderly way as the parameter $f/z_0$ becomes larger, which is the case for longer focal lengths and wavelengths and smaller focusing apertures. Since physical optics dictates that all beams must have some spatial extent, this result is useful in predicting the behavior of laser beam weapons focused on a target in atmospheric turbulence. Since laser beams are not diffraction limited, a complete description of their behavior would require the inclusion of higher order transverse modes in the beam width expression (4). Inclusion of these modes will be the subject of future work in this area.

We also show that our result reduces to the classical AOAs for plane and spherical waves respectively as $f \to \infty$ for the plane wave case and $f \to 0$ for the spherical wave case.

Figure 3 shows a small anomaly in the AOA for large values of $(z_0/f)$. At this time, we do not know whether this behavior is a result of a real physical phenomenon in which the effects of long range dominate AOA or if it is a deficiency in the simple model developed in [2]. We have shown by a simple analysis that this discrepancy does not result from our method of calculation.

## REFERENCES

1. R. S. Lawrence and J. W. Strohbehn, *Proc. IEEE*, Vol. 58, pp. 1523 – 1545, 1970
2. J. H. Churnside and R. J. Lataitis, *J. Opt. Soc. Am. A*, Vol. 4, pp. 1264 – 1272, July 1987
3. H. Kogelnik and T. Li, *Appl. Opt.*, Vol. 5, pp. 1550 – 1567, 1966
4. A. Yariv, *Quantum Electronics*, 3rd Edition, New York, John Wiley, 1989, Chapter 6