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Cost Cumulant-Based Control for a Class of Linear Quadratic Tracking Problems

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Abstract—The topic of cost-cumulant control is currently receiving substantial research from the theoretical community oriented toward stochastic control theory. For instance, the present paper extends the application of cost-cumulant controller design to control of a wide class of linear-quadratic tracking systems where output measurements of a tracker follow as closely as possible a desired trajectory via a complete statistical description of the associated integral-quadratic performance-measure. It is shown that the tracking problem can be solved in two parts: one, a feedback control whose optimization criterion representing a linear combination of finite cumulant indices of an integral-quadratic performance-measure associated to a linear tracking stochastic system over a finite horizon, is determined by a set of Riccati-type differential equations; and two, an affine control which takes into account of dynamics mismatched between a desired trajectory and tracker states, is found by solving an auxiliary set of differential equations (incorporating the desired trajectory) backward from a stable final time.

I. Preliminaries

An interesting extension of the cost-cumulant control theory [4]-[7] when both perfect and noisy state measurements are available, is to make a linear stochastic system track as closely as possible a desired trajectory via a complete statistical description of the associated finite-horizon integral-quadratic performance-measure. To the best knowledge of the author, this theoretical development appears to be the first of its kind and the optimal control problem being considered herein is actually quite general, and will enable control engineer not only to penalize for variations in, as well as for the levels of, the state variables and control variables, but also to characterize the probabilistic distribution of the performance-measure as needed in post controller-design analysis. Since this problem formulation is parameterized both by the number of cumulants and by the scalar coefficients in the linear combination, it defines a very general Linear-Quadratic-Gaussian (LQG) and Risk Sensitive problem classes. The special cases where only the first cost cumulant is minimized and whereas a denumerable linear combination of cost cumulants is minimized are, of course, the well known minimum-mean LQG problem and the Risk Sensitive control objective, respectively. Some practical applications for this theoretical development can be found in the references [2] and [3] where in tactical and combat situations, a vehicle with the goal seeking nature initially decides on an appropriate destination and then moves in an optimal fashion toward that destination, and tracking problems in economic stabilization policy, respectively.

Consider a linear stochastic tracking system governed by

\[
dx(t) = (A(t)x(t) + B(t)u(t))dt + G(t)dw(t), \quad x(t_0) = x_0, \quad t \in [0, T]
\]

(1)

\[
y(t) = C(t)x(t)
\]

(2)

where the deterministic coefficients \(A \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n})\), \(B \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times m})\), \(C \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{r \times n})\), and \(G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{r \times p})\). The system noise \(w(t) \in \mathbb{R}^p\) is the \(p\)-dimensional stationary Wiener process starting from \(t_0\), independent of the known \(x(t_0) = x_0\), and defined with \(\{F_{t \geq 0}\}\) being its filtration on a complete filtered probability space \((\Omega, \mathcal{F}, \{F_{t \geq 0}\}, \mathbb{P})\) over \([t_0, t_f]\) with the correlation

\[
E\left\{[w(\tau) - w(\xi)] [w(\tau) - w(\xi)]^T\right\} = \mathbb{W} |\tau - \xi|, \quad \mathbb{W} > 0.
\]

The control input \(u \in L^2_{\mathcal{F}_T}(\Omega; C([t_0, t_f]; \mathbb{R}^m))\) the subset of Hilbert space of \(\mathbb{R}^m\)-valued square-integrable process on \([t_0, t_f]\) that are adapted to the \(\sigma\)-field \(\mathcal{F}_T\) generated by \(w(t)\) to the specified system model is selected so that the measurement output \(y \in L^2_{\mathcal{F}_T}(\Omega; C([t_0, t_f]; \mathbb{R}^r))\) best matches the desired output \(z \in L^2([t_0, t_f]; \mathbb{R}^r)\) in the cost cumulant optimization criterion which will be clear shortly. Associated with the initial condition \((t_0, x_0; u) \in [t_0, t_f] \times \mathbb{R}^n \times L^2_{\mathcal{F}_T}(\Omega; C([t_0, t_f]; \mathbb{R}^m))\) is a traditional finite-horizon IQF random cost \(J : [t_0, t_f] \times \mathbb{R}^n \times L^2_{\mathcal{F}_T}(\Omega; C([t_0, t_f]; \mathbb{R}^m)) \mapsto \mathbb{R}^+\) such that

\[
J(t_0, x_0; u) = \left[z(t_f) - y(t_f)\right]^T Q_f \left[z(t_f) - y(t_f)\right]
\]

(3)

\[
+ \int_{t_0}^{t_f} \left[z(t) - y(t)\right]^T Q(t) \left[z(t) - y(t)\right] + u^T(t) R(t) u(t) dt
\]

in which the terminal penalty error weighting \(Q_f \in \mathbb{R}^{r \times r}\), the error weighting \(Q \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{r \times r})\), and the control input weighting \(R \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times m})\) are deterministic, symmetric, and positive semi-definite with \(R(t)\) invertible. In view of the linear system (1)-(2) and the quadratic cost (3), it is reasonable to assume the control input being generated from a class of linear-memoryless state-feedback strategies \(\gamma : [t_0, t_f] \times L^2_{\mathcal{F}_T}(\Omega; C([t_0, t_f]; \mathbb{R}^m)) \mapsto L^2_{\mathcal{F}_T}(\Omega; C([t_0, t_f]; \mathbb{R}^m))\), has a form of

\[
u(t) = \gamma(t, x(t)) = K(t)x(t) + u_z(t),
\]

(4)

where \(u_z \in \mathcal{C}([t_0, t_f]; \mathbb{R}^m)\) is an additional control signal which takes into consideration for dynamics mismatched between the tracking states \(x(t)\) and the desired trajectory.
\[ \psi(\alpha, x_\alpha; \theta) = \Psi(\alpha) + 2x_\alpha^T \eta(\alpha, \theta) + v(\alpha, \theta) \]  

with the terminal conditions \( T(tf, \theta) = \theta C^T(tf) Q_f C(tf), \)  
\( \tilde{\eta}(tf, \theta) = \theta C^T(tf) Q_f \tilde{z}(tf), \)  
\( v(tf, \theta) = \theta z^T(tf) Q_f \tilde{z}(tf). \)  

**Remark.** The expression for cost cumulants (10) in the tracking problem indicates that additional second and third affine terms are taken into account of dynamics mismatched in their trajectory-governing equations.

By definition, cost cumulants for the tracking problem can be generated by employing the MacLaurin series expansion for the cumulant-generating function

\[ \psi(\alpha, x_\alpha; \theta) = \sum_{i=1}^{\infty} \kappa_i(\alpha, x_\alpha) \frac{\theta^i}{i!}, \]

in which \( \kappa_i(\alpha, x_\alpha) \) are called cost cumulants. Furthermore, the series coefficients of the expansion are computed by

\[ \left. \frac{\partial^{(i)}}{\partial \theta^i} \psi(\alpha, x_\alpha; \theta) \right|_{\theta=0} = \frac{\partial^{(i)}}{\partial \theta^i} \Psi(\alpha) + 2x_\alpha^T \left. \frac{\partial^{(i)}}{\partial \theta^i} \eta(\alpha, \theta) \right|_{\theta=0} + \left. \frac{\partial^{(i)}}{\partial \theta^i} v(\alpha, \theta) \right|_{\theta=0}. \]

In view of the results (14) and (15), cost cumulants for the tracking problem are obtained as follows

\[ \kappa_i(\alpha, x_\alpha) = x_\alpha^T \left. \frac{\partial^{(i)}}{\partial \theta^i} \Psi(\alpha, \theta) \right|_{\theta=0} + 2x_\alpha^T \left. \frac{\partial^{(i)}}{\partial \theta^i} \eta(\alpha, \theta) \right|_{\theta=0} + \left. \frac{\partial^{(i)}}{\partial \theta^i} v(\alpha, \theta) \right|_{\theta=0}. \]

For any finite \( 1 \leq i < \infty. \) For notational convenience, it is necessary to denote \( H(\alpha, i) \triangleq \frac{\partial^{(i)}}{\partial \theta^i} \Psi(\alpha, \theta) \bigg|_{\theta=0}, \) and \( D(\alpha, i) \triangleq \frac{\partial^{(i)}}{\partial \theta^i} \eta(\alpha, \theta) \bigg|_{\theta=0}. \)

**Theorem 2:** Cost Cumulants in Tracking Problems.

The tracker dynamics governed by (5)-(6) attempt to track the desired trajectory \( z(t) \) with the IQF cost (7). For \( k \in \mathbb{Z}^+ \) fixed, the \( k \)th-cost cumulant of the Chi-square type random cost (7) is given by

\[ k_{\kappa}(t_0, t_1; K, u_z) = t_0^k H(t_0, k) x_0 + 2x_0^T \tilde{D}(t_0, k) + D(t_0, k), \]

where \( \{H(\alpha, i)\}_{i=1}^k, \{\tilde{D}(\alpha, i)\}_{i=1}^k, \) and \( \{D(\alpha, i)\}_{i=1}^k \) evaluated at \( \alpha = t_0 \) satisfy the matrix- and vector-valued differential equations (with the dependence of \( H(\alpha, i), \tilde{D}(\alpha, i), \) and \( (\alpha, i) \) upon \( u_z \) and \( K \) suppressed)

\[ \frac{d}{dt} H(\alpha, 1) = \Psi(\alpha) + 2x_\alpha^T \eta(\alpha, \theta) + v(\alpha, \theta) \]

\[ \frac{d}{dt} \tilde{H}(\alpha, 1) = -[A(\alpha) + B(\alpha)K(\alpha)]^T \Psi(\alpha) \]

\[ \frac{d}{dt} \eta(\alpha, \theta) = \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} H(\alpha, j) G(\alpha) W G^T(\alpha) \]
together with
\[ \frac{d}{da}\tilde{D}(a,1) = -[A(a) + B(a)K(a)]^T \tilde{D}(a,1) \]  
\[ -H(a,1)B(a)u_z(a) \]
\[ -K^T(a)R(a)u_z(a) + C^T(a)Q(a)z(a), \]
\[ \frac{d}{da}\tilde{D}(a,i) = -[A(a) + B(a)K(a)]^T \tilde{D}(a,i) \]
\[ -H(a,i)B(a)u_z(a), \quad 2 \leq i \leq k \]  
and
\[ \frac{d}{da}D(a,1) = -\text{Tr} \{H(a,1)G(a)WG^T(a)\} \]
\[ -2\tilde{D}^T(a,1)B(a)u_z(a) \]
\[ -u_z^T(a)R(a)u_z(a) - z^T(a)Q(a)z(a), \]
\[ \frac{d}{da}D(a,i) = -\text{Tr} \{H(a,i)G(a)WG^T(a)\} \]
\[ -2\tilde{D}^T(a,i)B(a)u_z(a), \quad 2 \leq i \leq k \]  
where the terminal conditions \(H(t,f,1) = C(t_f)Q_fC(t_f),\)
\(H(t_f,i) = 0\) for \(2 \leq i \leq k;\)
\(\tilde{D}(t_f,1) = -C(t_f)Q_fz(t_f),\)
\(\tilde{D}(t_f,i) = 0\) for \(2 \leq i \leq k\) and \(D(t_f,1) = z^T(t_f)Q_fz(t_f),\)
\(D(t_f,i) = 0\) for \(2 \leq i \leq k.\)

**II. PROBLEM STATEMENTS**

In preparing for the control statements of the tracking problem, let \(k\)-tuple variables \(\mathcal{H}, \mathcal{D}, \text{ and } \mathcal{D}\) be defined as \(\mathcal{H}(\cdot) = (H_1(\cdot), \ldots, H_k(\cdot)), \mathcal{D}(\cdot) = \left(\tilde{D}_1(\cdot), \ldots, \tilde{D}_k(\cdot)\right),\)
\(\mathcal{D}(\cdot) = (D_1(\cdot), \ldots, D_k(\cdot))\) for each element \(H_i \in C^4([t_0, t_f]; \mathbb{R}^{m \times n})\) of \(\mathcal{H}, D_1 \in C^4([t_0, t_f]; \mathbb{R}^n)\) of \(\mathcal{D}\) and \(D_1 \in C^4([t_0, t_f]; \mathbb{R})\) of \(\mathcal{D}\) having the representations \(H_i(\cdot) \equiv H(\cdot, i), \tilde{D}_i(\cdot) \equiv \tilde{D}(\cdot, i),\) and \(D_i(\cdot) \equiv D(\cdot, i)\) with the right members satisfying the dynamic equations (18)-(23) on the horizon \([t_0, t_f]).\) The problem formulation is greatly simplified if the convenient mappings are introduced
\[ \mathcal{F}_1 : [t_0, t_f] \times (\mathbb{R}^{m \times n})^k \times \mathbb{R}^{m \times n} \mapsto \mathbb{R}^{m \times n} \]
\[ \tilde{\mathcal{G}}_i : [t_0, t_f] \times (\mathbb{R}^{m \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^{m \times n} \mapsto \mathbb{R}^n \]
\[ \mathcal{G}_i : [t_0, t_f] \times (\mathbb{R}^{m \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^m \mapsto \mathbb{R} \]

where the actions are given by
\[ \mathcal{F}_1(\alpha, H, K) \equiv -[A(\alpha) + B(\alpha)K(\alpha)]^T H_1(\alpha) \]
\[ -H_1(\alpha) [A(\alpha) + B(\alpha)K(\alpha)] \]
\[ -C^T(\alpha)Q(\alpha)C(\alpha) - K^T(\alpha)R(\alpha)K(\alpha), \]
\[ \mathcal{F}_i(\alpha, H, K) \equiv -[A(\alpha) + B(\alpha)K(\alpha)]^T H_1(\alpha) \]
\[ -H_i(\alpha) [A(\alpha) + B(\alpha)K(\alpha)] \]
\[ -\sum_{j=1}^{i-1} \frac{2!}{(i-j)!} H_j(\alpha)G(a)WG^T(\alpha)H_{i-j}(\alpha), \]
\[ \tilde{\mathcal{G}}_1(\alpha, H, \tilde{D}, K, u_z) \equiv -[A(\alpha) + B(\alpha)K(\alpha)]^T \tilde{D}_1(\alpha) \]
\[ -H_1(\alpha)B(\alpha)u_z(\alpha) - K^T(\alpha)R(\alpha)u_z(\alpha) + C^T(\alpha)Q(\alpha)z(\alpha), \]
\[ \tilde{\mathcal{G}}_i(\alpha, H, \tilde{D}, K, u_z) \equiv -[A(\alpha) + B(\alpha)K(\alpha)]^T \tilde{D}_1(\alpha) \]
\[ -H_i(\alpha)B(\alpha)u_z(\alpha), \]
\[ \mathcal{G}_1(\alpha, H, \tilde{D}, u_z) \equiv -\text{Tr} \{H_1(\alpha)G(\alpha)WG^T(\alpha)\} \]
\[ -2\tilde{D}^T_1(\alpha)B(\alpha)u_z(\alpha) - u_z^T(\alpha)R(\alpha)u_z(\alpha) - z^T(\alpha)Q(\alpha)z(\alpha), \]
\[ \mathcal{G}_i(\alpha, H, \tilde{D}, u_z) \equiv -\text{Tr} \{H_i(\alpha)G(\alpha)WG^T(\alpha)\} \]
\[ -2\tilde{D}^T_i(\alpha)B(\alpha)u_z(\alpha). \]

Now there is no difficulty to establish the product mappings
\[ \mathcal{F}_1 \times \cdots \times \mathcal{F}_k : [t_0, t_f] \times (\mathbb{R}^{m \times n})^k \times \mathbb{R}^{m \times n} \mapsto (\mathbb{R}^{m \times n})^k \]
\[ \tilde{\mathcal{G}}_1 \times \cdots \times \tilde{\mathcal{G}}_k : [t_0, t_f] \times (\mathbb{R}^{m \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^{m \times n} \mapsto (\mathbb{R}^n)^k \]
\[ \mathcal{G}_1 \times \cdots \times \mathcal{G}_k : [t_0, t_f] \times (\mathbb{R}^{m \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^m \mapsto \mathbb{R}^k \]
along with the corresponding notations \(\mathcal{F} \equiv \mathcal{F}_1 \times \cdots \times \mathcal{F}_k, \)
\(\tilde{\mathcal{G}} \equiv \tilde{\mathcal{G}}_1 \times \cdots \times \tilde{\mathcal{G}}_k, \) and \(\mathcal{G} \equiv \mathcal{G}_1 \times \cdots \times \mathcal{G}_k.\) Thus, the dynamic equations of motion (18)-(23) can be rewritten as
\[ \frac{d}{da}\mathcal{H}(a) = \mathcal{F}(a, \mathcal{H}(a), K(a)), \quad \mathcal{H}(t_f) = \mathcal{H}_f, \]
\[ \frac{d}{da}\tilde{D}(a) = \tilde{\mathcal{G}}(a, \mathcal{H}(a), \tilde{D}(a), K(a), u_z(\alpha)), \quad \tilde{D}(t_f) = \tilde{D}_f, \]
\[ \frac{d}{da}\mathcal{D}(a) = \mathcal{G}(a, \mathcal{H}(a), \tilde{D}(a), u_z(\alpha)), \quad \mathcal{D}(t_f) = \mathcal{D}_f. \]

Note that the product system uniquely determines \(\mathcal{H}, \tilde{D}, \) and \(\mathcal{D}\) once the admissible affine control \(u_z\) and feedback gain \(K\) are specified. Hence, they are considered as \(\mathcal{H} \equiv \mathcal{H}(\cdot, K), \quad \tilde{D} \equiv \tilde{D}(\cdot, K, u_z),\) and \(\mathcal{D} \equiv \mathcal{D}(\cdot, K, u_z).\) The performance index in the cost-cumulant control problem can now be formulated in \(u_z\) and \(K.\)

**Definition 1:** Performance Index.

Fix \(k \in \mathbb{Z}^+\) and the sequence \(\mu = \{\mu_i \geq 0\}_{i=1}^k\) with \(\mu_1 > 0.\) Then, for the given \((t_0, x_0),\) the performance index
\[ \phi_{tk} : [t_0, t_f] \times (\mathbb{R}^{m \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^k \mapsto \mathbb{R}^+ \]
in cost-cumulant control for the tracking problem is defined as follows
\[ \phi_{tk} \left( t_0, \mathcal{H}(t_0), \tilde{D}(t_0), \mathcal{D}(t_0) \right) \]
\[ \triangleq \sum_{i=1}^k \mu_i \left[ x_0^T \mathcal{H}(t_0)x_0 + 2x_0^T \tilde{D}_i(t_0) + \mathcal{D}_i(t_0) \right] \]  
(24)
and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$, the set of admissible affine controls $U_{t_f, t_f, D_f, D_f^\mu}$ and feedback gains $K_{t_f, t_f, D_f, D_f^\mu}$ are respectively assumed to be the classes of $C([t_0, t_f]; \mathbb{R}^m)$ and $C([t_0, t_f]; \mathbb{R}^{m \times n})$ with values $u_z(\cdot) \in \mathbb{U}$ and $K(\cdot) \in K$ for which solutions to the dynamic equations with $H(t_f) = H_f$, $D(t_f) = D_f$, and $D(t_f) = D_f$ exist on the interval of optimization $[t_0, t_f]$. 

Definition 3: Optimization Problem.
Suppose that $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$ are fixed. Then the control optimization problem is defined as the minimization of (24) over $u_z(\cdot) \in U_{t_f, t_f, D_f, D_f^\mu}$ such that $K(\cdot) = K_{t_f, t_f, D_f, D_f^\mu}$ and subject to the dynamic equations of motion (25)-(27) for $\alpha \in [t_0, t_f]$.

Definition 4: Reachable Set.
Let reachable set $\mathcal{Q}$ be defined as $\mathcal{Q} = \{\{\varepsilon, \gamma, \hat{Z}, Z\} \in [t_0, t_f] \times (\mathbb{R}^{n \times n}) \times (\mathbb{R}^n) \times (\mathbb{R}^k)\}$ such that $K_{\varepsilon, \gamma, \hat{Z}, Z, \mu} \neq 0$ and $K_{\varepsilon, \gamma, \hat{Z}, Z, \mu} \neq 0$.

By adapting to the initial cost problem and the terminologies present in cost-cumulant control, the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the value function is motivated by the excellent treatment [1] and is given below.

Theorem 3: HJB Equation-Mayer Problem.
Let $\{\varepsilon, \gamma, \hat{Z}, Z\}$ be any interior point of the reachable set $\mathcal{Q}$ at which the value function $\mathcal{V}(\varepsilon, \gamma, \hat{Z}, Z)$ is differentiable. If there exist optimal affine control $u^*_z \in U_{\varepsilon, \gamma, \hat{Z}, Z, \mu}$ and feedback gain $K^* \in K_{\varepsilon, \gamma, \hat{Z}, Z, \mu}$, then the partial differential equation of dynamic programming

$$0 = \min_{u_z \in \mathcal{D}, K \in K} \left\{ \frac{\partial}{\partial \varepsilon} \mathcal{V}(\varepsilon, \gamma, \hat{Z}, Z) + \frac{\partial}{\partial \varepsilon} \mathcal{V}(\varepsilon, \gamma, \hat{Z}, Z) \text{vec}(\mathcal{F}(\varepsilon, \gamma, K)) + \frac{\partial}{\partial \varepsilon} \mathcal{V}(\varepsilon, \gamma, \hat{Z}, Z) \text{vec}(\mathcal{G}(\varepsilon, \gamma, \hat{Z}, K, u_z)) + \frac{\partial}{\partial \varepsilon} \mathcal{V}(\varepsilon, \gamma, \hat{Z}, Z) \text{vec}(\mathcal{G}(\varepsilon, \gamma, \hat{Z}, u_z)) \right\}$$

(28)
is satisfied together with the boundary value condition $\mathcal{V}(t_0, H_0, \bar{D}_0, D_0) = \phi_{tk}(t_0, H_0, \bar{D}_0, D_0)$.

Theorem 5: Fix $k \in \mathbb{Z}^+$ and let $\mathcal{V}(\varepsilon, \gamma, \hat{Z}, Z)$ be any interior point of $\mathcal{Q}$ at which the scalar-valued function

$$\mathcal{W}(\varepsilon, \gamma, \hat{Z}, Z) = \mathcal{W}(\varepsilon, \gamma, \hat{Z}, Z)$$

(29)
is differentiable. The time-varying parametric functions $\mathcal{E}_i \in C^1([t_0, t_f]; \mathbb{R}^{n \times n})$, $\mathcal{T}_i \in C^1([t_0, t_f]; \mathbb{R}^n)$ and $T_i \in C^1([t_0, t_f]; \mathbb{R})$ are yet to be determined. The derivative of $\mathcal{W}(\varepsilon, \gamma, \hat{Z}, Z)$ with respect to $\varepsilon$ is given

$$\frac{d}{d\varepsilon} \mathcal{W}(\varepsilon, \gamma, \hat{Z}, Z) = \mathcal{W}(\varepsilon, \gamma, \hat{Z}, Z)$$

(30)
is satisfied together with the boundary value condition $\mathcal{W}(t_0, H_0, \bar{D}_0, D_0) = \phi_{tk}(t_0, H_0, \bar{D}_0, D_0)$. Let $(t_f, H_f, \bar{D}_f, D_f)$ be in $\mathcal{Q}$ and $\mathcal{W}(\varepsilon, \gamma, \hat{Z}, Z)$ be continuously differentiable solution of the HJB equation (28) which satisfies the boundary condition $\mathcal{W}(t_0, H_0, \bar{D}_0, D_0) = \phi_{tk}(t_0, H_0, \bar{D}_0, D_0)$. Let $(t_f, H_f, \bar{D}_f, D_f)$ be in $\mathcal{Q}$ and $\mathcal{W}(\varepsilon, \gamma, \hat{Z}, Z)$ be continuously differentiable solution of the HJB equation (28) which satisfies the boundary condition $\mathcal{W}(t_0, H_0, \bar{D}_0, D_0) = \phi_{tk}(t_0, H_0, \bar{D}_0, D_0)$.

Replacing (31) into the HJB equation (28), it follows that

$$0 = \min_{u \in U} \left\{ x_0 T \sum_{i=1}^{k} \mu_i \left( F_i(\varepsilon, \mathcal{Y}, K) + \frac{d}{d\varepsilon} E_i(\varepsilon) \right) x_0 + 2x_0 T \sum_{i=1}^{k} \mu_i \left( \tilde{G}_i(\varepsilon, \mathcal{Y}, \tilde{K}) + \frac{d}{d\varepsilon} \tilde{I}_i(\varepsilon) \right) \right\}$$

(33)

Note that

$$\sum_{i=1}^{k} \mu_i F_i(\varepsilon, \mathcal{Y}, K) = -[A(\varepsilon) + B(\varepsilon)K]^T \sum_{i=1}^{k} \mu_i \mathcal{Y}_i - \sum_{i=1}^{k} \mu_i \mathcal{Y}_i [A(\varepsilon) + B(\varepsilon)K] - \mu_1 C^T(\varepsilon)Q(\varepsilon)C(\varepsilon)$$

$$- \mu_1 K^T R(\varepsilon)K - \sum_{i=2}^{k} \mu_i \sum_{j=1}^{i-1} \frac{2!}{j!(i-j)!} \mathcal{Y}_j G_j(\varepsilon)W G^T(\varepsilon) \mathcal{Y}_{i-j},$$

$$\sum_{i=1}^{k} \mu_i \tilde{G}_i(\varepsilon, \tilde{K}) = -[A(\varepsilon) + B(\varepsilon)K]^T \sum_{i=1}^{k} \mu_i \tilde{\mathcal{Y}}_i - \sum_{i=1}^{k} \mu_i \mathcal{Y}_i B(\varepsilon)u_z - \mu_1 K^T R(\varepsilon)u_z + \mu_1 C^T(\varepsilon)Q(\varepsilon)z(\varepsilon),$$

$$\sum_{i=1}^{k} \mu_i \tilde{G}_i(\varepsilon, \tilde{K}) = -\sum_{i=1}^{k} \mu_i \tilde{\mathcal{Y}}_i + \mu_i \tilde{\mathcal{Y}}_i B(\varepsilon)u_z - \mu_1 u_z^T R(\varepsilon)u_z - \mu_1 z^T(\varepsilon)Q(\varepsilon)z(\varepsilon).$$

Since $x_0$ and $M_0$ are arbitrary vector and rank-one matrix, the necessary condition for an extremum of (24) on $[t_0, t]$ is obtained by differentiating (33) with respect to $u_z$ and $K$

$$u_z(\varepsilon, \tilde{K}) = -R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{r=1}^{k} \tilde{\mu}_r \tilde{\mathcal{Y}}_r,$$

$$K(\varepsilon, \mathcal{Y}) = -R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{s=1}^{k} \tilde{\mu}_s \mathcal{Y}_s,$$

(34)

(35)

where $\tilde{\mu}_r \triangleq \mu_i / \mu_1$ and $\mu_1 > 0$. Substituting (34) and (35) into (33) leads to the value function

$$x_0 T \sum_{i=1}^{k} \mu_i \frac{d}{d\varepsilon} E_i(\varepsilon) - A^T(\varepsilon) \sum_{i=1}^{k} \mu_i \mathcal{Y}_i - \mu_1 C^T(\varepsilon)Q(\varepsilon)C(\varepsilon)$$

$$+ \sum_{r=1}^{k} \mu_r \mathcal{Y}_r B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{i=1}^{k} \mu_i \mathcal{Y}_i$$

$$+ \sum_{i=1}^{k} \mu_i \mathcal{Y}_i B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{s=1}^{k} \tilde{\mu}_s \mathcal{Y}_s$$

$$- \mu_1 \sum_{r=1}^{k} \tilde{\mu}_r \mathcal{Y}_r B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{s=1}^{k} \tilde{\mu}_s \mathcal{Y}_s$$

$$- \mu_1 \sum_{j=1}^{i-1} \frac{2!}{j!(i-j)!} \mathcal{Y}_j G_j(\varepsilon)W G^T(\varepsilon) \mathcal{Y}_{i-j},$$

(38)
The tracker dynamics governed by (5)-(6) attempt to track \( \mu \) which is fixed. Then, the control solution for the multi-cumulant tracking problem is implemented by

\[
u^*(t) = K^*(t)x^*(t) + u^*_z(t),
\]

where \( \mu \) represent different levels of influence as they depend on the overall cost distribution and \( \{\mathcal{H}_1^*(\alpha)\}_{r=1}^k \) are the solutions of the backward-in-time Riccati-type matrix differential equations.

\[
\frac{d}{d\alpha}\mathcal{H}_1^*(\alpha) = -[A(\alpha) + B(\alpha)K^*(\alpha)]^T\mathcal{H}_1^*(\alpha) - \mathcal{H}_1^*(\alpha)[A(\alpha) + B(\alpha)K^*(\alpha)] - C^T(\alpha)Q(\alpha)C(\alpha) - K^*T(\alpha)R(\alpha)K^*(\alpha),
\]

\[
\frac{d}{d\alpha}\mathcal{H}_r^*(\alpha) = -[A(\alpha) + B(\alpha)K^*(\alpha)]^T\mathcal{H}_r^*(\alpha) - \mathcal{H}_r^*(\alpha)[A(\alpha) + B(\alpha)K^*(\alpha)] - C_r^T(\alpha)Q(\alpha)z(\alpha) + \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} \mathcal{H}_r^*(\alpha)G(\alpha)W^T(\alpha)H_{r-s}(\alpha),
\]

and the auxiliary backward-in-time vector-valued differential equations

\[
\frac{d}{d\alpha}\mathcal{D}_1^*(\alpha) = -[A(\alpha) + B(\alpha)K^*(\alpha)]^T\mathcal{D}_1^*(\alpha) - \mathcal{H}_1(\alpha)B(\alpha)u^*_z(\alpha) - K^*T(\alpha)R(\alpha)u^*_z(\alpha) + C_r^T(\alpha)Q(\alpha)z(\alpha),
\]

\[
\frac{d}{d\alpha}\mathcal{D}_r^*(\alpha) = -[A(\alpha) + B(\alpha)K^*(\alpha)]^T\mathcal{D}_r^*(\alpha) - \mathcal{H}_r(\alpha)B(\alpha)u^*_z(\alpha) - K^*T(\alpha)R(\alpha)u^*_z(\alpha) + C_r^T(\alpha)Q(\alpha)z(\alpha),
\]

with the terminal boundary conditions \( \mathcal{H}_1(t_f) = C^T(t_f)Q_fC(t_f), \mathcal{H}_r(t_f) = 0 \) for \( 2 \leq r \leq k \), \( \mathcal{D}_1(t_f) = -C^T(t_f)Q_fz(t_f), \mathcal{D}_r(t_f) = 0 \) for \( 2 \leq r \leq k \). The boundary condition of \( W(\varepsilon, \mathcal{Y}, \mathcal{Z}) \) implies that

\[
x_0^T \sum_{i=1}^{k} \mu_i (\mathcal{H}_{i0} + \mathcal{E}_i(t_0)) x_0 + 2x_0^T \sum_{i=1}^{k} \mu_i (\mathcal{D}_{i0} + \mathcal{W}_i(t_0)) = x_0^T \sum_{i=1}^{k} \mu_i \mathcal{H}_{i0} x_0 + 2x_0^T \sum_{i=1}^{k} \mu_i \mathcal{D}_{i0} + \sum_{i=1}^{k} \mu_i \mathcal{D}_{i0}.
\]

Therefore, the extremizing affine control (34) and state-feedback gain (35) minimizing (24) become optimal

\[
\nu^*_z(\varepsilon) = -R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{r=1}^{k} \mu_r \mathcal{D}_r^*(\varepsilon), \quad K^*(\varepsilon) = -R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{r=1}^{k} \mu_r \mathcal{H}_r^*(\varepsilon).
\]

**Theorem 6:** Cost-Cumulant Control Solution.

The tracker dynamics governed by (5)-(6) attempt to track the desired trajectory \( z(t) \) with the Chi-square random cost (7). Assume both \( k \in \mathbb{Z}^+ \) and the sequence \( \mu = \{\mu_i \geq 0\}_{i=1}^{k} \) with \( \mu_1 > 0 \) are fixed. Then, the control solution for the

\[\text{References}\]


