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14. ABSTRACT

A conceptual framework for the design of transmit signals for MIMO radar systems as developed for communication signals has been formulated. This was achieved by constructing a generalized matched filter in analogy of the usual case. The corresponding maximum likelihood receiver was studied in detail and shown this to be optimal in accordance with the Cramer-Rao lower-bound criterion. Mathematics for defining and generating coding coefficients was developed. Tracking of a single target has been treated within this framework in terms of state equations. A tracker defined by Kalman predictor has been introduced. A channel/target model involving both linear and nonlinear dynamics has been studied.

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Objectives

Concepts of Space-Time Coding (STC) for communication signals have been well researched and implemented in the domain of MIMO telecommunication systems. The aim of this project has been to explore how far such signal design aspects could be carried over to MIMO radar systems for exploiting the additional degrees of freedom introduced by STC. Examples of MIMO radars are phased arrays and netted radars in the context of network centric warfare. The fundamental problem here is to answer what is an optimal radar receiver in the context of signal processing and how to solve the mathematical intricacies of coding. Once this issue is addressed, one would like to examine the performance of such a receiver for various radar related applications, for example for target tracking.

Status of effort

We have presented in the Conferences all our research results. We will be extending this work in further depth and have added a section in this regard on future direction at the end of this report for perusal. It is our objective to consolidate all existing work in an archival journal paper to be submitted in 2006. There remain several remaining unsolved problems, mainly in respect to the solution to the fundamental optimisation problems arising from the work.

Abstract

The conceptual framework for designing transmit signals for MIMO radar systems in terms of the techniques of STC as developed for communication signals has been formulated. This is achieved by constructing a generalized matched filter in analogy of the usual case. We have studied in detail the corresponding maximum likelihood receiver and shown this to be optimal in accordance with the Cramer-Rao lower bound criterion. We have developed the mathematics for defining and generating the coding coefficients. Tracking of a single target has been treated within this framework in terms of state equations. A tracker defined by Kalman predictor has been introduced. A channel/target model involving both linear and nonlinear dynamics has been studied. We have briefly touched on the implication of STC for transmit beam forming.

Personnel Supported

The research has been carried out by the PI (Professor L.B White) and his associate (Dr. P.S Ray).

Publications

The following conference papers (refereed) have appeared to date:

Proceedings Second International Waveform Diversity and Design Conference, Lihue-Hawaii (January 2006) - "Optimal Code Design for MIMO Radar – Optimal Receivers": *L.B.White & P.S.Ray.*

Proceedings 38th Asilomar Conference on Signals, Systems & Computers, Pacific Grove, CA, pp.973-977 (7-10 Nov 2004) *Invited Paper* - "Signal Design for MIMO Diversity System": *L.B.White & P.S.Ray.*

Proceedings of the First Workshop on Progress in Radar Research 2005 (PIRR05) held at the Defence Science and Technology Organisation, Edinburgh, South Australia, November 2005. - "Optimal Waveform Design for MIMO Tracking Radar", *L. B. White and P. S. Ray. (Refereed on abstract only)*

Proceedings First International Conference on Waveform Diversity & Design, Edinburgh (November 2004) - "Receiver Design for MIMO Tracking Radar": *L.B.White & P.S.Ray*

Interactions

As noted above the results of the research have been presented in the First and Second International Conference on Waveform Diversity and Design (Dr. P.S.Ray) and in the Asilomar Conference (Prof. L.B.White) which was an invited session. There has been considerable scientific interaction with international experts at the Conferences as described in the following.

First, Dr. Ray had a brief discussion with Dr. Mike Wicks (AFRL) who advised to keep him up-to-date with our further progress in the research. He has assured his support for any purpose when required. He endorsed a "short" visit by Dr. Ray to Hanscom (AFRL) in order to create an opportunity for his participation at the upcoming SAM 2006 Workshop (Sensor Array and Multi-Channel Signal Processing). This is subject to the immediate availability of funding from AFRL, which is being explored by Dr. Rangaswamy (AFRL).

Dr. Marshall Greenspan, Technical Director of Northrop Grumman Corporation, indicated to Dr. Ray that he intended to draw the attention of some group in the Ohio State University to the expertise in Adelaide in MIMO coding techniques for possible future collaboration. They are planning some activity on MIMO systems for short range applications.

A promising development has come from Professor Chris Baker in University College London. He wanted us to submit a program of research on coding of MIMO radars for collaborative work. His group will undertake experimental investigations in this area as they have hardware designing expertise. We are following up this in due course.

Following the Asilomar conference in 2004, Dr Rick Blum from Lehigh University, USA invited Prof White to be an associate investigator on a MURI application for MIMO radar system design. This application was unsuccessful but did serve the purpose of familiarising the research teams at each institution about each others' work.

New

The work undertaken here falls under the category of "enabling research" and as such it is early to assess at present its impact in the field. We can only vouch for its originality by our publications. We believe that our work is the first to use a model based approach to extend earlier work by David Kershaw and Rob Evans (IEEE Trans,. on Information Theory, 1994) to the MIMO case. We also believe that we are the first to link the use of unitary transmitter codes to obtain a simple correlation based receiver which achieves close to maximum likelihood performance. This result has the added implication that obtaining transmit directionality by use of non-unitary codes may lead to either the requirement for an excessively complicated optimal, or perhaps poorly performing sub-optimal MIMO receiver.

Honors/Awards

This research program is at its infancy at the present stage and as such this criterion seems not to be applicable.

Archival Documentation

The research publications in refereed Conferences are enclosed.

Software and/or Hardware

We have developed experimental software in matlab to simulate a single target MIMO transmitter and receiver baseband system.

Future Directions

At this stage I will note the following areas for immediate investigation:

> We need to sharpen our results for the tracking application. A preliminary study was undertaken in this direction in the paper presented at the Edinburgh Conference (2004). In particular, there are issue of the effect of Doppler side-lobes on the receiver parameter estimation which we need to address properly. Also, it is not clear whether the resulting code optimisation problem eeven has a feasible solution, and some imposed code constraints may need to be relaxed.

> We have just touched in our present work the implication of MIMO coding for the transmit beam forming. We need to develop this aspect fully. The incorporated explicit transmitter beamforming into our system model has the advantage of permitting transmitter bean control whilst still retaining near ML receiver performance enabled by the use of orthogonal transmitter codes.

> The questions raised in the conferences prompt us to take into account clutter analysis for the receiver performance, a significant issue in the radar context. We have neglected this so-far in our work, as we have focussed on airborne targets.

> We have as yet to deal with multiple target tracking, where we conjecture that the additional degrees of freedom offered by MIMO systems will lead to considerable benefits.

The fundamentals of this research fall in the category of statistical signal processing. We have already initiated some discussions with Professor Eric Moulines (ENST, Paris) in this context. We intend to introduce Monte Carlo techniques for multiple target simulations and the associated optimization technique in future. Profesor Moulins's group has international expertise in this area. Prof White will be hosted by Prof Moulines during his sabbatical in the first half of 2007. We will be seeking funding to permit Dr Ray to visit for an extended period so that work on multiple target tracking can be advanced.

Receiver Design for MIMO Tracking Radar

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Abstract— **The paper is the first in a series of three which address the problem of radar transmitter and receiver design for MIMO (multiple transmit and receive antennae) tracking systems. This paper considers the receiver design problem. The method described consists of a generalised matched filter receiver which produces approximate maximum likelihood estimates of the target's delay and doppler, and the spatial wavenumber (angle of arrival) and its time derivative. The paper argues that this approach is novel in the sense that it appropriately combines the usual single antenna matched filter receiver, and the beamforming required for the multiple antenna case. Simulation results are used to assess the performance of the proposed method.**

I. Introduction

There has been considerable recent interest in the use of transmit and receive antenna arrays to improve the detection and tracking performance of radar systems, in particular the recent emergence of a number of dedicated workshops addressing this problem. Whilst the waveform design problem has been considered for single transmit antenna tracking systems (see eg [2]), there has been little attention paid to the corresponding problem for MIMO (Multiple-Input, Multiple-Output) systems. Recently, the corresponding problem for MIMO *communications* systems has been considered [3], although there are significant differences between the communications and radar problems.

The use of antenna arrays permits not only the estimation of the targets' range and radial velocity, but also its spatial wavenumber which is related to the target position. The time derivative of the wavenumber can also be estimated giving information on the angular velocity of the target with respect to the antenna array. An approximation to the MIMO antenna data likelihood for a single target model, leads to a generalised matched filter receiver architecture consisting of N matched filters for each of the N receiver antennae. The outputs of these N^2 matched filters are coherently combined to yield a statistic which can be maximised to yield the required estimates. Due to the approximation made in deriving the statistic, bias will generally be present in the estimates, especially for small antenna arrays. Local maxima can also be a problem, especially when estimating the velocity parameters.

In this paper, we have made no attempt to choose the transmitted signal waveforms in order to optimise the receiver performance. In [6], we derive approximate Cramer-Rao bounds for the receiver presented here, and show how the transmitted waveforms can be chosen to minimise a weighted mean-square error measure for the estimated parameters. In [7], the tracking problem is addressed, including adaptive waveform design to optimise tracking performance.

II. Signal Model

Consider an linear equispaced array of N sensors located along the y axis as shown in figure 1. The position of the n-th sensor is $(n-1)d\hat{y}$ where d is the element spacing and $\hat{y}$ is the unit vector in the y direction. We will transmit successive blocks of M pulses, each of duration T seconds from each antenna. These signals have the form (after upmixing on carrier of frequency ω (rad/sec))

$$s_{n,k}(t) = e^{i\omega t} \sum_{m=0}^{M-1} x_{n,m}(k) \, g_n(t - (kM + m)T) \, , \quad (1)$$

for $n = 0, \ldots, N - 1$, where g_n is the unit energy pulse shaping function for transmitter n supported on $[0, T]$ and $x_{n,m}(k)$ is a complex code sequence transmitted by transmitter n for signal block k. The baseband quadrature received signal at sensor n due to a single target reflection is given by

$$r_{n,k}(t) = \sigma(k) \sum_{\ell=0}^{N-1} e^{it(\nu_n(k)+\nu_\ell(k))} s_{\ell,k}(t - (\tau_n(k) + \tau_\ell(k)))$$
$$e^{-i\omega(\tau_n(k)+\tau_\ell(k))} + w_{n,k}(t) \quad (2)$$

where $\sigma(k)$ is an unknown complex quantity incorporating the effects of medium attenuation and target reflectivity. These variables are modelled as zero mean complex Gaussian process independent from block to block. Here $\tau_n(k)$ and $\nu_n(k)$ are the (one-way) delay and doppler from sensor n to the target over the signal block k, and $w_{n,k}(t)$ is complex Gaussian zero mean white receiver noise. We assume that the duration of each signal block is small compared to the target motion is sufficiently slow with respect to the block length so that the delay and doppler are constant over each signal block.

We assume that there is a single target located at polar co-ordinates $(r_0(k), \theta_0(k), \phi_0(k))$ during signal block k with respect to the reference sensor located at the origin. The delay $\tau_n(k)$ and doppler shift $\nu_n(k)$ as seen by sensor n are related for each n to the reference delay and doppler seen by the sensor at the origin (to linear terms, and assuming that the target distance is much larger than the array
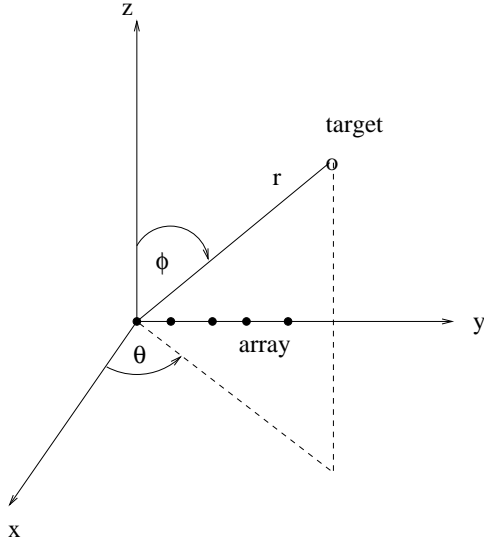
Fig. 1. Array and Target Geometry

dimension) by

$$
\begin{aligned}
\tau_n(k) &= \tau_0(k) - \frac{n d \rho_0(k)}{c} \\
\nu_n(k) &= \nu_0(k) - \frac{\omega n d \dot\rho_0(k)}{c} \ ,
\end{aligned} \tag{3}
$$

where

$$
\begin{aligned}
\tau_0(k) &= \frac{r_0(k)}{c} \\
\nu_0(k) &= \frac{\omega \dot r_0(k)}{c}
\end{aligned} \tag{4}
$$

are the one-way delay and doppler to the reference sensor. The wavenumber variable for this array geometry is $\rho_0(k) = \sin(\theta_0(k)) \ \sin(\phi_0(k))$. The target state variables are denoted $\theta(k) = [\tau_0(k), \ \nu_0(k), \ \rho_0(k), \ \dot\rho_0(k)]$.

III. Receiver Architecture

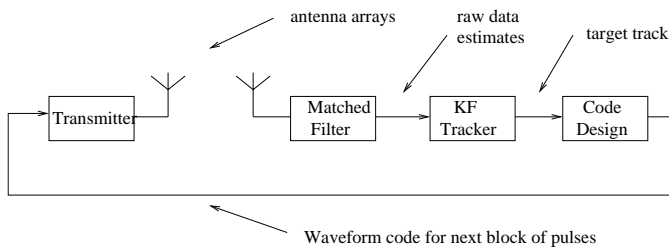The overall system architecture is shown in figure 2.

Fig. 2. Radar Tracker System Architecture

In this paper, we only address the design of the generalised matched filter, with performance evaluation (wrt the Cramer-Rao bound) and code design discussed in [6]. The code design problem for tracking systems is addressed in [7].

We now describe the form of the generalised matched filter. The log likelihood function associated with the measurements from the N receive sensors for block k is

$$
\begin{aligned}
\ell_k = -\int \sum_{n=0}^{N-1} \Big| r_{n,k}(t) - \sigma \sum_{j=0}^{N-1} s_{j,k}\left(t - \tau_n(k) - \tau_j(k)\right) \\
e^{-i\omega(\tau_n(k)+\tau_\ell(k))} \ e^{it(\nu_n(k)+\nu_j(k))} \Big|^2 \ dt
\end{aligned} \tag{5}
$$

where $\tau_n(k)$ and $\nu_n(k)$ are the (one-way) delay and doppler associated with sensor n. Such a function is difficult to maximise over the unknown parameters due to the cross terms present in this superimposed signal model. Thus we neglect these cross terms and use instead the statistic [1]

$$
\begin{aligned}
\hat\ell_k = \sum_{n,j=0}^{N-1} a \ \mathrm{Re} \ \Big\{ e^{-i\phi} \int r_{n,k}(t) \ s_{j,k}^*\left(t - \tau_n(k) - \tau_j(k)\right) \\
\times \ e^{-i\omega(\tau_n(k)+\tau_\ell(k))} \ e^{-it(\nu_n(k)+\nu_j(k))} \ dt \Big\} - N/2 \ a^2 \ E_k
\end{aligned} \tag{6}
$$

where E_k is the total transmitted energy of the signal during block k, and $\sigma = a \ e^{i\phi}$. Maximising directly over a and ϕ yields the (approximate) MLEs

$$
\begin{aligned}
\hat a &= \frac{1}{NE_k} \Big| \sum_{n,j} \chi_{n,j}^{(k)}(\theta) \Big| \\
\hat\phi &= \arctan \left[\frac{\mathrm{Im} \sum_{n,j} \chi_{n,j}^{(k)}(\theta)}{\mathrm{Re} \sum_{n,j} \chi_{n,j}^{(k)}(\theta)} \right] \ ,
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
\chi_{n,j}^{(k)}(\theta) = \int r_{n,k}(t) \ s_{j,k}^*\left(t - \tau_n(k) - \tau_j(k)\right) \\
\times \ e^{-i\omega(\tau_n(k)+\tau_\ell(k))} \ e^{-it(\nu_n(k)+\nu_j(k))} \ dt
\end{aligned} \tag{8}
$$

is the (phase compensated) cross ambiguity function between the received signal on antenna n and the transmitted signal on antenna j over block k. Here the delay and doppler terms $\tau_n(k)$ and $\nu_n(k)$ are related to the state variables θ via (3). Substituting in (6) yields

$$
\hat\ell_k(\theta) = \frac{1}{2NE_k} \Big| \sum_{n,j=0}^{N-1} \chi_{n,j}^{(k)}(\theta) \Big|^2 \ . \tag{9}
$$

The approximate MLEs for the delay-doppler and reflection parameters [2] are obtained by maximising this function, ie

$$
\begin{aligned}
\hat\theta(k) &= \mathrm{argmax}_{\theta \in \Theta(k)} \Big| \sum_{n,j=0}^{N-1} \chi_{n,j}^{(k)}(\theta) \Big| \\
\hat a(k) &= \frac{1}{NE_k} \Big| \sum_{n,j=0}^{N-1} \chi_{n,j}^{(k)}(\hat\theta(k)) \Big| \ ,
\end{aligned} \tag{10}
$$

---

[1] The cross terms become insignificant as $N \to \infty$.
[2] We don't use the reflectivity phase information here.

SNR (dB)	Bias				Std. Dev.			
	r	$\dot{r}$	ρ	$\dot{\rho}$	r	$\dot{r}$	ρ	$\dot{\rho}$
∞	0.14	41	0	0	0	0	0	0
40	-15	0.02	0	0	74	6.2	0	0
30	-28	31	0	0	175	14	0	0
20	-92	-0.15	0	0	350	3.9	0	0
10	-193	6.6	0	0	381	16	0	0

TABLE I

BIAS AND STD. DEV. OF ESTIMATES VS SNR

Pos'n	Bias				Std. Dev.			
	r	$\dot{r}$	ρ	$\dot{\rho}$	r	$\dot{r}$	ρ	$\dot{\rho}$
1	-92	-0.15	0	0	350	3.9	0	0
2	-178	-3.8	0	0.1	119	2.8	0	0.2
3	-120	-0.9	-1.5	0	226	3.0	1.1	0

TABLE II

BIAS AND VARIANCE OF ESTIMATES FOR 3 TARGET POSITIONS

N	Bias				Std. Dev.			
	r	$\dot{r}$	ρ	$\dot{\rho}$	r	$\dot{r}$	ρ	$\dot{\rho}$
3	-92	-0.15	0	0	350	3.9	0	0
7	-67	1.7	0	0	265	5.8	0	0
13	-18	0.2	0	0	143	4.7	0	0

TABLE III

BIAS AND VARIANCE OF ESTIMATES FOR DIFFERENT ARRAY SIZES

where $\Theta(k) \subset \mathbb{R}^4$ is the so-called *validation gate*. The statistic (9) represents the magnitude of the coherent sum of N^2 matched filters. As we shall later see, neglecting the cross terms in the likelihood function results in estimates which can be significantly biased, especially for small arrays. A correction scheme for reducing the bias in superimposed signal problems such as the one presently at hand is described in [4].

IV. SIMULATION EXAMPLES

A single target starts at the point in the $x - z$ plane at $r = 10$ km, and $\phi = \pi/4$, and moves at speed 167 m/s towards the origin. At a distance of 1 km from the origin, the target turns right and heads in the y direction at a speed of 200 m/s. Our first 3 examples consider the target parameter estimation at three points in the trajectory ; one near the start, one at the point of turning, and one near the end. For these examples, we chose an array of 3 elements, spaced one-half wavelength. The carrier frequency was 5 GHz, and pulse duration was 500 μsec. Identical raised-cosine pulse shaping was used for each antenna. The temporal block length was 8 pulses, and the pulse repetition interval (PRI) was 67 msec. The target reflectivity had unit variance. We show in table I, the bias and standard deviation estimates for the four state parameters for several values of signal-to-noise ratio (SNR) [3] for the target at position 1.

Table II shows the bias and variance of the estimates for the 3 target positions at SNR = 20 dB.

In order to assess the effect of array length on the statistical properties of the estimates, we repeated the experiments for various array lengths at an SNR of 20 dB. In all cases, the transmitted code was a random unitary

[3]We define SNR as the ratio of target reflectivity variance to noise variance

code, held constant for each of 100 independent realisations which were averaged to obtain the results presented in table III. For array lengths of 3 and 7, the block length was 8, whilst for array length of 13, the block length was 14 pulses. Target position 1 was used.

Figure 3 shows the ambiguity function for the transmitted signal on antenna 1.
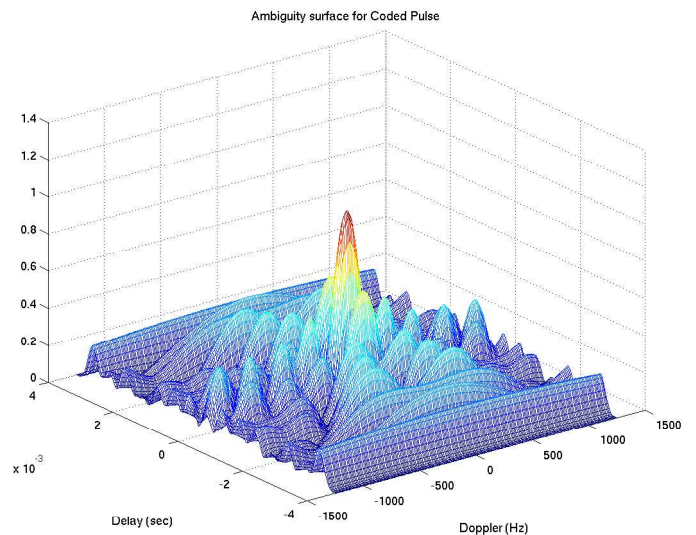

Fig. 3. Coded Signal Ambiguity Function

The second experiment deals with estimation of the target parameters over the complete trajectory. Here, a new and randomly chosen unitary code was used for each signal block. Figure 4 shows the true and estimated target range over the trajectory. Note that the range estimates are the raw estimates from the generalised MF ; there is no post-detection tracking used. Figure 5 shows the radial velocity estimates. The SNR was 20 dB and we used 3 elements and a block length of 8.

A. Discussion

The above results show that for small array sizes, the target parameter estimates are generally biased. This is due to the fact that certain cross terms are neglected in the formulation of the estimator. These terms are significant for small arrays. The results shown in table III however suggest that the bias problem may diminish as
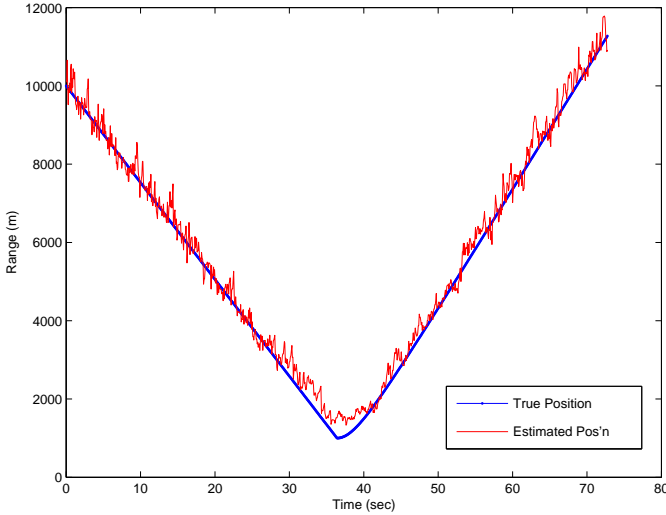
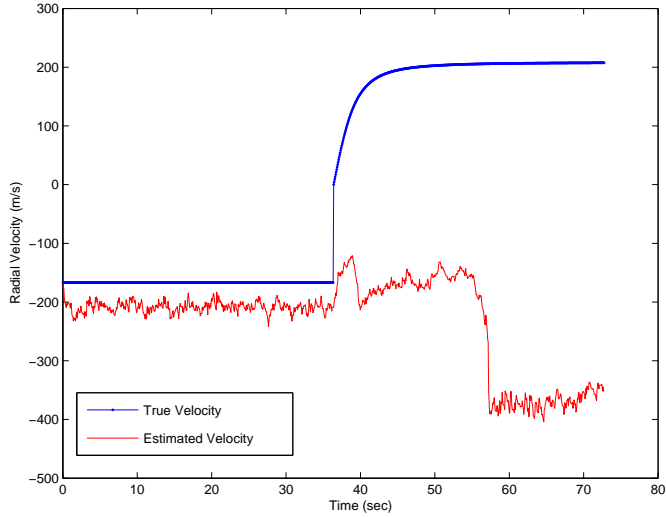Fig. 4.   Range Estimation (3 elements, SNR = 20 dB)

Fig. 5.   Radial Velocity Estimation (3 elements, SNR = 20 dB)

the array size becomes larger. All experiments show that the range estimates generally improve for higher SNR and larger arrays, but the velocity estimation is problematic. The problem with estimation of the velocity (and sometimes the wavenumber parameters) is due to a combination of bias and the presence of local maxima in the estimation statistic. This behaviour is particularly evident in figure 5, where the lack of effective doppler gating results in the parameter estimates moving to an incorrect local maximum. All these results should be considered with some degree of caution as we have made no attempt to optimise the transmitted codes, in particular, as such, no beamforming has been performed.

V. Conclusions and Further Work

In this paper, we have proposed a new approximate maximum likelihood estimator for the parameters of a single target when a multiple antenna radar is used. The pro-

posed estimator combines both the usual delay-doppler estimation with array beamforming, by the use of multiple coded transmitter waveforms. We have examined the performance of the estimator using random unitary transmitter codes. The performance of the receiver in the absence of any attempt to optimise the codes is problematic, however results to be presented in [6], show significant improvement, mainly due to the inherent use of near-optimal beamforming as a result of the optimisation of transmitter code selection.

In future work [6], we derive an approximate Cramer-Rao bound for the receiver presented here, and use this bound to optimise the choice of transmitter codes. We also investigate incorporating bias correction using the scheme proposed in [4]. The paper [7] is concerned with optimising tracker performance when a Kalman filter based tracker is used to process the raw parameter estimated from the approximate ML receiver.

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Signal Design for MIMO Diversity Systems

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Abstract— **This paper addresses the problem of waveform design for general diversity systems. The paper firstly introduces a general model for such systems, and then considers linear and nonlinear cases. Examples of each case are given - in the linear case, the MIMO communications design problem ; in the nonlinear case, the MIMO radar waveform design problem. In the latter case, a simulation example is provided which illustrates the potential benefits which might be obtained. Finally, the paper concludes with a brief discussion of the MIMO radar tracking problem.**

I. INTRODUCTION

There has been considerable interest in the last several years among wireless telecommunications researchers in the use of diversity techniques to improve the capacity and performance of such systems. In particular, the use of multiple transmit and receive antennae has been demonstrated to offer significant performance improvement in wireless systems. More recently, the idea of exploiting diversity in radar systems has also drawn attention. At this conference, there were a number of important papers presented which focused on various fundamental aspects of Multiple-Input, Multiple-Output (MIMO) radar systems. Also, a conference dedicated to radar diversity and waveform design was recently held in Edinburgh, UK, where many aspects of MIMO radar were presented. We can expect that the area of MIMO radar will receive an increasing amount of attention by researchers and research sponsors.

The purpose of the recent work in the area of MIMO radar being conducted at the University of Adelaide is fourfold : (i) Firstly we seek to understand how models used in MIMO communications systems and associated transmitter and receiver design methods, might be relevant to the radar case. (ii) Secondly, we seek to study how to design appropriate receiver structures for both the radar and communications cases, and importantly, be able to offer some guidance as to the statistical performance of such receivers. (iii) Thirdly, we seek to address the transmitter waveform design problem in each case. We envisage a system whereby the transmitted waveforms are adaptively derived from current system state information under some performance criterion eg tracking MSE. (iv) Fourthly, we are interested in how feedback information is to be sent back to the transmitter in the case when such information must be sent over capacity limited channels.

In [1], we considered channel modelling and receiver design for MIMO communications. The proposed receiver used one-step predictions for the channel gains incorporated into a Per-Survivor Processing based detection scheme. Both [2] and [3] dealt with MIMO transmit code design for the communications problem. In these papers, communications based criteria such as maximal distance between codes and maximal diversity were applied. In [4], channel tracking MSE was used as a code design criterion, and was observed to offer better performance at low SNRs. It was demonstrated in [5] that channel estimation MSE is not the optimal criterion from the point of view of symbol detection. We started to address the MIMO radar problem in [6] where channel modelling and receiver design were considered in the single target case. Importantly, the receiver structure considered incorporated a number of constraints on the transmit codes in order to permit the calculation of the Cramer-Rao bound on target parameter estimation performance. This is required for parameterisation of the tracker following the parameter estimation step.

The purpose of this paper is to provide a general description of the MIMO code design problem, and to consider two special cases. The first is the case where the model is linear in the target/channel parameters and the MSE estimation performance is time-invariant. An example is the communications problem as discussed in [4]. The second example is the MIMO radar problem as considered in [6] in which the target parameters appear in the model in a nonlinear way. Here we extend that work to specify how the codes may be adaptively chosen to optimise the target parameter estimation. We then mention how the tracking problem is set up, with results to be provided in work to be published [7]. Finally, we conclude with some open research questions.

II. MIMO SYSTEM MODELS

We assume a channel/target model with linear dynamics, and generally, non-linear measurements of the form

$$\begin{aligned} x_{k+1} &= A\,x_k + B\,u_k \\ y_k &= f_k(x_k) + v_k \ , \end{aligned} \qquad (1)$$

where $x_k \in \mathbb{C}^N$ is the target/channel state, $u_k \in \mathbb{C}^P$ is the white state noise process, $y_k \in \mathbb{C}^M$ is the white measurement noise process, and $v_k \in \mathbb{C}^M$ denotes the measurement noise process. We assume that the matrix A is stable, and that the noise processes u_k and v_k are zero mean Gaussian with known covariance matrices Q and R respectively.

The matrices A and B specify the target/channel dynamics whilst the functions $f_k(.)$ describes the mapping from the target state to the observed measurements.

A. The Linear Case

In the linear case, $f_k(x) = C_k\,x_k$, where C_k are $M \times N$ matrices. An example of the linear case is in the MIMO communications problem, [4] with n transmit antennae and ℓ receive antennae. The states x_k are the complex gains [1] associated with a Rayleigh/Ricean fading channel with n inputs and ℓ outputs, and have dimension $N = n\ell$. The measurement vector y_k has length $M = m\ell$ where m is the temporal block length. The matrix C_k has the form $C_k = I_\ell \otimes X_k\,P$ where X_k is a sequence of unitary $m \times n$ matrices which are chosen from a set $\mathcal{U}$ of cardinality 2^p in order to encode p input message bits per temporal block. Here P is a fixed precoder matrix of size $n \times n$. Our design task is to choose M based on the *a priori* statistics of the channel, in order to minimise in an appropriate sense, the channel estimation error. The MLE of the state term $C_k\,x_k$ given y_k is simply y_k itself, so in reference to the block diagram (figure 1), the MLE block is trivial. We return to this problem in section III.

B. The Nonlinear Case

In the nonlinear case, we construct an approximate MLE of x_k given each measurement y_k. Such an estimate neglects the *a priori* dynamics of the state, these being taken into account in the tracker. We will assume that these estimates are given by $\tilde{x}_k = g(y_k)$ for some function $g(.)$ which is generally nonlinear. For example, $g(.)$ might represent a generalised matched-filter correlator such as will be described in the MIMO radar example in section IV. Following the idea of Kershaw and Evans [8], we characterise the statistical performance of the estimator by assuming that it is unbiased and efficient, so that $\tilde{x}_k = x_k + w_k$ where $\mathrm{E}\{w_k\} = 0$ and $\mathrm{E}\{w_k w_k^T\} = F^{-1}$ where F is the Fisher information given by

$$F = -\left[\frac{\partial^2 p(y_k|x_k)}{\partial x_k \partial x_k^T}\right] \ . \tag{2}$$

This simplifying assumption is made because it is generally not possible to determine the exact statistical performance for any exact or approximate MLE in the nonlinear case. In general, F depends on the state x_k as well as the underlying code C_k (specified in the form of $f_k(.)$). One possible objective might be to choose the code to maximise the objective function $J(C_k, x_k) = \mathrm{Tr}(FWF) = \|FW^{1/2}\|_F^2$, where W is a diagonal, state-dependent matrix. Of course, since we don't know x_k we need to replace its value by an estimate (perhaps from the tracker) and perform the optimisation in a block by block adaptive manner. We return to this problem in section IV

[1] In the following, we use real variables for simplicity reasons, so the model for the complex case is built from the real and imaginary parts

III. MIMO COMMUNICATIONS EXAMPLE

In a communications problem, the overall code design objective is to minimise the resulting symbol detection error probability. This is an unsolved problem in general, although results exist for the temporally uncorrelated and asymptotically high SNR case (eg [9], [10]). Here we take the approach of minimising the channel estimation mean-square error, although it is known that this does not generally lead to optimal symbol detection performance [5]. However the codes so designed can offer improvement over the existing designs in some cases, eg at low SNR. The optimal tracker is given by the Kalman predictor (causality is generally required for any receiver incorporating the channel estimates)

$$\hat{x}_{k+1|k} = A\ (I - G_k C_k)\,\hat{x}_{k|k-1} + G_k\,y_k \tag{3}$$

where the gain G_k is given by

$$G_k = \Sigma_k C_k^T \left(C_k \Sigma_k C_k^T + R\right)^{-1} \ , \tag{4}$$

where Σ_k satisfies the Riccati equation

$$\Sigma_{k+1} = A\left(\Sigma_k - \Sigma_k C_k^T \left(C_k \Sigma_k C_k^T + R\right)^{-1} C_k \Sigma_k\right) A^T \\ + BQB^T \ . \tag{5}$$

We can show that $\mathrm{Tr}(\Sigma_k)$ is asymptotically constant, and is independent of the choice of code X_k and only depends on the (fixed) precoder P. Indeed, $\mathrm{Tr}(\Sigma_k) \to \mathrm{Tr}(\Sigma)$ where Σ is the solution to the *algebraic* Riccati equation

$$\Sigma = A\left(\Sigma - \Sigma P^T \left(P\Sigma P^T + R\right)^{-1} P\Sigma\right) A^T + BQB^T \ . \tag{6}$$

In [4], we show how to compute the Jacobian (first derivative) of Σ with respect to P. We then define a gradient projection scheme for minimising $\mathrm{Tr}(\Sigma)$ over P subject to the power constraint $\mathrm{Tr}(PP^T) = 1$. These results rely on methods of matrix differential calculus and perturbation theorems for eigenvectors. The reader is referred to [4] for details and some simulation results.

IV. MIMO RADAR EXAMPLE

In this section, we describe the signal model and receiver for a MIMO single target problem. In the MIMO radar problem, the baseband signal transmitted on antenna j for temporal block k has the form

$$s_{j,k}(t) = \sum_{q=0}^{m-1} X_{j,q}(k)\ g_j(t - (km + q)T) \ , \tag{7}$$

where the $g_j(.)$ are unit energy pulse shaping functions supported on $[0,\ T]$, and $X(k)$ is the space-time code sent for block k. The baseband received signal on antenna r is

$$y_{r,k}(t) = a(k) \sum_{j=0}^{n-1} e^{it(\nu_r(k)+\nu_j(k))} \, s_{j,k}(t - \tau_r(k) - \tau_j(k))$$
$$\times e^{i\phi_{r,j}} + w_{r,j}(k) \; . \tag{8}$$

Here $\tau_j(k)$ and $\nu_j(k)$ are the one-way delay and doppler between antenna j and the target assumed constant over the duration of each block, and $a(k)$ is the magnitude of the target reflectivity coefficient together with the path loss. The terms $\phi_{r,j}$ are uniform iid variables on $[-\pi, \pi]$. The delay and doppler parameters contained in (8) are functions of the delay and doppler respectively to some reference antenna, together with the array geometry and target motion. More precisely, for a uniform linear array with spacing d metres, to linear terms, [2]

$$\begin{aligned} \tau_j &= \tau_0 - jd\rho/c \\ \nu_j &= \nu_0 - jd\omega\dot{\rho}/c \; , \end{aligned} \tag{9}$$

where $\tau_0 = r/c$ and $\nu_0 = \dot{r}\omega/c$ are the reference delay and doppler, ρ is the spatial wavenumber, and $\dot{\rho}$ its time derivative. Here r and $\dot{r}$ are the radial distance from the reference antenna to the target and its time derivative respectively, ω is the carrier frequency (rad/s), and c is the propagation speed. We will denote the state of the system (target) by the vector $x_k = [a(k), \; r(k), \; \dot{r}(k), \; \rho(k), \; \dot{\rho}(k)]^T$. Notice that (8) is a nonlinear mapping from the target state to the observed signal.

A. Approximate MLE

In [6], we described a generalised matched filter receiver for this MIMO radar model. The data likelihood contains cross-terms between the individual receive antenna signal models which make exact MLEs impractical to obtain. These cross terms are known to tend to zero as the data length goes to infinity. However, the receiver proposed in [6] imposes a number of constraints on the transmitter MIMO codes, among which is that they be unitary. This results in the suppression of the aforementioned cross terms, and thus this receiver is a better approximation to the ML one.

More specifically, the data log likelihood for a block of received data is given by

$$\ell = -\int \sum_{n=0}^{N-1} \left| r_n(t) - a \sum_{j=0}^{N-1} s_j\left(t - \tau_n - \tau_j\right) e^{i\phi_{n,j}} \right.$$
$$\left. \times e^{it(\nu_n + \nu_j)} \right|^2 \, dt \; . \tag{10}$$

Neglecting the cross terms, we use instead the approximate log likelihood

$$\hat{\ell} = a \sum_{n,j=0}^{N-1} \mathrm{Re} \left\{ e^{-i\phi_{n,j}} \chi_{n,k}(\tau_n + \tau_k, \nu_n + \nu_k) \right\} - \frac{N^2 a^2}{2} \tag{11}$$

which is maximised by maximising the detection statistic

$$\begin{aligned} \eta(r, \dot{r}, \rho, \dot{\rho}) &= \sum_{j,k=0}^{N-1} |\chi_{j,k} \left[(r - (j+k)d\rho)/c, \right. \\ & \left. (\dot{r} - (j+k)d\dot{\rho})\omega/c \right]|^2 \; , \end{aligned} \tag{12}$$

where

$$\chi_{j,k}(\tau, \nu) = \int r_j(t) \, s_k^*(t - \tau) \, e^{-it\nu} \, dt \; , \tag{13}$$

are the outputs of the generalised matched filters for each receive antenna r_j, corresponding to each transmitted signal s_k. Approximate ML estimates for r, $\dot{r}$, ρ and $\dot{\rho}$ are obtained by maximising (12), and an estimate of a is obtained from that maximum value of η. Phase estimates can also be obtained but are not used here.

In [6], we were able to evaluate the Fisher information matrix corresponding to (8) by imposing a number of constraints on the signal codes. Indeed, these constraints have the form of moments

$$\begin{aligned} XX^H &= I \\ X\Gamma_j X^H &= \Lambda_j \; , \end{aligned} \tag{14}$$

for $j = 1, 2$, where the Γ_j are specified diagonal, positive matrices which depend on the pulse shaping functions, and the Λ_j are free diagonal (positive) matrices. In this case, the Fisher information F is independent of the target state, and only depend on the choice of code X via the free diagonal terms of Λ_1 and Λ_2. Thus the optimisation task is to maximise $J(\Lambda_1, \Lambda_2, x) = \|F(\Lambda_1, \Lambda_2) \, W^{1/2}(x)\|_F^2$, where W is a possibly state (ie x) dependent positive diagonal weighting subject to the constraints (14). We used a projection gradient approach, but the issues concerning the feasibility of the constraints (14) and the convergence of the scheme remain open questions which we aim to address in [7].

We simulated the system described, with the target located 10km from the antenna array, moving towards the array at speed 267 m/s, at wavenumber 0.1, and wavenumber derivative 0.01. The results shown in table 1 show the proportional errors obtained over 50 simulation runs each consisting of 100 pulse blocks. We used 3 antennas and a temporal block length of 4 pulses. Raised cosine shaping with pulse duration 50 msec was used. The SNR was 20 dB (total energy per pulse divided by noise variance per sensor). We compared 2 code sets - one random unitary codes,

and the other optimal codes chosen according to the above criterion with weighting matrix $W = \text{diag}(0.01, 1, 10, 10)$. A significant improvement is noted when the optimal codes are used.

Parameter	range	radial vel.	ρ	$\dot{\rho}$
Opt bias (%)	2.5	0.02	0.6	1.5
Opt stdev (%)	1.7	0.004	1.4	1.5
Ran bias (%)	4.3	0.07	0.8	2.5
Ran stdev (%)	3.7	0.03	2.6	2.2

TABLE I

BIAS AND STANDARD DEVIATION ESTIMATES FOR OPTIMAL AND RANDOM CODES

B. Discussion

As shown by Furhmann [11], the use of unitary codes results in no directivity in the transmitted energy which whilst desirable initially in the absence of *a priori* information about the target state, is clearly sub-optimal when estimates of the target state are available. However, we found it necessary to impose a number of simplifying constraints on the codes, including the unitary one, in order to evaluate, even approximately, the statistical performance of the (approximate) MLE. More work is thus required in the area of receiver performance evaluation so that the unitary constraints might be relaxed. Alternatively, we could explicitly add beamforming into the code design problem. These are areas for ongoing research.

V. SIGNAL DESIGN FOR TRACKING

For tracking problems, we feed the "raw" estimates of the target state $\tilde{x}_k$ to a Kalman filter derived from the state space model

$$
\begin{aligned}
x_{k+1} &= A\,x_k + B\,u_k \\
\tilde{x}_k &= x_k + w_k
\end{aligned}
\tag{15}
$$

where $w_k \sim \mathcal{N}(0, F^{-1})$. The resulting tracking error covariance is given (approximately) by the Riccati difference equation

$$
\begin{aligned}
\Sigma_{k+1|k} &= A\Sigma_{k|k}A^T + BQB^T \\
\Sigma_{k+1|k+1} &= \Sigma_{k+1|k} - \Sigma_{k+1|k}\left(\Sigma_{k+1|k} + F^{-1}\right)^{-1}\Sigma_{k+1|k}
\end{aligned}
\tag{16}
$$

where F depends on the chosen code as described in the previous section. Thus we seek to select X_{k+1} by minimising $\text{Tr}\{\Sigma_{k+1|k+1}\}$ with x_{k+1} replaced by the current one step prediction state estimate $\hat{x}_{k+1|k}$. Matrix differential calculus can be used to obtain the Jacobian of $\text{Tr}(\Sigma_{k+1|k+1})$ wrt the Λ_1 and Λ_2 matrices, and an alternating convex projection gradient algorithm thus derived. We describe this approach in [7].

VI. CONCLUSIONS AND FUTURE RESEARCH

In this paper, we have introduced a general model for waveform design in MIMO diversity systems. We have considered both linear and non-linear models, and have indicated methodologies for optimal code design in each case. Examples of designs for MIMO communications systems (linear), and for MIMO radar systems (nonlinear) have been given. In the latter case, we have included a simulation example illustrating the potential benefit of designing the transmitter code to optimise the performance of an approximate ML receiver. Finally we have indicated how to design optimal codes for tracking systems.

Several open questions remain, including receiver performance characterisation for non-unitary codes, convergence of numerical schemes for code optimisation, using codes chosen from discrete sets, performance analysis of tracking systems and the implementation of feedback information in the system.

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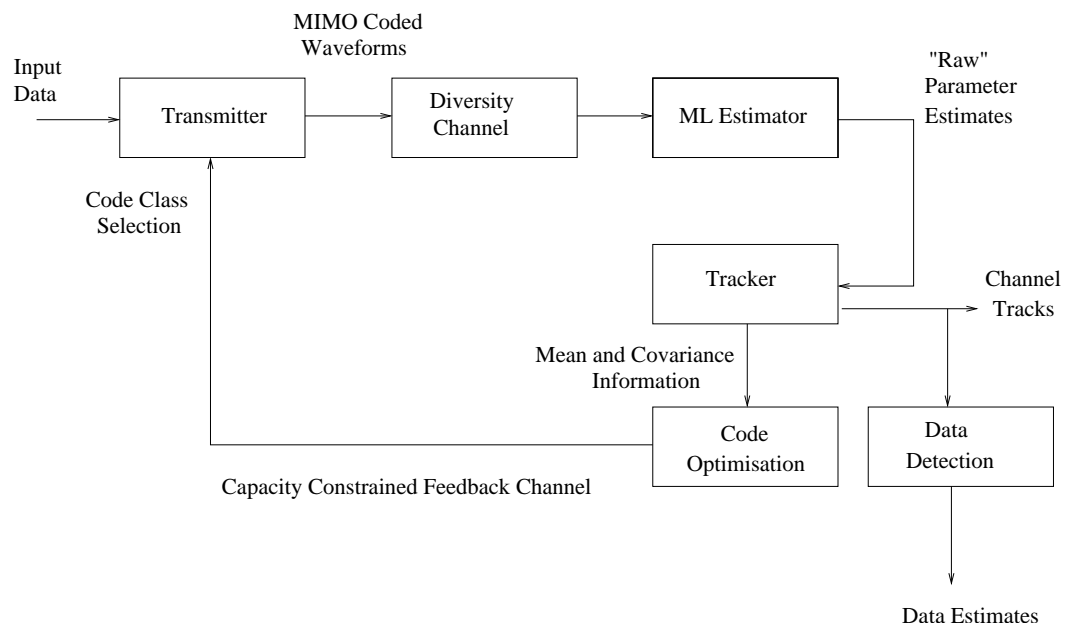
Fig. 1. Block Diagram for a general diversity system

Optimal Code Design for MIMO Radar - Optimal Receivers

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Abstract—This paper addresses the problem of transmitter design for a single target MIMO radar system. The transmitted space-time codes are unitary apart from the inclusion of a transmit beamformer. The paper demonstrates that the use of unitary codes leads to a simple coherent correlations-based maximum-likelihood receiver yielding unbiased estimates of the target range and wavenumber, and their time derivatives. The transmitter codes are chosen to maximise the Fisher information for the model. This leads to an alternating projection gradient adaptive algorithm for dynamic code design. The paper argues that proper choice of beamformer phases can lead to efficient estimators, thus validating the optimisation methodology.

I. INTRODUCTION

In recent years, there has been a rapid increase in interest in the use of multiple input - multiple output (MIMO) systems for radar detection, estimation and tracking. Our recent work has focused on receiver design [1], and transmitter design [2]. In each of these papers, the transmitter codes were constrained to be orthogonal, leading to a simple derivation of the maximum likelihood (ML) receiver. Additional orthogonality constraints were included to permit the derivation of the corresponding Fisher information for the model. Following the methodology proposed by Kershaw and Evans [7] in the SISO case, we attempt to optimise system performance by maximising the Fisher information of the model by appropriate choice of codes. In the SISO case, the ML estimators for range and Doppler are efficient for finite sample sizes, thus validating this approach. However, it is unclear whether this property holds for the MIMO case, where the time derivatives of range and Doppler are also estimated. Another important issue is that the use of unitary codes precludes directivity being introduced in the transmitter beamformer [6]. This is obviously a clear deficiency in a MIMO design approach.

In this paper, we include explicit (linear) transmit beamforming in addition to unitary space-time coding. However, retaining orthogonality between transmitted waveforms (so that a simple ML receiver can be designed), means the resulting Fisher information is independent of the beamformer phases. Thus it is unlikely, in general, that the resulting estimates are efficient. A natural question arises as to whether efficiency can be attained by "steering" the transmitter beam towards the expected target location. In a more general tracking context [3], the tracker can assist with this operation.

The layout of the paper is as follows. Firstly we introduce the coherent signal model for single target MIMO radar, and thus derive a correlation based ML receiver. We then provide a characterisation of the receiver performance in terms of the transmit codes (including beamformer). We then introduce an procedure for the design of optimal codes. Simulation results will be presented at the meeting, and are also available at [4].

II. SIGNAL MODEL

We consider a radar system with N transmit and N receive antennae. The baseband transmitted signal for temporal block k and transmit antenna n is given by

$$s_{n,k}(t) = \sum_{m=0}^{M-1} Y_{n,m}(k) \, g_n(t - (k\overline{M} + m)T) \,, \qquad (1)$$

where $M \geq N$ is the number of pulses in each temporal block, T is the pulse duration and g_n is the baseband pulse shaping function for antenna n, which is assumed real and twice differentiable, supported on $[0, T]$, and with $g_n(0) = g_n(T) = 0$. The quantity $\overline{M}$ denotes the pulse repetition interval (PRI). Here, $Y(k) \in \mathbb{C}^{N \times M}$ is the transmitted space-time code for block k, given by

$$Y_{n,m}(k) \quad = \quad W_n(k) \, X_{n,m}(k) \,, \qquad (2)$$

where $X(k) \in \mathbb{C}^{N \times M}$ is an *unitary* code (ie $XX^H = I_N$), and $W_n(k)$ denotes the transmitter beamformer weights for block k. The signals are modulated onto a carrier with frequency f_0 (Hz).

We assume a single target located at a radial distance r with respect to the origin located at a reference element of the transmit array. For simplicity of presentation, we assume in the following, that the transmit and receive arrays are uniformly spaced linear arrays located along the y axis, and are co-located. The case where there is a constant distance between the arrays is easily dealt with. We address the case where transmit and receive arrays are in relative motion in [5]. To

linear terms, the two-way delay and Doppler from element n to element m are

$$\begin{aligned} \tau_{n,m}(k) &= \frac{2r(k) - (n+m)d\rho(k)}{c} \\ \nu_{n,m}(k) &= \frac{(2\dot{r}(k) - (n+m)d\dot{\rho}(k))\,f_0}{c} \,, \end{aligned} \tag{3}$$

where d is the array spacing, c the propagation speed, ρ is the target wavenumber, with $\dot{\rho}$ being its time derivative, and $\dot{r}$ is the radial target velocity. For the purposes of target tracking, we shall define the target state to be the vector $\theta = (a, \phi, r, \dot{r}, \rho, \dot{\rho})$ which is a function of the temporal block index k. The variables (a, ϕ) are defined below and $(r, \dot{r}, \rho, \dot{\rho})$ are defined in [1].

The basebanded received signal at antenna m for transmit block k for a *fully coherent* system [1] is

$$\begin{aligned} r_{m,k}(t) &= a(k)\,e^{i\phi(k)} \sum_{n=0}^{N-1} s_{n,k}\left(t - \tau_{n,m}(k)\right) \\ &\times\; e^{2\pi i t \nu_{n,m}(k)}\, e^{-2\pi i f_0 \tau_{n,m}(k)} + \xi_{m,k}(t) \,, \end{aligned} \tag{4}$$

where $a(k) \geq 0$ is the target reflectivity amplitude, and $\phi(k)$ denotes the associated phase shift. The quantities $\xi_{m,k}(t)$ represent receiver noise, and are assumed to be zero mean, white, Gaussian processes with identical variance σ^2, and independent between receivers. The approximate log likelihood (neglecting cross terms) is (dropping the k index for clarity) [8]

$$\ell \;\approx\; \frac{a}{\sigma^2}\,\mathrm{Re}\left\{e^{-i\phi}\sum_{n,m=0}^{N-1}\chi_{n,m}\left(\tau_{n,m}, \nu_{n,m}\right)\right\} - \frac{a^2 N}{2\sigma^2} \,, \tag{5}$$

where we assume that $\|s_{n,k}\|^2 = p_{n,k}$, with $\sum_n p_{n,k} = 1$ for all k, and

$$\chi_{n,m}(\tau, \nu) = e^{2\pi i f_0 \tau} \int r_m(t)\, s_n^*(t - \tau)\, e^{-2\pi i \nu t}\, dt \,, \tag{6}$$

denotes the matched filter for transmit waveform n received on antenna m. Maximising (5) over a and ϕ (assumed deterministic and unknown) we obtain the sufficient statistic (generalised matched filter), written as a function of the target state

$$\eta(r, \dot{r}, \rho, \dot{\rho}) = \frac{1}{2N\sigma^2}\left|\sum_{n,m=0}^{N-1}\chi_{n,m}\left(\tau(\theta), \nu(\theta)\right)\right|^2 \,, \tag{7}$$

which is an coherent sum of the outputs of the N^2 matched filters. In order to estimate the target state, we perform a *constrained* maximisation of (7) subject to the constraints given in (3). This generally requires a search over an appropriately

[1]We assume all transmitters and receivers are phase locked to a reference clock.

chosen subset of the 4 dimensional parameter space. The mean value of the statistic is

$$\mathrm{E}\,\eta(r, \dot{r}, \rho, \dot{\rho}) = \frac{a^2 N}{2\sigma^2} + 1/2 \,. \tag{8}$$

We thus observe a linear increase in the mean detection statistic with the number of antenna provided the target signal to noise ratio is sufficiently large.

III. CHARACTERISING RECEIVER PERFORMANCE

In earlier work, [1] we derived the Fisher information matrix for the receiver described above when the transmit codes were unitary. Part of this analysis also showed that the cross terms neglected in the derivation of (7) are suppressed by the use of unitary codes. Thus our receiver achieves close to ML performance in this case. It is easy to demonstrate that the inclusion of the transmit beamformer W does not alter this property. The beamformer is required because no directivity can be achieved (apart from that inherent in the array elements) with unitary codes [6]. As shown in [7], for the SISO case, the Fisher information depends essentially on the first two time and frequency moments of the transmitted signal. In order to find tractable form for the Fisher information in the MIMO case, we need to make a number of orthogonality assumptions on the transmit signals and their moments as in [1]. It is a natural question to ask as to whether it is useful to minimise a lower bound on receiver performance (Cramer-Rao bound). Clearly, if the receiver is consistent, then it is a sound approach. In the SISO case, it is known [8], that the matched filter (ambiguity function peak) yields unbiased and consistent estimators for delay and Doppler (if side lobes are negligible). However, it is not clear that maximisation of (7) provides consistent estimators of θ for finite sample sizes. We have, however demonstrated that these estimators are unbiased.

IV. OPTIMISING RECEIVER PERFORMANCE

We shall use as our measure of receiver performance, the cost function

$$J(Y; \theta) = \log \det F(Y; \theta) \,, \tag{9}$$

where F is the Fisher information matrix

$$F = -\mathrm{E}\left\{\frac{\partial^2 \ell(Y; \theta)}{\partial \theta^2}\right\} \,, \tag{10}$$

which is strictly positive definite. This quantity is a function of the quantities $\tau_{n,m}$, and thus of the state variables r and ρ, as well as the signal waveforms, and thus of the code $Y_{n,m}$. This cost function is chosen because maximisation of J implies minimisation of the Cramer-Rao lower bound on the parameter estimation uncertainty. There are four constraints to be met. Firstly, the beamformer weight magnitudes specify the transmit powers for each antenna, ie $|W_n|^2 = p_n = q_n^2$, with $\sum_n q_n^2 = 1$. We also have the code constraints

$$
\begin{aligned}
x_k^H \, x_n &= \delta_{n,k} \\
x_k^H \, \Gamma_t \, x_n &= 0, \ \forall \ n \neq k \\
x_k^H \, \Gamma_{tt} \, x_n &= 0, \ \forall \ n \neq k \ ,
\end{aligned}
\tag{11}
$$

where Γ_t and Γ_{tt} are diagonal matrices specified by the signalling pulse, and the x_n are the rows of X (ie the sequence transmitted on antenna n. We can write

$$
F(q, X; \theta) = \sum_n q_n^2 \left(A_n(\theta) + B_n(\theta) x_n^H \Gamma_t x_n + C_n(\theta) x_n^H \Gamma_{tt} x_n \right) \ ,
$$

where $A_n(\theta), B_n(\theta), C_n(\theta)$ are 6×6 matrices whose entries can be determined from the form of the second derivatives of ℓ wrt a, ϕ and θ. We thus have

$$
\begin{aligned}
\mathrm{Vec} \ dF \ &= \ 2\mathrm{Re} \sum_n q_n^2 \left((\Gamma_t x_n \otimes B_n(\theta)) + (\Gamma_{tt} x_n \otimes C_n(\theta)) \right) \\
&\qquad \times d\overline{x_n} \\
&+2 \sum_n q_n \ \mathrm{Vec} \left(A_n(\theta) + B_n(\theta) x_n^H \Gamma_t x_n + C_n(\theta) x_n^H \Gamma_{tt} x_n \right) \tag{12} \\
&\qquad \times dq_n \ . \tag{13}
\end{aligned}
$$

Thus [9]

$$
dJ(q, X; \theta) \ = \ \mathrm{Tr} F^{-1} \ \mathrm{Vec} \ dF \ , \tag{14}
$$

permits the gradient of J wrt q and X to be determined. We can thus define a projection gradient scheme for numerical constrained maximisation of J using the forms of the projections given in appendix III, applied to the constraints (13). Let $\hat{X}_j$ and $\hat{q}_j$ denote estimates of the parameters at iteration j, then we perform the updates

$$
\begin{aligned}
\mathrm{Vec} \ \tilde{X}_j \ &= \ \mathrm{Vec} \ \hat{X}_j + \mu \ \mathrm{Tr} \ F^{-1}(\hat{q}_j, \hat{X}_j; \hat{\theta}) \\
&\qquad \times \frac{\partial \mathrm{Vec} \ F(\hat{q}_j, \hat{X}_j; \hat{\theta})}{\partial \mathrm{Vec} \ X} \\
\hat{q}_{j+1} \ &= \ \hat{q}_j + \mu \ \mathrm{Tr} \ F^{-1}(\hat{q}_j, \hat{X}_j; \hat{\theta}) \\
&\qquad \times \frac{\partial \mathrm{Vec} \ F(\hat{q}_j, \hat{X}_j; \hat{\theta})}{\partial q} \\
\hat{X}_{j+1} \ &= \ \left(\mathcal{P}_u \, \mathcal{P}_{\Gamma_t} \, \mathcal{P}_{\gamma_{tt}} \right)^\ell \ \tilde{X}_j \ , \tag{15}
\end{aligned}
$$

where μ is a small scalar step size, and θ is an estimate of the target state which is generally supplied by the post-receiver tracker [3]. The projection steps are repeated ℓ times until there is little change in the resulting matrix.

V. Conclusion

This paper has derived a simple maximum likelihood receiver for MIMO radar systems with unitary space-time codes and explicit transmitter beamforming. The transmitter codes are chosen to maximise the Fisher information associated with the MIMO model. An gradient projection algorithm has been proposed to solve this constrained optimisation problem. An

important assumption underlying the approach is that reduction of the CRB leads to improved performance. Whilst this is true in the SISO case, because the delay-Doppler estimates are efficient in that case, it is not clear that this property holds in the orthogonal coded MIMO case. We have seen that the CRB is independent of the transmit beamformer phase, suggesting lack of efficiency of the estimates. Thus explicit transmit beamforming is argued to be essential in this context. In simulation results to be presented at the meeting (and available at [4]), we investigate this issue in detail.

VI. Acknowledgement

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Appendix I
Derivation of Receiver CRB - Outline

In order to compute the Fisher information for the model (5), we need to evaluate the first and second order derivatives of $\chi_{n,m}(\tau, \nu)$. These are

$$\frac{\partial \chi_{n,m}}{\partial \nu} = -2\pi i e^{2\pi i f_0 \tau} \int t\, r_m(t)\, s_n^*(t-\tau)\, e^{-2\pi i \nu t}\, dt$$

$$\frac{\partial \chi_{n,m}}{\partial \tau} = -e^{2\pi i f_0 \tau} \int r_m(t)\, \frac{\partial s_n^*}{\partial t}(t-\tau)\, e^{-2\pi i \nu t}\, dt$$
$$+ 2\pi i f_0 \chi_{n,m}(\tau,\nu)$$

$$\frac{\partial^2 \chi_{n,m}}{\partial \nu^2} = -4\pi^2 e^{2\pi i f_0 \tau} \int t^2\, r_m(t)\, s_n^*(t-\tau)\, e^{-2\pi i \nu t}\, dt$$

$$\frac{\partial^2 \chi_{n,m}}{\partial \tau^2} = e^{2\pi i f_0 \tau} \int r_m(t)\, \frac{\partial^2 s_n^*}{\partial t^2}(t-\tau)\, e^{-2\pi i \nu t}\, dt$$
$$- 2\pi i f_0 e^{2\pi i f_0 \tau} \int r_m(t)\, \frac{\partial s_n^*}{\partial t}(t-\tau)\, e^{-2\pi i \nu t}\, dt$$
$$+ 2\pi i f_0 \frac{\partial \chi_{n,m}}{\partial \tau}(\tau,\nu)$$

$$\frac{\partial^2 \chi_{n,m}}{\partial \tau \partial \nu} = 2\pi i e^{2\pi i f_0 \tau} \int t\, r_m(t)\, \frac{\partial s_n^*}{\partial t}(t-\tau)\, e^{-2\pi i \nu t}\, dt$$
$$+ 2\pi i f_0 \frac{\partial \chi_{n,m}}{\partial \nu}(\tau,\nu) \ . \quad (16)$$

We then substitute for $r_m(t)$ from (4), and take expected values. For example,

$$E\, \chi_{n,m}\,(\tau_{n,m},\nu_{n,m}) = a\, e^{i\phi} \sum_{k=0}^{N-1} e^{2\pi i f_0 (\tau_{n,m} - \tau_{k,m})}$$
$$\int s_k(t-\tau_{k,m})\, s_n^*(t-\tau_{n,m})\, e^{-2\pi i t(\nu_{k,m} - \nu_{n,m})}\, dt \ . \quad (17)$$

We assume that the differential time and frequency shifts across the array are negligible compared to the signal duration and bandwidth, and that signals transmitted on different antennas are orthogonal. Thus

$$\int s_k(t-\tau_{k,m})\, s_n^*(t-\tau_{n,m})\, e^{-2\pi i t(\nu_{k,m} - \nu_{n,m})}\, dt$$
$$\approx \int |s_n(t)|^2\, dt\, \delta_{k,n} \ , \quad (18)$$

and thus

$$E\, \chi_{n,m}\,(\tau_{n,m},\nu_{n,m}) \approx a\, e^{i\phi}\, p_n \ . \quad (19)$$

This implies that the expected value of the test statistic
This process is repeated with the terms in (16), assuming the orthogonality properties detailed in appendix II to give :

$$E\, \frac{\partial \chi_{n,m}}{\partial \nu}(\tau_{n,m},\nu_{n,m}) \approx -2\pi i a e^{i\phi} \left(\|s_n\|_t^2 + \tau_{n,m}\, p_n \right)$$

$$E\, \frac{\partial \chi_{n,m}}{\partial \tau}(\tau_{n,m},\nu_{n,m}) \approx 2\pi i f_0\, a e^{i\phi}\, p_n$$

$$E\, \frac{\partial^2 \chi_{n,m}}{\partial \nu^2}(\tau_{n,m},\nu_{n,m}) \approx -4\pi^2 a e^{i\phi} \times$$
$$\left(\|s_n\|_{t^2}^2 + 2\tau_{n,m} \|s_n\|_t^2 + \tau_{n,m}^2 p_n \right)$$

$$E\, \frac{\partial^2 \chi_{n,m}}{\partial \nu \partial \tau}(\tau_{n,m},\nu_{n,m}) \approx a e^{i\phi} \Big(4\pi^2 f_0 \|s_n\|_t^2$$
$$- 2\pi i \langle s_n,\ s_n' \rangle_t \Big)$$

$$E\, \frac{\partial^2 \chi_{n,m}}{\partial \tau^2}(\tau_{n,m},\nu_{n,m}) \approx a e^{i\phi} \left(-4\pi^2 f_0^2 p_n + \langle s_n,\ s_n'' \rangle \right)$$
$$(20)$$

Thus we can determine the Fisher information terms :

$$-E\, \frac{\partial^2 \ell}{\partial r^2} = \frac{4a^2 N}{c^2 \sigma^2} \left(4\pi^2 f_0^2 + c_g \right)$$

$$-E\, \frac{\partial^2 \ell}{\partial \dot{r}^2} = \frac{16 a^2 \pi^2 f_0^2}{c^2 \sigma^2} \sum_{n,m} U_{n,m}$$

$$-E\, \frac{\partial^2 \ell}{\partial \rho^2} = \frac{a^2 d^2}{c^2 \sigma^2} \left(4\pi^2 f_0^2 + c_g \right) \left(N \sum_n n^2\, p_n \right.$$
$$\left. + N(N-1) \sum_n n\, p_n + \frac{N(N-1)(2N-1)}{6} \right)$$

$$-E\, \frac{\partial^2 \ell}{\partial \dot{\rho}^2} = \frac{4\pi^2 a^2 d^2 f_0^2}{c^2 \sigma^2} \sum_{n,m} (n+m)^2\, U_{n,m} \ ,$$
$$(21)$$

where c_g is a constant depending only on the pulse g. In (20) and (21),

$$U_{n,m} = \|s_n\|_{t^2}^2 + 2\tau_{n,m} \|s_n\|_t^2 + \tau_{n,m}^2 p_n$$

$$\langle s_k,\ s_n \rangle_t = \int t\, s_k(t)\, s_n^*(t)\, dt = \left(Y \Gamma_t Y^H \right)_{k,n}$$

$$\langle s_k,\ s_n \rangle_{t^2} = \int t^2\, s_k(t)\, s_n^*(t)\, dt = \left(Y \Gamma_{tt} Y^H \right)_{k,n} \quad (22)$$

We can also compute the mixed second order derivatives of ℓ which are not shown here. We note that all mixed second order partials wrt a are zero.

We can also show that the expected value of all first order derivatives of ℓ evaluated at the true parameter values, are zero thus demonstrating that estimates are unbiased.

Appendix II
Waveform Constraints

We need to have $< s_n, s_k > = \delta_{n,k} p_n$, where $p_n > 0$ is the energy transmitted on antenna n each block. This leads directly to the constraint

$$Y\,Y^H \;\; = \;\; W\,W^H = \Lambda = \text{diag}\,(p_0,\,\ldots,\,p_{N-1})\;\;,\quad(23)$$

where we assume that the baseband pulses $g_n(t)$ are identical for each n, are supported on $[0,T]$, and have unit norm. Here W is an $N \times N$ diagonal matrix with the array weights on its diagonal, and Λ is a diagonal, positive definite matrix which specifies the transmit powers p_n. Unity total transmit energy implies that $\text{Tr}(\Lambda) = 1$. The second constraint is that

$$\int t\,s_n(t)\,s_k^*(t)\,dt = q_n\,\delta_{n,k} \Leftrightarrow Y\,\Gamma_t\,Y^H = \Lambda_t\;,\quad(24)$$

where Γ_t is a diagonal, positive definite matrix with elements

$$[\Gamma_t]_{nn} = \int t\,|g(t-nT)|^2\;dt = \int t\,|g(t)|^2\;dt + nT\;,\quad(25)$$

and Λ_t is a diagonal, positive definite matrix. The next constraint is

$$\int t^2\,s_n(t)\,s_k^*(t)\,dt = u_n\,\delta_{n,k} \Leftrightarrow Y\,\Gamma_{tt}\,Y^H = \Lambda_{tt}\;,\quad(26)$$

where Γ_{tt} is a diagonal, positive definite matrix with elements

$$[\Gamma_{tt}]_{nn} \;\; = \;\; \int t^2\,|g(t-nT)|^2\;dt\;,\quad(27)$$

and Λ_{tt} is a diagonal, positive definite matrix.

The terms

$$\int s_n(t)\,\frac{\partial s_k^*(t)}{\partial t}\;dt \;\; = \;\; \sum_{\ell,m} Y_{k,\ell}\,Y_{n,m}^*$$
$$\times \int g(t-\ell T)\,\frac{\partial g^*}{\partial t}(t-mT)\;dt\,(28)$$

are zero for all k,n because if g is real, and $g(0)=g(T)=0$, the integral on the rhs of (28) is identically 0 for all ℓ and m.

Consider now

$$-\int s_n(t)\,\frac{\partial^2 s_k^*(t)}{\partial t^2}\;dt \;\; = \;\; \left(Y\,\Gamma_{ff}\,Y^H\right)_{n,k}$$
$$(29)$$

where Γ_{ff} is a diagonal matrix with elements

$$[\Gamma_{ff}]_{n,n} \;\; = \;\; -\int g(t)\,\frac{\partial^2 g^*}{\partial t^2}(t)\;dt = \left\|\frac{\partial g}{\partial t}\right\|^2\;,\quad(30)$$

under the assumption that $g(0) = g(T) = 0$. Notice that Γ_{ff} is proportional to the identity matrix, so the required orthogonality constraint is met by virtue of (23).

The final constraint is that

$$\int t\,s_n(t)\,\frac{\partial s_k^*}{\partial t}(t)\;dt\quad(31)$$

is zero whenever $k \neq n$. This follows from the property

$$\int t\,g(t-\ell T)\,\frac{\partial g^*}{\partial t}(t-mT)\;dt = 0\quad(32)$$

for all $\ell \neq m$. When $\ell = m$, we have that

$$\int t\,g(t-mT)\,\frac{\partial g^*}{\partial t}(t-mT)\;dt = \int t\,g(t)\,\frac{\partial g^*}{\partial t}(t)\;dt$$
$$= -1 - \int t\,g^*(t)\,\frac{\partial g}{\partial t}(t)\;dt\,(33)$$

leading to the fact that the integral on the lhs equals -1/2. Thus this constraint is also met by virtue of (23).

APPENDIX III
APPROXIMATION BY ORTHOGONAL MATRICES

Suppose $Y \in \mathbb{C}^{N \times M}$ with $M \geq N$ is given and has full rank N. We seek $X \in \mathbb{C}^{N \times M}$ satisfying $X\Gamma X^H = D$ where $\Gamma > 0$ is Hermitian and is given, and $D > 0$ is a free diagonal matrix, such that $\|X - Y\|$ is minimised. Let

$$Y = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_N^T \end{bmatrix},\qquad X = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_N^T \end{bmatrix}.\quad(34)$$

Consider now the constraint $X\Gamma X^H = D$. Since $\Gamma > 0$, we can find $V \in \mathbb{C}^{M \times M}$ such that $\Gamma = VV^H$ with $V > 0$. Then $XV = \Delta\,U$ where $\Delta = D^{1/2}$ is a free diagonal matrix, and U is any $N \times M$ matrix satisfying $UU^H = I$. Let $U = [\,u_1\;\cdots\;u_N\,]$ with each $u_i \in \mathbb{C}^M$. Thus $X = \Delta\,U\,V^{-1}$ and

$$\|Y - X\|^2 = \text{Tr}\left(Y\,Y^H - \Delta^*\,Y\,V^{-H}\,U^H\right)$$
$$- \text{Tr}\left(\Delta\,U\,V^{-1}\,Y^H + |\Delta|^2\,U\,\Gamma^{-1}\,U^H\right)\;.\quad(35)$$

We minimise this quantity over choice of diagonal Δ. We find the optimal $\Delta = \text{diag}(\delta_1,\ldots,\delta_N)$ with

$$\delta_i = \frac{\left[U\,V^{-1}\,Y^H\right]_{ii}}{\left[U\,\Gamma^{-1}\,U^H\right]_{ii}}\;.\quad(36)$$

Then

$$\|Y - X\|^2 = \text{Tr}\left(Y\,Y^H\right) - \sum_{i=1}^N \frac{\left|\left[U\,V^{-1}\,Y^H\right]_{ii}\right|^2}{\left[U\,\Gamma^{-1}\,U^H\right]_{ii}}\;,\quad(37)$$

which we now minimise over unitary $N \times M$ matrices U. A Lagrangian approach yields the necessary condition $U = \Lambda\,Y\,V^{-H}$, where Λ is a free Hermitian $N \times N$ matrix. Thus $I = U\,U^H = \Lambda\,Y\,\Gamma^{-1}\,Y^H\,\Lambda^H$, or $\Lambda^{-1}\,\Lambda^{-H} = Y\,\Gamma^{-1}\,Y^H$. We can thus choose $\Lambda = G^{-1}$ where G is the unique symmetric square root of $Y\,\Gamma^{-1}\,Y^H$. Then $X = $

$\Delta\ G^{-1}\ Y\ \Gamma^{-1}$. We can also show that X is unique and is independent of the choice of V. We denote the mapping (projection) by $X = \mathcal{P}_\Gamma(Y)$. Simulations suggest that $\mathcal{P}_\Gamma$ is a contraction mapping on $\mathbb{R}^{N \times M}$ with Frobenius norm. In the unitary case, where $\Gamma = D = I$, the optimal solution is $X = G^{-1}\ Y$ where G is the (unique) symmetric square root of YY^H.

Optimal Code Design for MIMO Radar

Optimal Receiver

Langford B White and Pinaki S Ray

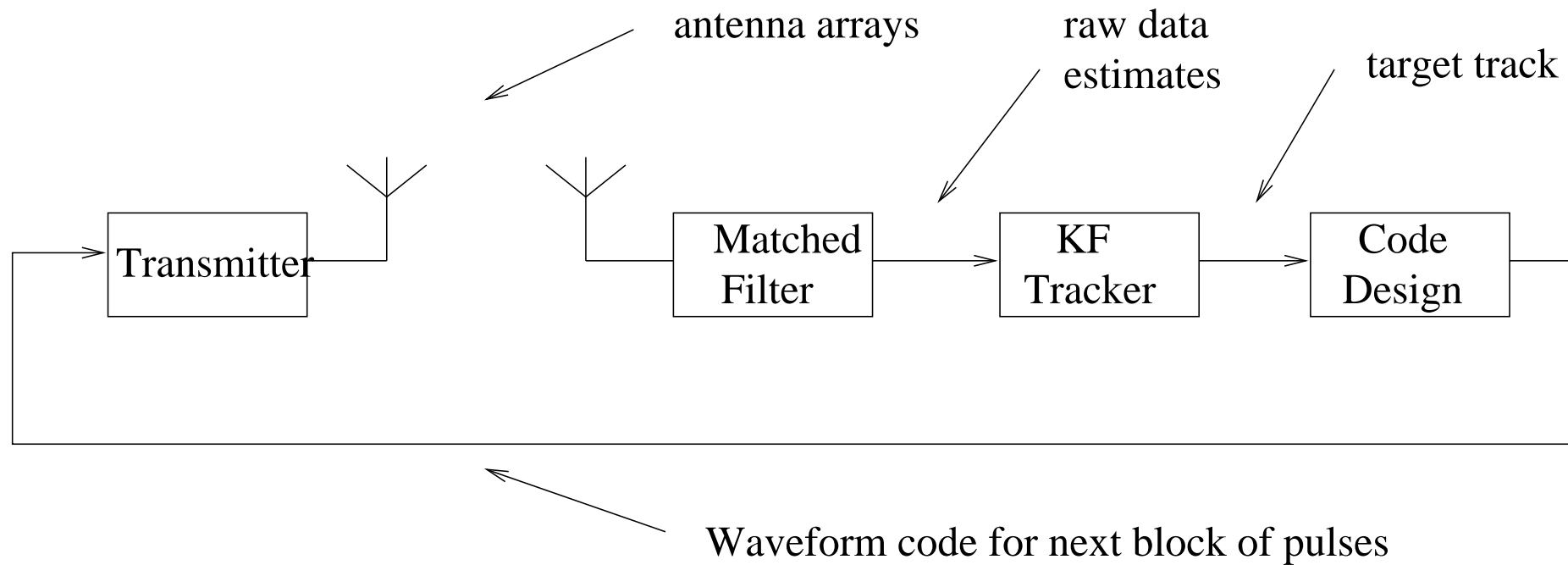School of Electrical and Electronic Engineering
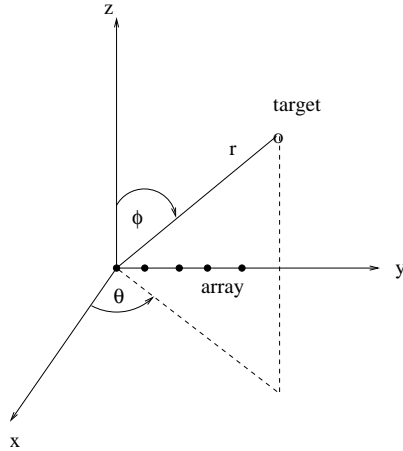
The University of Adelaide

Adelaide, Australia

Outline of the Program

- ⟶ Fundamental Signal Design Issues for MIMO Radars

- ⟶ MIMO System exploits spatial and temporal diversiy

- ⟶ Pulse Coding – Code Design to yield a simple ML receiver (correlation based)

- ⟶ Permit Optimisation of receiver performance based on *CRLB*: Key Issue Efficiency of the estimates

- ⟶ Optimisation leads to constraints relating to moments and inner products of signal waveforms

System Architecture

antenna arrays

raw data estimates

target track

| Transmitter | | Matched Filter | KF Tracker | Code Design |

Waveform code for next block of pulses

Target Dynamics

State varables: $\Theta(k) = (r(k), \dot{r}(k), \rho(k), \dot{\rho}(k))^T$

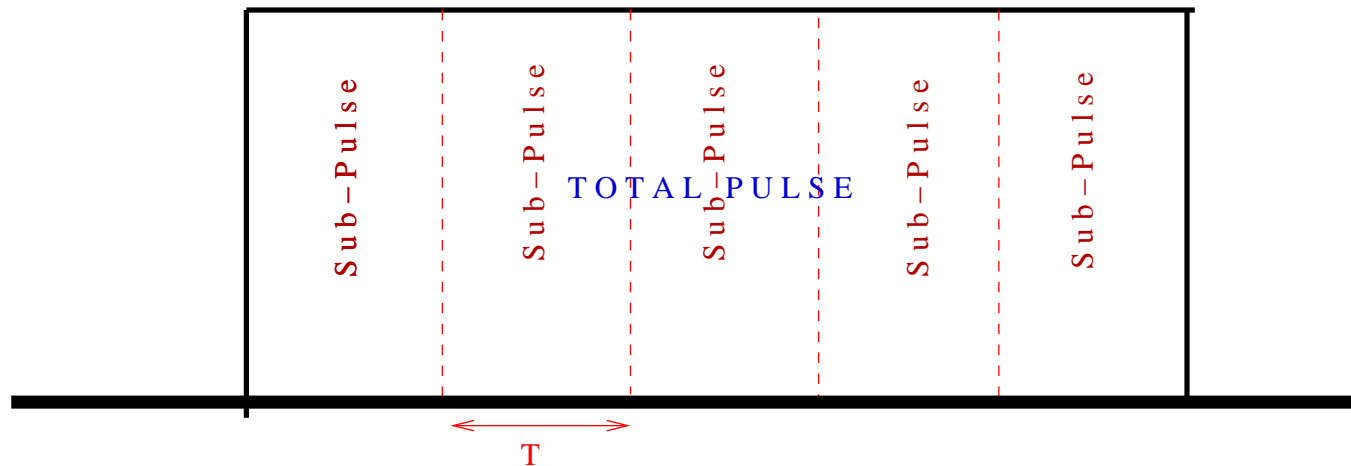with $\rho(k) = \sin\theta(k) \sin\phi(k)$

Time evolution:

$$\Theta(k+1) = A\,\Theta(k) + B\,w(k)$$

where $W(k)$ Gaussian white noise process.

Coding of Pulse

Technique of Pulse Coding

Sub−Pulse   #   M

Pulse Length = M x T

Sub−Pulse Waveform: g(t) − "Pulse Shaping Function"

g − Raised Cosine Function

| supp g(t) | = T

Transmit Signal Model

Baseband signal for N array elements

$$s_{n,k}(t) = \sum_{m=0}^{M-1} Y_{n,m}(k)\, g_n(t - (kM + m)T)$$

where $M \geq N$ and k pulse-block number.

$$Y(k) \in \mathbb{C}^{N \times M} : \text{transmitted space-time code}$$

$$Y_{n,m}(k) = W_n(k)\, X_{n,m}(k)$$

Space - Time Coding

Transmitted space-time code for pulse-block k

$$Y(k) \ \in \ \mathbb{C}^{N \times M}$$

$$Y_{n,m}(k) \ = \ W_n(k) \, X_{n,m}(k)$$

Indices $-$ n array element m sub-pulse number in k-th block

$$X(k) \in \mathbb{C}^{N \times M}: \ \textit{Orthogonal} \ \text{code} \ \text{ie} \ XX^H = \tfrac{1}{N} \, \mathbb{I}_N$$

$W_n(k)$: Array Beamformer Weights for pulse-block k

NB. Orthogonal code enables directivity in Beamformer in contrast to Unitary code.

Delay and Doppler across array

The two-way delay and Doppler terms for signal transmission from element n to element m after reflection from a single target are

$$\tau_{n,m}(k) \;=\; \frac{2\,r(k) - (n+m)\,d\,\rho(k)}{c}$$

$$\nu_{n,m}(k) \;=\; \frac{(2\,\dot{r}(k) - (n+m)\,d\,\dot{\rho}(k))\,f_0}{c}$$

f_0 is the carrier frequency

Received Signal at Array

The basebanded received signal at antenna m for transmit block k for a *fully coherent* system [a] is

$$r_{m,k}(t) \; = \; a(k) \, e^{i\phi(k)} \sum_{n=0}^{N-1} s_{n,k} \left(t - \tau_{n,m}(k) \right)$$

$$\times \; e^{2\pi i t \nu_{n,m}(k)} \, e^{-2\pi i f_0 \tau_{n,m}(k)} + \xi_{m,k}(t)$$

$a(k) \geq 0$ target reflectivity amplitude and $\phi(k)$ associated phase shift. $\xi_{m,k}(t)$ receiver noise Gaussian processes with identical variance σ^2.

[a]We assume all transmitters and receivers are phase locked to a reference clock.

Likelihood Function

The method of correlation between the transmitted and received signal leads to the log likelihood function as

$$\ell \;\approx\; \frac{a}{\sigma^2}\,\text{Re}\left\{ e^{-i\phi}\sum_{n,m=0}^{N-1}\chi_{n,m}\left(\tau_{n,m},\nu_{n,m}\right)\right\} - \frac{a^2 N}{2\sigma^2}$$

where we assume $\|s_{n,k}\|^2 = p_{n,k}$ with $\sum_n p_{n,k} = 1 \;\forall\; k$ and

$$\chi_{n,m}(\tau,\nu) \;=\; e^{2\pi i f_0 \tau}\int r_m(t)\,s_n^*(t-\tau)\,e^{-2\pi i \nu t}\,dt$$

which is the *generalised* matched filter.
This is an approximate form since the cross terms are neglected.

Maximum Likelihood Estimate

Maximisation of the likelihood function ℓ gives the sufficient statistic (the generalised matched flier) as a function of the target state in the form

$$\eta(r, \dot{r}, \rho, \dot{\rho}) = \frac{1}{2N\sigma^2} \left| \sum_{n,m=0}^{N-1} \chi_{n,m}\left(\tau(\theta), \nu(\theta)\right) \right|^2 \qquad (1)$$

This form is a *coherent* sum of the outputs of N^2 matched filters. We optimise η subject to the delay and Doppler constraints for the given target. The mean value of the statistic is

$$\mathrm{E}\,\eta(r, \dot{r}, \rho, \dot{\rho}) = \frac{a^2 N}{2\sigma^2} + 1/2 \,. \qquad (2)$$

Thus a linear increase of the mean detection statistic as a function of the number of antennae elements N is obtained.

Receiver Performance

Cost function

$$J(Y; \theta) = \log \det F(Y; \theta)$$

where F is the Fisher information matrix

$$F = -\mathrm{E} \left\{ \frac{\partial^2 \ell(Y; \theta)}{\partial \theta^2} \right\},$$

which is strictly positive definite. This quantity is a function of the quantities $\tau_{n,m}$, and thus of the state variables r and ρ, as well as the signal waveforms, and thus of the code $Y_{n,m}$.

Performance Optimisation

The cost function J is chosen because its maximisation implies minimisation of the Cramer-Rao lower bound on the parameter estimation uncertainty. There are four constraints to be met.Firstly, the beamformer weight magnitudes specify the transmit powers for each antenna, ie $|W_n|^2 = p_n = q_n^2$, with $\sum_n q_n^2 = 1$. We also have the code constraints

$$
\begin{aligned}
x_k^H \, x_n &= \delta_{n,k} \\
x_k^H \, \Gamma_t \, x_n &= 0, \ \forall \, n \neq k \\
x_k^H \, \Gamma_{tt} \, x_n &= 0, \ \forall \, n \neq k \, ,
\end{aligned}
$$

where Γ_t and Γ_{tt} are diagonal matrices specified by the signalling pulse, and the x_n are the rows of X (ie the sequence transmitted on antenna n).

Conclusion

- space-time coding leads to a new design concept for adaptive radar

- estimates are unbiased so that CRLB can be utilised for performance measure

- processing can be done at individual antenna elements – reduces load on central processor for real-time implmentation

- this formulation can be generalised for netcentric radars

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Acknowledgement

Open Problems

- Are the constraints compatible ?

- How do we find the best code ?

- Is the receiver efficient? – Kershaw and Evans proves this for SISO.

- How does one maximize η in the presence of local maxima?