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Theoretical Analysis of Image Processing Using Parameter-Tuning Stochastic Resonance Technique

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14. ABSTRACT
Parameter-tuning stochastic resonance (PSR) technique provides a new approach for signal processing. This paper will first fill the gap in the performance analysis of the nonlinear PSR-based detector by comparing it with the matched filter detector by comparing it with the matched filter detector under both ideal conditions (white Gaussian noise, and perfect synchronization) and no-ideal conditions (colored noise, desynchronization, and low sampling rate) to identify its strengths and weaknesses.

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Theoretical Analysis of Image Processing Using Parameter-Tuning Stochastic Resonance Technique

Bohou Xu, Zhong-Ping Jiang, Xingxing Wu, and Daniel W. Repperger

Abstract— **Parameter-tuning stochastic resonance has been successfully applied to the one-dimensional signal processing. This paper explores the feasibility to extend this technique for image processing. Based on the two-dimensional nonlinear bistable dynamic system, the equation satisfied by the system output probability density function is derived for the first time. The corresponding equation for the one-dimensional system is the famous Fokker-Planck-Kolmogorov (FPK) equation. The stationary solution, eigenvalues and eigenfunctions of this equation are then investigated. The upper bound of the system response speed and the related calculation algorithm which are necessary for the applications of this technique to image processing are also proposed in this paper. Finally, the potential applications of this approach in image processing and some future research are suggested.**

Index Terms— **Stochastic Systems, Stochastic Resonance, Filtering, Nonlinear Systems, Image Processing**

I. INTRODUCTION

Image processing has been widely applied in different areas, such as diagnosing tumors in medical images, detecting and identifying hostile targets in military images. Over the years, many effective image processing algorithms have been proposed to meet the increasing requirements on image qualities. For the images corrupted by noise, most of the denosing algorithms will try to remove or suppress the noise from the systems, because the noise is usually thought to be annoying. Stochastic resonance, on the contrary, is a phenomenon that the noise can be used to enhance rather than hinder the system performance. The concept of stochastic resonance was first proposed by Benzi in 1981 [1]. Since then, stochastic resonance has been applied in a wide-range of areas, such as physics, chemistry, biomedical sciences, and engineering [2]. It has been successfully used to improve the balance control for elderly people [3]. The profoundly deaf people can improve their speech understanding with the aid of noise [4]. For the signal processing area, the stochastic resonance technique has been applied to the signal detection [5], signal transmission [6], and signal estimation [7]. In order to realize stochastic resonance to make the noise beneficial to the system, the synchronization between the input signal and the noise must occur. Basically, there are two approaches to realize the stochastic resonance effect. The traditional method is to add an optimal amount of noise into the system [2]. Parameter-tuning stochastic resonance proposed by us is the other way [8]-[11][16]. It realizes the stochastic resonance effect by tuning system parameters to their optimal values without adding any additional noise into the system. We also reveal that the parameter-tuning stochastic resonance is superior to the traditional method [8], especially when the initial noise intensity is already beyond its resonant region. Parameter-tuning stochastic resonance has also been used to recover the noisy multi-frequency signals [9], reduce the bit-error rate (BER) of the transmission of baseband binary signals [8]. Image processing is another potential application area of stochastic resonance technique. There are some initial research work on it [12][13]. The initial research results demonstrate that it is promising and feasible to develop the innovative and effective image processing algorithms using stochastic resonance technique. All these approaches, however, are based on simulations and are lacking rigorous theoretical analysis. It is difficult to implement and extend to other image processing fields. Another problem with these approaches is that they need to add an optimal additional noise into the images, which is impossible for some image processing tasks. All these motivate us to investigate the application of parameter-tuning stochastic resonance in image processing based on the systematic and theoretical analysis. This paper will mainly focus on the theoretical investigation of the feasibility of this approach. In order to apply the current parameter-tuning technique, the two-dimensional image signals can be first converted to one-dimensional signals. Unfortunately, our research demonstrates this method is not effective. The only possible method is then to treat the two-dimensional image signals directly without conversion and develop all the theoretical results in an analytic way as one-dimensional parameter-tuning stochastic resonance. For one-dimensional systems, the parameter-tuning stochastic resonance is based on the derivation of the Fokker-Planck-Kolmogorov (FPK) equation satisfied by the system output probability density function and also the derivation of the system response speed and its calculation algorithm. Similarly, for two-dimensional image signals, the feasibility investigation means whether the equation and its solutions satisfied by system output probability density function can be derived and whether the system response time can be calculated. So far, no related research result could be found in this area.

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The rest of this paper is organized as follows. In Section II, we will first propose a two-dimensional nonlinear bistable dynamic system and then derive the equation satisfied by the system output probability density function. Section III will give the stationary solution of this equation. The system response speed for this two-dimensional system will be investigated in Section IV. The potential applications of this approach in image processing are mentioned in Section V. Finally, Section VI closes the paper with brief concluding remarks and future research directions.

II. TWO-DIMENSIONAL NONLINEAR BISTABLE SYSTEM AND RELATED FPK EQUATION

The one-dimensional nonlinear bistable stochastic resonance system can be described by the following equation [2]

$$\dot{x}(t) = ax(t) - bx^3(t) + s(t) + \eta(t), \tag{1}$$

where a and b are system parameters, $s(t)$ is the input signal, and $\eta(t)$ is an additive Gaussian white noise with zero mean average and autocorrelation of $\langle \eta(t)\eta(0)\rangle = 2D\delta(t)$.

For this nonlinear dynamic system, the output signal-to-noise ratio will be maximized when an optimal amount of noise is added into the system. This is the stochastic resonance phenomenon.

Similarly, we can propose the following two-dimensional nonlinear bistable dynamic system

$$\frac{\partial^2 w}{\partial x \partial y} = f(w) + \Gamma(x, y), \tag{2}$$

where $w = w(x, y)$ is the state variable (system output), $f(w) = aw - bw^3 + h$, $\Gamma(x, y)$ is additive white Gaussian noise, and h is the input signal.

The corresponding difference equation is

$$\begin{aligned} w(x+\Delta x, y+\Delta y) =& w(x+\Delta x, y) + w(x, y+\Delta y) - w(x, y) \\ &+ f(w)\Delta x\Delta y + \Gamma(x, y)\Delta x\Delta y. \end{aligned} \tag{3}$$

In order to derive the FPK equation satisfied by the system output probability density function, one possible approach is to use the similar method as deriving one-dimensional FPK equation [14].

We have

$$\begin{aligned} &\rho(w, x+\Delta x, y+\Delta y) \\ &= \int P(w, x+\Delta x, y+\Delta y | u, x, y)\rho(u, x, y)du, \end{aligned} \tag{4}$$

where $\rho(w, x, y)$ is the probability density function of the system output w and $P(w, x + \Delta x, y + \Delta y|u, x, y)$ is the conditional probability density function.

Also,

$$\begin{aligned} &P(w, x+\Delta x, y+\Delta y|u, x, y) \\ &= \int \delta(v-w)P(v, x+\Delta x, y+\Delta y|u, x, y)dv \\ &= \int \sum_{n=0}^{\infty} \frac{(v-u)^n}{n!} \frac{\partial^n \delta(u-w)}{\partial u^n} P(v, x+\Delta x, y+\Delta y|u, x, y)dv \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} M_n(u, x, \Delta x, y, \Delta y)\frac{\partial^n \delta(u-w)}{\partial u^n}, \end{aligned} \tag{5}$$

where

$$M_0 = 1, \tag{6}$$

$$\begin{aligned} &M_n(u, x, \Delta x, y, \Delta y) \\ &= \int (v-u)^n P(v, x+\Delta x, y+\Delta y|u, x, y)dv, n = 1, 2, \ldots \end{aligned} \tag{7}$$

Then

$$\begin{aligned} &\rho(w, x+\Delta x, y+\Delta y) - \rho(w, x, y) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \int M_n(u, x, \Delta x, y, \Delta y)\rho(u, x, y)\frac{\partial^n \delta(u-w)}{\partial u^n} du \\ &= \sum_{n=1}^{\infty} \frac{1}{n!}(-\frac{\partial}{\partial w})^n [M_n(w, x, \Delta x, y, \Delta y)\rho(w, x, y)]. \end{aligned} \tag{8}$$

Similarly, we can obtain

$$\begin{aligned} &\rho(w, x+\Delta x, y) - \rho(w, x, y) \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{n!}(-\frac{\partial}{\partial w})^n [M_n(w, x, \Delta x, y, 0)\rho(w, x, y)]. \end{aligned} \tag{9}$$

$$\begin{aligned} &\rho(w, x, y+\Delta y) - \rho(w, x, y) \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{n!}(-\frac{\partial}{\partial w})^n [M_n(w, x, 0, y, \Delta y)\rho(w, x, y)]. \end{aligned} \tag{10}$$

From this, we can derive the corresponding FPK equation

$$\frac{\partial^2 \rho(w, x, y)}{\partial x \partial y} = \sum_{n=1}^{\infty} \frac{1}{n!}(-\frac{\partial}{\partial w})^n [C_n(w, x, y)\rho(w, x, y)], \tag{11}$$

where

$$\begin{aligned} C_n(w, x, y) =& \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{1}{\Delta x \Delta y}[M_n(w, x, \Delta x, y, \Delta y) \\ &- M_n(w, x, \Delta x, y, 0) - M_n(w, x, 0, y, \Delta y)]. \end{aligned} \tag{12}$$

For $n = 1$, C_1 can be calculated as

$$\begin{aligned} C_1(w, x, y)\Delta x\Delta y =& \langle w(x+\Delta x, y+\Delta y) - w(x+\Delta x, y) \\ &- w(x, y+\Delta y) + w(x, y)\rangle \\ =& f(w)\Delta x\Delta y. \end{aligned} \tag{13}$$

where the angular brackets denote the ensemble average.

Unfortunately, it is impossible to calculate C_n for $n \geq 2$, because

$$\begin{aligned} C_2(w, x, y)\Delta x\Delta y =& \langle [w(x+\Delta x, y+\Delta y) - w(x, y)]^2\rangle \\ &- \langle [w(x+\Delta x, y) - w(x, y)]^2\rangle \\ &- \langle [w(x, y+\Delta y) - w(x, y)]^2\rangle. \end{aligned} \tag{14}$$

The system described by (2) provides no enough information to calculate the C_m for $m \geq 2$.

To overcome the above difficulty, we reduce the partial differential equation (2) to an ordinary differential equation along the line $x = x_0 + t\Delta x, y = y_0 + t\Delta y$ as

$$\frac{d^2 w}{dt^2} = \Delta x\Delta y f(w) + \Delta x\Delta y \Gamma(x_0 + t\Delta x, y_0 + t\Delta y), \tag{15}$$

where

$$\langle \Gamma(x, y)\Gamma(x_1, y_1)\rangle = 2D\delta(x - x_1, y - y_1). \tag{16}$$

The equation (15) can then be rewritten as

$$\begin{aligned}\frac{dw}{dt}=&\Delta xv,\\ \frac{dv}{dt}=&\Delta yf(w)+\Delta y\Gamma(\Delta xt,\Delta yt)\\ =&\Delta yf(w)+\Delta y\Gamma_1(t),\end{aligned} \quad (17)$$

where

$$\langle\Gamma_1(t)\Gamma_1(t_1)\rangle=2D\delta(t-t_1). \quad (18)$$

Based on the concept in [14], we can prove the probability density function $\rho(w,v,t)$ satisfies the following FPK equation

$$\begin{aligned}\frac{\partial\rho(w,v,t)}{\partial t}=&-\frac{\partial}{\partial w}[\Delta xv\rho(w,v,t)]-\frac{\partial}{\partial v}[\Delta yf(w)\rho(w,v,t)]\\ &+D\Delta y^2\frac{\partial^2\rho(w,v,t)}{\partial v^2}.\end{aligned} \quad (19)$$

Equation (19) is independent on (x_0,y_0) explicitly, the solution of this equation will be valid for any (x_0,y_0).

Equation (19), however, does not have a stationary solution, because it is a hyperbolic equation without any damping [17]. In order to overcome this problem, equation (2) is now changed to

$$\frac{\partial^2 w}{\partial x\partial y}=-\gamma\frac{\partial w}{\partial x}+f(w)+\Gamma(x,y), \quad (20)$$

where γ is a positive damping coefficient.

In the similar way, we can derive the FPK equation for system (20)

$$\begin{aligned}\frac{\partial\rho(w,v,t)}{\partial t}=&-\frac{\partial}{\partial w}[\Delta xv\rho(w,v,t)]\\ &-\frac{\partial}{\partial v}[\Delta y(-\gamma v+f(w))\rho(w,v,t)]\\ &+D\Delta y^2\frac{\partial^2\rho(w,v,t)}{\partial v^2}.\end{aligned} \quad (21)$$

In the following section, we will show that the stationary solution of (21) exists. System described by (20) will be used as the two-dimensional bistable dynamic stochastic resonance system to process the image signals.

III. STATIONARY SOLUTION OF TWO-DIMENSIONAL FPK EQUATIONS

Equation (21) is linear for the probability density function $\rho(w,v,t)$, so it is possible to be solved by eigenfunction expanding method.

Let $\rho_0(w,v)$ be the stationary solution. It will satisfy the following equation

$$\begin{aligned}-\frac{\partial}{\partial w}[\Delta xv\rho_0(w,v)]-&\frac{\partial}{\partial v}[\Delta y(-\gamma v+f(w))\rho_0(w,v)]\\ &+D\Delta y^2\frac{\partial^2\rho_0(w,v)}{\partial v^2}=0.\end{aligned} \quad (22)$$

Assume $\rho_0(w,v)=e^{-a_0v^2}\varphi(w)$, we have

$$\begin{aligned}&-\Delta xve^{-a_0v^2}\varphi'(w)+\gamma\Delta ye^{-a_0v^2}\varphi(w)\\ &+2a_0\Delta y(-\gamma v+f(w))ve^{-a_0v^2}\varphi(w)\\ &+D\Delta y^2(-2a_0+4a_0^2v^2)e^{-a_0v^2}\varphi(w)=0.\end{aligned} \quad (23)$$

In order to meet (23), the following should be satisfied by the coefficient of v^i, for $i=0,1,2$

$$v^0:\quad \gamma\Delta y-2a_0D\Delta y^2=0, \quad (24)$$

$$v^1:\quad -\Delta x\varphi'(w)+2a_0\Delta yf(w)\varphi(w)=0, \quad (25)$$

$$v^2:\quad -2a_0\gamma\Delta y+4a_0^2D\Delta y^2=0. \quad (26)$$

By solving (24), (25), and (26), we can get

$$a_0=\frac{\gamma}{2D\Delta y}, \quad (27)$$

$$\varphi(w)=N_0\exp[\frac{\gamma}{D\Delta x}\int_0^w f(w)dw], \quad (28)$$

$$\rho_0(w,v)=N_0\exp[-\frac{\gamma}{2D\Delta y}v^2+\frac{\gamma}{D\Delta x}\int_0^w f(w)dw], \quad (29)$$

where N_0 can be determined by

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_0(w,v)dwdv=1. \quad (30)$$

For convenience, (29) can be rewritten as

$$\rho_0(w,v)=e^{-\phi(w,v)}, \quad (31)$$

where

$$\phi(w,v)=\frac{\gamma}{2D\Delta y}v^2-\frac{\gamma}{D\Delta x}\int_0^w f(w)dw-\ln N_0. \quad (32)$$

IV. SYSTEM RESPONSE SPEED OF TWO-DIMENSIONAL BISTABLE SYSTEMS

Similar to one-dimensional parameter-tuning stochastic resonance, investigating the characteristics of system response speed and developing its calculation algorithm are very important tasks for the two-dimensional parameter-tuning stochastic resonance. It will determine whether the one-dimensional parameter-tuning stochastic resonance can be extended to the two-dimensional case.

Let

$$D_1=v\Delta x, \quad (33)$$

$$D_2=\Delta y[-\gamma v+f(w)], \quad (34)$$

$$D_{22}=D\Delta y^2. \quad (35)$$

Equation (21) can be rewritten as

$$\begin{aligned}\frac{\partial\rho}{\partial t}=&-\frac{\partial}{\partial w}(D_1\rho)-\frac{\partial}{\partial v}(D_2\rho)+D_{22}\frac{\partial^2\rho}{\partial v^2}\\ =&L_{FP}(\rho),\end{aligned} \quad (36)$$

where

$$L_{FP}(\rho)=-\frac{\partial}{\partial w}(D_1\rho)-\frac{\partial}{\partial v}(D_2\rho)+D_{22}\frac{\partial^2\rho}{\partial v^2}. \quad (37)$$

Assume $\rho = \xi(w,v)e^{-\lambda t}$, equation (36) then becomes

$$\lambda\xi(w,v) = -L_{FP}\xi(w,v). \tag{38}$$

Similar to the one-dimensional case, let

$$\xi(w,v) = \psi(w,v)e^{-\phi/2}, \tag{39}$$

where ϕ is defined in (32) and $e^{-\phi}$ is the stationary solution of (21).

In this case, equation (38) becomes

$$\lambda\psi = -L\psi, \tag{40}$$

where L is a differential operator, and

$$\begin{aligned} L\psi=&e^{\phi/2}L_{FP}(e^{-\phi/2}\psi)\\ =&-e^{\phi/2}\frac{\partial}{\partial w}(D_1e^{-\phi/2}\psi) - e^{\phi/2}\frac{\partial}{\partial v}(D_2e^{-\phi/2}\psi)\\ &+e^{\phi/2}D_{22}\frac{\partial^2}{\partial v^2}(e^{-\phi/2}\psi). \end{aligned} \tag{41}$$

We can assume

$$\psi(w,v) = O(e^{-\phi/2}), \quad \text{as} \quad w \to \pm\infty \quad \text{or} \quad v \to \pm\infty, \tag{42}$$

because

$$\lim_{w\to\pm\infty} \rho(w,v,t)= \lim_{v\to\pm\infty} \rho(w,v,t) = 0, \tag{43}$$

$$\iint e^{-\phi}dwdv=\text{const} \neq 0. \tag{44}$$

We denote the conjugate operator of L as L^c. Let $L_s = (L+L^c)/2$, and $L_{as} = (L-L^c)/2$, that is $L = L_s + L_{as}$, we can obtain the following theorems.

Theorem 1: The differential operator L in (40) can be decomposed into symmetric part L_s and anti-symmetric part L_{as}. Also, L_s is negative semi-definite.

Proof: First, we will calculate the conjugate operator of L. It is denoted as L^c. Assume

$$\lim_{w\to\pm\infty} \psi(w,v)= \lim_{w\to\pm\infty} \eta(w,v) = 0, \tag{45}$$

$$\lim_{v\to\pm\infty} \psi(w,v)= \lim_{v\to\pm\infty} \eta(w,v) = 0. \tag{46}$$

From

$$\begin{aligned} &\iint \eta\, Ldv\psi dw\\ &=\iint \eta[-e^{\phi/2}\frac{\partial}{\partial w}(D_1e^{-\phi/2}\psi) - e^{\phi/2}\frac{\partial}{\partial v}(D_2e^{-\phi/2}\psi)\\ &+e^{\phi/2}D_{22}\frac{\partial^2}{\partial v^2}(e^{-\phi/2}\psi)]dwdv\\ &=\iint \psi[D_1e^{-\phi/2}\frac{\partial}{\partial w}(e^{\phi/2}\eta) + D_2e^{-\phi/2}\frac{\partial}{\partial v}(e^{\phi/2}\eta)\\ &+D_{22}e^{-\phi/2}\frac{\partial^2}{\partial v^2}(e^{\phi/2}\eta)]dwdv, \end{aligned} \tag{47}$$

we can define the conjugate operator of L as

$$\begin{aligned} L^c=&e^{-\phi/2}[D_1\frac{\partial}{\partial w}(e^{\phi/2}) + D_2\frac{\partial}{\partial v}(e^{\phi/2})\\ &+D_{22}\frac{\partial^2}{\partial v^2}(e^{\phi/2})]. \end{aligned} \tag{48}$$

Now, we can calculate the symmetric part L_s and the anti-symmetric part L_{as}. Because

$$-\frac{\partial}{\partial w}(D_1e^{-\phi}) - \frac{\partial}{\partial v}(D_2e^{-\phi}) + \frac{\partial}{\partial v}(D_{22}\frac{\partial}{\partial v}e^{-\phi}) = 0. \tag{49}$$

we can get

$$\begin{aligned} L_s\psi=&\frac{(L+L^c)}{2}\psi\\ =&\frac{1}{2}[-e^{\phi/2}\frac{\partial}{\partial w}(D_1e^{-\phi/2}\psi) - e^{\phi/2}\frac{\partial}{\partial v}(D_2e^{-\phi/2}\psi)\\ &+e^{\phi/2}D_{22}\frac{\partial^2}{\partial v^2}(e^{-\phi/2}\psi) + e^{-\phi/2}D_1\frac{\partial}{\partial w}(e^{\phi/2}\psi)\\ &+e^{-\phi/2}D_2\frac{\partial}{\partial v}(e^{\phi/2}\psi) + e^{-\phi/2}D_{22}\frac{\partial^2}{\partial v^2}(e^{\phi/2}\psi)]\\ =&\frac{1}{2}\psi e^{\phi}[-\frac{\partial}{\partial w}(D_1e^{-\phi}) - \frac{\partial}{\partial v}(D_2e^{-\phi}) + D_{22}\frac{\partial^2}{\partial v^2}(e^{-\phi})]\\ &+e^{\phi/2}\frac{\partial}{\partial v}[D_{22}e^{-\phi}\frac{\partial}{\partial v}(e^{\phi/2}\psi)]\\ =&e^{\phi/2}\frac{\partial}{\partial v}[D_{22}e^{-\phi}\frac{\partial}{\partial v}(e^{\phi/2}\psi)]. \end{aligned} \tag{50}$$

Also

$$\begin{aligned} L_{as}\psi=&\frac{1}{2}(L-L^c)\psi = (L-L_s)\psi\\ =&e^{\phi/2}\{-\frac{\partial}{\partial w}(D_1e^{-\phi/2}\psi) - \frac{\partial}{\partial v}(D_2e^{-\phi/2}\psi)\\ &+\frac{\partial}{\partial v}[D_{22}\frac{\partial}{\partial v}(e^{-\phi/2}\psi)]\} - e^{\phi/2}\frac{\partial}{\partial v}[D_{22}e^{-\phi}\frac{\partial}{\partial v}(e^{\phi/2}\psi)]\\ =&-e^{\phi/2}\{\frac{\partial}{\partial w}(D_1e^{-\phi/2}\psi) + \frac{\partial}{\partial v}(D_2e^{-\phi/2}\psi)\\ &+\frac{\partial}{\partial v}[D_{22}\frac{\partial\phi}{\partial v}e^{-\phi/2}\psi]\}. \end{aligned} \tag{51}$$

To conclude, we get

$$L_s=e^{\phi/2}\frac{\partial}{\partial v}[D_{22}e^{-\phi}\frac{\partial}{\partial v}(e^{\phi/2})], \tag{52}$$

$$\begin{aligned} L_{as}=&-e^{\phi/2}[\frac{\partial}{\partial w}(D_1e^{-\phi/2}) + \frac{\partial}{\partial v}(D_2e^{-\phi/2})\\ &+\frac{\partial}{\partial v}(D_{22}\frac{\partial\phi}{\partial v}e^{-\phi/2})]. \end{aligned} \tag{53}$$

For any ψ and η satisfying (45) and (46), we have

$$\begin{aligned} &\iint \eta L_s\psi dwdv\\ &= \iint e^{\phi/2}\eta\frac{\partial}{\partial v}[D_{22}e^{-\phi}\frac{\partial}{\partial v}(e^{\phi/2}\psi)]dwdv\\ &= -\iint D_{22}e^{-\phi}\frac{\partial}{\partial v}(e^{\phi/2}\psi)\frac{\partial}{\partial v}(e^{\phi/2}\eta)dwdv\\ &= \iint e^{\phi/2}\psi\frac{\partial}{\partial v}[D_{22}e^{-\phi}\frac{\partial}{\partial v}(e^{\phi/2}\eta)]dwdv\\ &= \iint \psi L_s\eta dwdv, \end{aligned} \tag{54}$$

and

$$\iint \eta L_{as}\psi dwdv$$
$$= -\iint e^{\phi/2}\eta[\frac{\partial}{\partial w}(e^{-\phi/2}\psi) + \frac{\partial}{\partial v}(D_2e^{-\phi/2}\psi)$$
$$+\frac{\partial}{\partial v}(D_{22}\frac{\partial\phi}{\partial v}e^{-\phi/2}\psi)]dwdv$$
$$= \iint [e^{-\phi/2}\psi D_1\frac{\partial}{\partial w}(e^{\phi/2}\eta) + e^{-\phi/2}\psi D_2\frac{\partial}{\partial v}(e^{\phi/2}\eta)$$
$$+D_{22}\frac{\partial\phi}{\partial v}e^{-\phi/2}\psi\frac{\partial}{\partial v}(e^{\phi/2}\eta)]dwdv$$
$$= \iint e^{\phi/2}\psi[\frac{\partial}{\partial w}(D_1e^{-\phi/2}\eta) + \frac{\partial}{\partial v}(D_2e^{-\phi/2}\eta)$$
$$+\frac{\partial}{\partial v}(D_{22}\frac{\partial\phi}{\partial v}e^{-\phi/2}\eta)]dwdv$$
$$= -\iint \psi L_{as}\eta\, dwdv. \quad (55)$$

This means the operator L_s is symmetric and L_{as} is anti-symmetric. Now, we will prove L_s is also negative semi-definite.

We have

$$\iint \psi L_s\psi dwdv = \iint \psi e^{\phi/2}\frac{\partial}{\partial v}[D_{22}e^{-\phi}\frac{\partial}{\partial v}(e^{\phi/2}\psi)]dwdv$$
$$= -\iint D_{22}e^{-\phi}(\frac{\partial}{\partial v}e^{\phi/2}\psi)^2dwdv \le 0. \quad (56)$$

From this, we can conclude that L_s is semi-negative definite. This completes the proof. ■

Theorem 2: If λ is any non-zero eigenvalue of the operator $-L$, and λ_1^s is the least non-zero eigenvalue of the operator $-L_s$, then $Re\lambda \ge \lambda_1^s$.

Proof: Assume $\{\lambda_i^s, \psi_i^s\}$, for $i = 0, 1, ...$, are the eigenvalues and eigenfunctions of $-L_s$, that is

$$-L_s\psi_i^s = \lambda_i^s\psi_i^s, \quad i = 0, 1, ... \quad (57)$$

Because $-L_s$ is symmetric and semi-positive definite, we have

$$0 = \lambda_0^s < \lambda_1^s \le \lambda_2^s \le ..., \quad (58)$$
$$\iint \psi_i^s\psi_j^s dwdv = \delta_{ij}. \quad (59)$$

Moreover, we assume that λ_j and ψ_j are the eigenvalue and eigenfunction of $-L$. That is $-L\psi_j = \lambda_j\psi_j$. λ_j might be complex. Then, we get

$$\lambda_j = -\frac{\iint \psi_j^* L\psi_j dwdv}{\iint \psi_j^*\psi_j dwdv} \quad (60)$$

where ψ_j^* is the complex conjugate of ψ_j.

Because $\{\psi_i^s\}$ is complete in H-space, ψ_j can be expanded as

$$\psi_j = \sum_{i=0}^{\infty} C_{ji}\psi_i^s. \quad (61)$$

Substitute (61) into (60), we get

$$\lambda_j = -\frac{\iint \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} C_{ji}^*\psi_i^s L\psi_k^s C_{jk}dwdv}{\sum_{i=0}^{\infty}|C_{ji}|^2}. \quad (62)$$

We can also obtain the following

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} C_{ji}^*C_{jk}\iint \psi_i^s L\psi_k^s dwdv$$
$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} C_{ji}^*C_{jk}(-\lambda_i^s\delta_{ik} + \iint \psi_i^s L_{as}\psi_k^s dwdv). \quad (63)$$

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} C_{ji}^*C_{jk}\psi_i^s L_{as}\psi_k^s + \overline{\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} C_{ji}^*C_{jk}\psi_i^s L_{as}\psi_k^s}$$
$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} C_{ji}^*C_{jk}\psi_i^s L_{as}\psi_k^s + \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} C_{ji}C_{jk}^*\psi_i^s L_{as}\psi_k^s$$
$$= 0. \quad (64)$$

From above, we have

$$Re(\lambda_j) = \frac{\sum_{i=0}^{\infty}|C_{ji}|^2\lambda_i^s}{\sum_{i=0}^{\infty}|C_{ji}|^2} \quad (65)$$

Assume $0 < Re(\lambda_1) \le Re(\lambda_2) \le ...$, then

$$Re(\lambda_1) = \frac{\sum_{i=1}^{\infty}|C_{1i}|^2\lambda_i^s}{\sum_{i=1}^{\infty}|C_{1i}|^2} \ge \lambda_1^s. \quad (66)$$

Here, $C_{10} = 0$. We then prove that $Re(\lambda) \ge \lambda_1^s$. ■

From Theorem 2, we can regard λ_1^s as a lower bound of the system response speed of $-L$ and take λ_1^s as an approximation of the system response speed of system (20). We now investigate its calculation algorithm.

The eigenvalue problem of operator L_s, $\lambda^s\psi = -L_s\psi$, can be rewritten into the variational form (Rayleigh quotient)[15]

$$\lambda^s = \underset{\psi\ne 0}{\textbf{st.}} -\frac{\iint \psi L_s\psi dwdv}{\iint \psi^2 dwdv}$$
$$= \underset{\psi\ne 0}{\textbf{st.}} -\frac{\iint \psi e^{\phi/2}\frac{\partial}{\partial v}[D_{22}e^{-\phi}\frac{\partial}{\partial v}(e^{\phi/2}\psi)]dwdv}{\iint \psi^2 dwdv}$$
$$= \underset{\psi\ne 0}{\textbf{st.}} \frac{\iint D_{22}e^{-\phi}[\frac{\partial}{\partial v}(e^{\phi/2}\psi)]^2dwdv}{\iint \psi^2 dwdv}$$
$$= \underset{\zeta\ne 0}{\textbf{st.}} \frac{\iint D_{22}e^{-\phi}(\frac{\partial\zeta}{\partial v})^2dwdv}{\iint e^{-\phi}\zeta^2 dwdv} \quad (67)$$

where $\zeta = e^{\phi/2}\psi$, and "**st.**" denotes the stationary value of the functional.

Obviously, $\lambda^s \ge 0$. It will take the zero value, if ζ is constant (This means $\psi = Ce^{-\phi/2}$, and it is a stationary solution). The above problem can be solved using the subspace iteration method.

The calculation algorithm of λ_1^s is outlined as follows

1) Determine the integral range in (67);
2) Divide the integral range into $n \times m$ parts and use the finite element method to transform the variational problem (67) into a general-matrix eigenvalue problem

$$(\lambda[M] - [K])\{\zeta\} = 0. \quad (68)$$

where $[M]$ is a positive definite matrix and $[K]$ is a non-negative matrix.

3) Use subspace against-iteration method to get the minimum positive eigenvalue λ_1^s.

Besides the stationary solution, we can now get the expanding solution of (21)

$$\rho(w,v,t) = \rho_0(w,v) + C_1\psi_1(w,v)e^{-\lambda_1 t-\phi/2} + C_2\psi_2(w,v)e^{-\lambda_2 t-\phi/2} + \dots \quad (69)$$

where ρ_0 is the stationary solution described by (31), and $\{\lambda_i, \psi_i\}$, for $i = 1,2,\dots$, are the eigenvalues and eigenfunction of $-L$. It is obvious that $\rho(w,v,t) \approx \rho_0(w,v)$, when $\lambda_1^s t \gg 1$. This can be satisfied by adjusting the system parameters a and b which are defined in (2). In this case, we can use $\rho_0(w,v)$ to approximate $\rho(w,v,t)$ when applying this technique to image processing.

V. POTENTIAL APPLICATIONS IN IMAGE PROCESSING

The derivation of (69) demonstrates the feasibility to extend the concepts of parameter-tuning stochastic resonance to the two-dimensional case. The two-dimensional bistable system (20) can be used as a nonlinear filter to process the noisy two-dimensional image signal to improve the image quality. The probability characteristics of the image signal after the filter is described by (20). It can then be used to derive other performance measures based on different image processing tasks, such as image signal-to-noise ratio, probability of target detection error, etc. The performance measures can then be taken as the object function to be optimized by tuning the system parameter a and b, which is the concept of parameter-tuning stochastic resonance. This method has the potential to be applied in image processing tasks in which the image signals are corrupted by the noise. The system parameters will be tuned properly to synchronize the signals and the noise to convert the noise to be a positive factor to improve the image qualities.

VI. CONCLUSION AND FUTURE WORK

Through theoretical analysis, this paper reveals that the one-dimensional parameter-tuning stochastic resonance can be extended to the two-dimensional case and it is feasible to use it for image processing. The Fokker-Planck-Kolmogorov (FPK) equation for the two-dimensional nonlinear bistable dynamic stochastic resonance system can be derived. Its stationary solution is also given in this paper. Also, the system response speed for this two-dimensional bistable system, which is another important part of parameter-tuning stochastic resonance, can be calculated with our proposed algorithm. All these demonstrate that all the critical techniques of the parameter-tuning stochastic resonance are now available. They are ready to be applied to process two-dimensional image signals. The two-dimensional parameter-tuning stochastic resonance provides an innovative and promising approach for image processing. It will find wide-range applications, just like that of one-dimensional parameter-tuning stochastic resonance. Next, we will apply this new approach to real image processing tasks. We will start from the binary images with white Gaussian noise and then extend to gray images with colored noise.

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