NRL Report 5216

PROJECT VANGUARD REPORT NO. 34

APPLICATION OF THE SIMPLIFIED PHASE PLANE TO THE ANALYSIS AND DESIGN OF MISSILE JET-RELAY CONTROL SYSTEMS

J. L. Hieatt

Vehicle Branch
Project Vanguard

September 30, 1958

FR-5216

U. S. NAVAL RESEARCH LABORATORY
Washington, D.C.

CONTENTS

ABSTRACT

The application of phase plane analysis to a missile jet relay control system is reviewed, explained, and illustrated. Only a single lead network is considered under conditions in which aerodynamic damping is negligible. The assumption that all forces occur as step inputs is used to simplify analysis techniques and does not constitute a significant limitation on the application of this method for design purposes.

To aid in system design, an analytic method for the solution of phase plane plots is developed. The analytic technique reduces the time necessary to evaluate a system design by minimizing the necessity for construction of phase plane diagrams.

In conclusion a summary of general principles which may be used to optimize a missile jet relay control system is presented.

PROBLEM STATUS

This is an interim report on one phase of the problem; work is continuing.

AUTHORIZATION

NRL Problem A02-18

Manuscript submitted 2 September 1958

INTRODUCTION

The jet relay control system has wide application in the field of missile control and guidance. A jet relay control system is generally composed of a gyro reference unit, a control amplifier, pilot and slave relays, and solenoids which actuate reaction jets. The operation of this system may be described by considering control in a single axis. The gyro output signal, which indicates missile angular deviation from a reference (gyro error), is applied to the control amplifier. After modification by the amplifier the signal is applied to a balanced-output amplifier. The output of this amplifier is biased so that the magnitude of the error signal must exceed a predetermined value, plus or minus, to cause pilot relay operation. A heavy-duty relay is slaved to the pilot relay and when the error exceeds the amplifier "dead space," the action of the slave relay energizes a solenoid which turns on a reaction jet. The action of the jet creates a correcting moment which rotates the vehicle in such a direction as to reduce gyro error and cause relay drop-out (Fig. 1).

This type of control system is most effective and economical where relatively small correcting forces are necessary—usually under flight conditions wherein aerodynamic disturbances are small or negligible.

Since this method of control is nonlinear, frequency response analysis techniques are not applicable, and a system using phase plane analysis has been generally adopted. This report will describe a phase plane method for analysis of missile jet relay control systems and will develop analytic design criteria based on this analysis. The technique described in this report makes the simplifying assumption, valid for design purposes, that the forces acting on the vehicle occur as step functions. The phase plane diagram is a plot of the velocity of the controlled variable as a function of its displacement. This technique is only valid when the differential equation describing the system dynamics may be reduced to one of the second order. Missile dynamics in the absence of aerodynamic damping meet this criterion.

An example of a phase plane diagram is given in Fig. 2. Consider that some disturbance of short duration has resulted in an angular

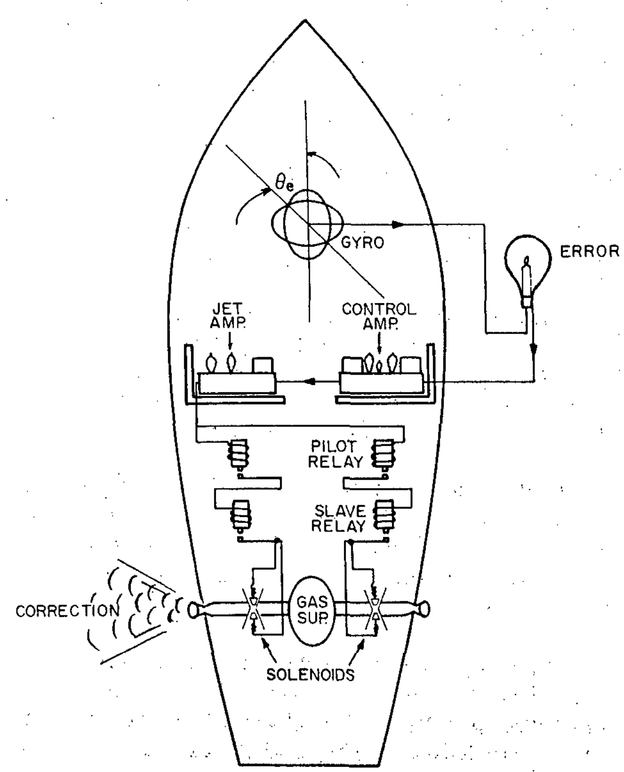

Figure 1 - Typical jet relay control system

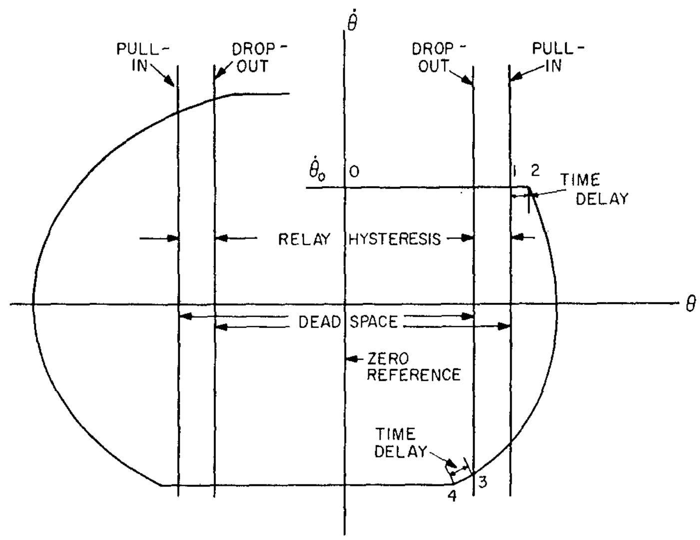

Figure 2 - Simple phase plane

velocity $\dot\theta_0$ of the missile. The angular deviation of the missile from the zero reference will increase with time and will generate on the phase plane a trajectory, the line 0, 1 and 2. When the angular deviation of the missile has reached the end of the dead space, represented by the intersection of the trajectory and the pull-in line, point 1, the control relay will be energized. The relay actuates a solenoid and jet thrust is developed. Since there is a time delay between relay energization and full thrust, the angular deviation of the missile will proceed along the constant-velocity path during this time, from point 1 to point 2. It will be shown later that when the reaction jet exerts thrust the vehicle trajectory on the phase plane will be parabolic. The acceleration due to the jet action will reverse the velocity along the parabola from point 2 to point 3. Since relays exhibit hysteresis, the angular deviation at which the relay is de-energized will differ from the pull-in line as represented by the drop-out line.

There will be a time delay between the de-energization of the relay and the decay of the thrust, during which the trajectory will continue along the parabola from point 3 to point 4. The system then has obtained a constant angular velocity opposite in sense to the initial condition and the sequences will be repeated. The "limit cycle" is established when the phase plane pattern is exactly repeated.

In the above simplified example the pull-in and drop-out lines were represented by vertical lines which indicate that the control amplifier did not modify the error signal. In practice, lead networks are employed in the control amplifier in order to damp out transient disturbances and to increase the period of hunting for purposes of fuel conservation.

This report will develop the equations of the relay pull-in and drop-out lines for a single lead network, the general trajectory of vehicle motion on a phase plane plot, the relationship between the phase plane plot and time, analytical methods for determining the system limit cycles, and general rules for optimization in system design.

EQUATIONS FOR VEHICLE TRAJECTORY ON THE PHASE PLANE

From an examination of missile dynamics in the absence of aerodynamic forces, it can be seen that the angular deviation θ of the missile heading with respect to the roll, pitch, or yaw axis is given by the following equation:

$$\theta = \theta_0 + \dot{\theta}_0 t + \frac{1}{2} \alpha_g t^2 , \qquad (1)$$

and differentiating this with respect to t gives

$$\dot{\theta} = \dot{\theta}_0 + \alpha_g t , \qquad (2)$$

where

α_g = the system angular acceleration,

θ_0 = the initial angular displacement, and

$\dot{\theta}_0$ = the initial angular velocity.

By combining these equations to eliminate time, the angular deviation may be expressed as a function of angular velocity:

$$\theta = \theta_0 + \frac{\dot{\theta}\dot{\theta}_0 - \dot{\theta}_0{}^2}{\alpha_g} + \frac{(\dot{\theta} - \dot{\theta}_0)^2}{2\alpha_g} . \qquad (3)$$

For the simplification of the analysis, all quantities should be nondimensionalized, since then a parabolic template may be made for analyzing the system for various parameter variations. As a base for nondimensionalizing all quantities having time dimensions, $t_b = 1$ second is used. This defines a nondimensional time scale $\tau = t/t_b$.

To nondimensionalize the other quantities, the control acceleration, α, is used. It follows that angular displacement, velocity, and acceleration will then be defined as

$$\Omega = \frac{\theta}{\alpha t_b{}^2}, \qquad \dot{\Omega} = \frac{\dot{\theta}}{\alpha t_b}, \qquad \ddot{\Omega} = \frac{\alpha_g}{\alpha} .$$

Hence Eq. (3) may be written in the nondimensional form:

$$\Omega = \Omega_0 + \frac{\alpha}{\alpha_g t_b} (\dot{\Omega}\dot{\Omega}_0 - \dot{\Omega}_0{}^2) + \frac{1}{2} \frac{\alpha}{\alpha_g t_b{}^2} (\dot{\Omega} - \dot{\Omega}_0)^2 , \qquad (4)$$

which may be simplified to:

$$\Omega = \Omega_0 + \frac{1}{2} \frac{\alpha}{\alpha_g} (\dot{\Omega}^2 - \dot{\Omega}_0{}^2) . \qquad (5)$$

This equation represents a parabola which is symmetric about the Ω axis and with vertex at

$$\Omega = \Omega_0 - \frac{1}{2} \frac{\alpha}{\alpha_g} \dot{\Omega}_0^2 .$$

There are four general conditions which may govern a vehicle trajectory in the phase plane:

(a) No disturbance, no correction, $\alpha_g = 0$. In this case the vehicle trajectory is represented by a constant-angular-velocity line

$$\dot{\Omega} = K.$$

The following three conditions result in a parabolic trajectory:

(b) No disturbance, correction applied, $\alpha_g = \alpha$. The parabola will be

$$\Omega = \Omega_o \pm \frac{1}{2} (\dot{\Omega}^2 - \dot{\Omega}_o^2).$$

(c) Disturbance, no correction, $\alpha_g = \alpha_D$ = acceleration due to the disturbance. The parabola will be

$$\Omega = \Omega_o \pm \frac{1}{2} \frac{\alpha}{\alpha_D} (\dot{\Omega}^2 - \dot{\Omega}_o^2).$$

(d) Disturbance, correction applied, $\alpha_g = (\alpha_D + \alpha)$ where the sign of α will be opposite to the sign of Ω so that it will oppose the vehicle angular motion. The parabola will be

$$\Omega = \Omega_o \pm \frac{1}{2} \frac{|\alpha|}{|\alpha_D + \alpha|} (\dot{\Omega}^2 - \dot{\Omega}_o^2).$$

Before proceeding with examples of these conditions, however, it is necessary to derive equations for the relay drop-out and pull-in lines and develop expressions for determination of time on the phase plane.

GENERAL EQUATIONS OF RELAY DROP-OUT AND PULL-IN LINES (SINGLE LEAD NETWORK) ON THE PHASE PLANE

To determine relay drop-out and pull-in lines on a phase plane diagram it is necessary to develop an expression for the output voltage of the control amplifier as a function of its input voltage. This expression then will relate the controls response to vehicle motion.

For most relay control systems the control amplifier may be closely approximated by a single lead network. The general form for the output of a single lead network in terms of LaPlace transforms is:

$$E_o(S) = \frac{1}{\gamma} \frac{(1 + TS)}{\left(1 + \frac{T}{\gamma} S\right)} E_i(S) , \tag{6}$$

where

$E_o(S)$ = output voltage,

$E_i(S)$ = input voltage, and

γ, T = constants of lead network.

This expression, written as a differential equation, becomes

$$\gamma E_o(t) + T\frac{dE_o(t)}{dt} = E_i(t) + T\frac{dE_i(t)}{dt}. \tag{7}$$

This is a first order linear differential equation which has a general solution as follows:

$$E_o(t) = e^{-\frac{\gamma}{T}t}\left\{\int_0^t \left[\dot{E}_i(t) + \frac{1}{T} E_i(t)\right] e^{\frac{\gamma}{T}t}\, dt\right\}. \tag{8}$$

The integration may be performed by parts and the equation becomes

$$E_o(t) = \frac{1}{\gamma} E_i(t) + \frac{T}{\gamma}\frac{(\gamma - 1)}{\gamma} \dot{E}_i(t) - \frac{T}{\gamma}\frac{(\gamma - 1)}{\gamma} e^{-\frac{\gamma}{T}t} \int_0^t \ddot{E}_i(t)\, e^{\frac{\gamma}{T}t}\, dt . \tag{9}$$

The above method of expansion could be continued, but for most control problems a sufficient degree of accuracy may be obtained by assuming that all vehicle angular accelerations occur in the form of step functions, i.e., $\ddot{E}_i(t)$ is either constant or zero. Thus Eq. (9) becomes

$$E_o(t) = \frac{1}{\gamma}\left[E_i(t) + \frac{T}{\gamma}(\gamma - 1)\,\dot{E}_i(t) - \frac{T^2}{\gamma^2}(\gamma - 1)\,\ddot{E}_i(t)\left(1 - e^{-\frac{\gamma}{T}t}\right)\right]. \tag{10}$$

Since the input voltage is directly proportional to vehicle angular motion, i.e., $E_i(t) = K\theta(t)$, Eq. (10) may be rewritten in terms of θ, vehicle angular deviation:

$$\frac{\gamma F_o(t)}{K} = \theta_R(t) = \theta(t) + \frac{T}{\gamma}(\gamma-1)\,\dot{\theta}(t) - \frac{T^2}{\gamma^2}(\gamma-1)\,\ddot{\theta}(t)\left(1-e^{-\frac{\gamma}{T}t}\right). \tag{11}$$

where $\theta_R(t)$ is that value of θ corresponding to the relay drop-out or pull-in point when there is no lead circuit in the system. Equation (11) is the exact solution for the relay pull-in and drop-out lines for a constant angular acceleration. In most control problems the transient term may be considered zero, and the approximate equation for the pull-in and drop-out lines may be used:

$$\theta_R(t) = \theta(t) + \frac{T}{\gamma}(\gamma-1)\,\dot{\theta}(t) - \frac{T^2}{\gamma^2}(\gamma-1)\,\ddot{\theta}(t)\,. \tag{12}$$

Evaluation of the transient term is discussed in Appendix A. Since the transient correction is generally insignificant it will be neglected in the remainder of this report.

The general equation for relay pull-in and drop-out lines may be nondimensionalized as follows:

$$\Omega_R = \Omega + \frac{m\dot{\Omega}}{t_b} - \frac{mc\,\ddot{\Omega}}{t_b^2}\,, \tag{13}$$

where

$$m = \frac{T}{\gamma}(\gamma-1)\,, \qquad c = \frac{T}{\gamma}, \qquad t_b = 1 \text{ sec}\,, \qquad \ddot{\Omega} = \frac{\alpha_g}{\alpha}.$$

Let Ω_{RP} be the nondimensional angular position for relay pull-in determined by the dead space. Then the relay drop-out value Ω_{RD} is given by

$$\Omega_{RD} = \Omega_{RP}(H)\,. \tag{14}$$

where H is the value of the relay hysteresis.

It can be seen that the positions of the relay pull-in and drop-out lines depend on the system angular acceleration; this important point will be demonstrated in an example later. Moreover, since the system angular acceleration is considered a constant at any time, the relay pull-in and drop-out lines are represented by straight lines on the phase plane plot, with slopes m and intercepts determined by Ω_R, $\ddot{\Omega}$, m and c.

There are four general conditions which affect the equations for relay pull-in and drop-out:

(a) No disturbance, no correction, $\alpha_g = 0$. The equation for the relay pull-in line in this case will be

$$\pm\,\Omega_{RP} = \Omega + m\dot{\Omega}\,. \tag{15}$$

(b) No disturbance, correction applied, $\alpha_g = \alpha$ (note that α is always opposite in sign to Ω_R). The equation for the relay drop-out line will be

$$\pm\,\Omega_{RD} = \Omega + m\dot{\Omega} \pm mc\,. \tag{16}$$

(c) Disturbance, no correction, $\alpha_g = \alpha_D$ (consider $\alpha_D > 0$). The equation for the relay pull-in line will be

$$\pm\,\Omega_{RP} = \Omega + m\dot{\Omega} \mp mc\left|\frac{\alpha_D}{\alpha}\right|\,. \tag{17}$$

(d) Disturbance, correction applied, $\alpha_g = \alpha_D + \alpha$ (consider $\alpha_D > 0$). The equation for the relay drop-out line will be

$$\pm \Omega_{RD} = \Omega + m\dot{\Omega} \pm mc \frac{|\alpha_D + \alpha|}{|\alpha|}, \tag{18}$$

where the sign of α will be opposite to the sign of Ω_{RP}.

RELATION BETWEEN THE PHASE PLANE PLOT AND TIME

Time is a parameter necessary in constructing the phase plane diagram, but it does not appear explicitly on the plot. Therefore a relationship between time and the phase plane parameters must be established.

Again, four general conditions affect the relationship between time and the phase plane parameters.

(a) No disturbance, no correction, $\alpha_g = 0$. Under these conditions, the trajectory is represented by a constant-angular-velocity line, $\Omega = K$. Then

$$\Delta\tau = \frac{\Delta\Omega}{K} = \frac{\Delta t}{t_b} ;$$

therefore

$$\Delta t = t_b \frac{\Delta\Omega}{K} . \tag{19}$$

(b) No disturbance, correction applied, $\alpha_g = \alpha$. In this case

$$\dot{\theta}_2 = \dot{\theta}_1 + \alpha(t_2 - t_1)$$

$$= \dot{\theta}_1 + \alpha\,\Delta t ,$$

thus

$$\Delta t = \frac{\Delta\dot{\theta}}{\alpha} .$$

This expression is then nondimensionalized as follows:

$$\frac{\dot{\theta}}{\alpha t_b} = \dot{\Omega}$$

$$\Delta t = \Delta\dot{\Omega}\, t_b . \tag{20}$$

(c) Disturbance, no correction, $\alpha_g = \alpha_D$. In this case

$$\Delta\dot{\theta} = \alpha_D\,\Delta t$$

$$\Delta t = \frac{\Delta\dot{\Omega}\,\alpha\, t_b}{\alpha_D} . \tag{21}$$

(d) Disturbance, correction applied, $\alpha_g = \alpha_D + \alpha$. By considerations similar to the above,

$$\Delta t = \frac{\Delta\dot{\Omega}\, t_b\, |\alpha|}{|\alpha_D + \alpha|} . \tag{22}$$

EXAMPLE PHASE PLANE ANALYSES

The use of the equations developed for phase plane analysis can best be demonstrated by examples. The following examples are not intended to illustrate design, but to present technique.

EXAMPLE 1. SYSTEM WITH INITIAL DISTURBANCE ONLY

Consider the case where a control system is subjected to an initial disturbance which lasts for a finite time; after this time no disturbance is present (Fig. 3). The problem is to determine the maximum excursion experienced by the system due to the disturbance, and the system efficiency in the final limit cycle.

The initial disturbance is expressed in terms of a disturbing acceleration. The system is considered at rest at $\Omega = \dot{\Omega} = 0$; and Ω_{RP}, Ω_{RD}, α, m, c, and the system time delays are known quantities.

Step 1. Plot on the phase plane the pull-in and drop-out lines.

(a) Since the initial pull-in may occur with a disturbance and no correction,

(Line C) $$\Omega_{RP} = \Omega + m\dot{\Omega} - mc\left(\frac{\alpha_D}{\alpha}\right) . \qquad (23)$$

(b) Until the plot has been started it will not be known whether or not the disturbance will still exist at drop-out. Plot the drop-out lines for no disturbance.

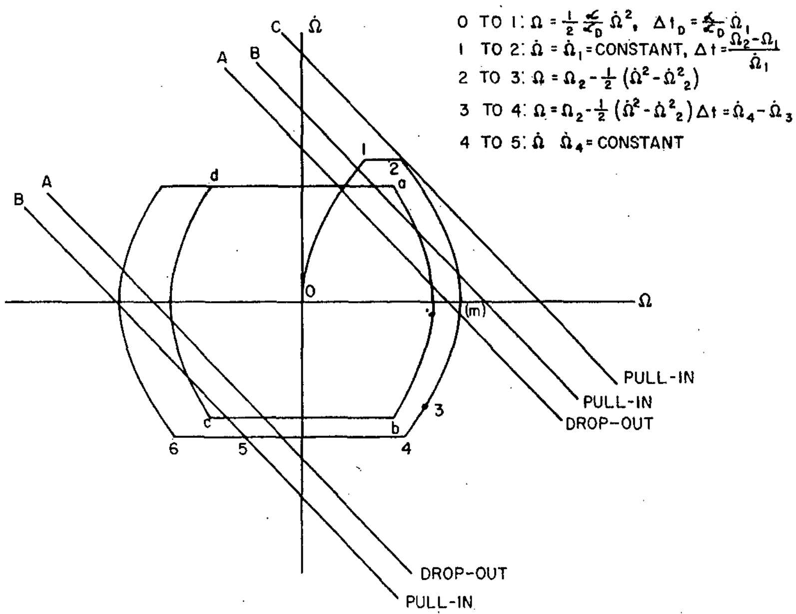

Figure 3 - Phase plane with initial disturbance only

(Lines A) $$\pm \Omega_{RD} = \Omega + m\dot{\Omega} \pm mc \,. \tag{24}$$

(c) Plot the pull-in lines needed for the limit cycle, the case where no disturbance exists.

(Line B) $$\pm \Omega_{RP} = \Omega + m\dot{\Omega} \,. \tag{25}$$

Step 2. Determine the length of time the initial disturbance will exist on the phase plane and plot the initial trajectory.

(a) The initial trajectory will be the parabola

$$\Omega = \frac{1}{2}\frac{\alpha}{\alpha_D}\dot{\Omega}^2 \,. \tag{26}$$

(b) This parabola will exist for the duration of the disturbance if the pull-in line (line C) is not intersected during that time. The time duration Δt_D of the disturbance is known; the parabola will extend to a velocity $\dot{\Omega}_1$ defined by the equation

$$\Delta t_D = \frac{\alpha}{\alpha_D}\dot{\Omega}_1 \,. \tag{27}$$

From Fig. 3 it may be seen that the pull-in line, Line C, is not intersected during the time of the disturbance, point 0 to point 1. When the disturbance stops the pull-in line shifts and becomes line B since there is no acceleration. In this case pull-in occurs at the instant the disturbance stops, since the trajectory already has intersected line B; i.e., pull-in is not initiated from line B but at the end of the disturbance.

Step 3. Because of the time delay in relay action, solenoid action, and jet thrust buildup, the control acceleration will not occur for some finite time after the initiation of pull-in, when the disturbance stops. This time delay Δt, is known for the system; during this time delay vehicle angular motion is along a constant-velocity path, point 1 to point 2. The length of this path is determined by the equation

$$\Omega_2 - \Omega_1 = \Delta\Omega = \dot{\Omega}_1 \Delta t \,. \tag{28}$$

Step 4. At point 2, the vehicle angular motion is due to the control angular acceleration α, and the trajectory in the phase plane may be expressed as

$$\Omega = \Omega_2 - \frac{1}{2}(\dot{\Omega}^2 - \dot{\Omega}_2^2) \,. \tag{29}$$

In practice this portion of the trajectory is drawn using a parabolic template

$$\Omega = \frac{1}{2}\dot{\Omega}^2 \,. \tag{30}$$

This template is moved with vertex on the Ω axis until the curve intersects the trajectory at point 2. The trajectory is then drawn along the constant-acceleration path, point 2 to point 3.

Step 5. The maximum vehicle angular excursion is the intersection of the parabola, from point 2 to point 3, with the Ω axis:

$$\theta_m = \Omega_m \alpha \,, \tag{31}$$

since $t_b = 1$ second.

Step 6. Point 3 is the intersection of the trajectory and the drop-out line. Again, during the time delay in the relay, solenoid, and thrust decay, the trajectory in the phase plane will continue along the parabolic path. The point at which thrust is zero, point 4, is determined from the equation

$$\Delta t = \dot{\Omega}_4 - \dot{\Omega}_3 , \tag{32}$$

where Δt is the time delay, and since $t_b = 1$ second. This procedure is then continued until the pattern repeats itself, at which time the limit cycle has been reached.

With the limit cycle a, b, c, and d established, it is possible to determine the system efficiency. A convenient way of expressing the efficiency is by the ratio, jet-on time per cycle divided by the period of cycle. Hereafter this ratio will be used as the definition of efficiency. It should be carefully noted that the higher this ratio, called the "Efficiency Number," the lower the efficiency of the system:

$$\text{eff. no.} = \frac{\text{jet-on time per cycle}}{\text{period of cycle}} . \tag{33}$$

With the efficiency established, the total jet-on time for any flight duration may be determined by multiplying the efficiency by the flight time.

Step 7. Determine the jet-on time by taking the value of Ω_o and $\dot{\Omega}_o$ at a, b, c, or d. In the limit, the pattern should be symmetric and the absolute value of these quantities should be the same at any of the four points. In this case

$$\text{jet-on time} = 2(2\dot{\Omega}_o) , \tag{34}$$

$$\text{jet-off time} = 2\left(2\frac{\Omega_o}{\dot{\Omega}_o}\right). \tag{35}$$

EXAMPLE 2. SYSTEM WITH A CONTINUOUS DISTURBANCE

Consider the case in which a system is subjected to a continuous disturbance. If the acceleration due to the disturbance is large enough, the trajectory of the vehicle in the phase plane will oscillate about the positive pull-in and drop-out lines, as shown in Fig. 4. There is, however, a possibility that the vehicle trajectory will intersect the negative pull-in line, resulting in a phase plane unlike that of Fig. 4. Because of this possibility the negative pull-in and drop-out lines should be plotted. The problem, as in Example 1, is to determine the maximum excursion experienced by the system due to the disturbance, and the system efficiency in the final limit cycle.

Again the disturbance is expressed in terms of a disturbing acceleration. The system is considered at rest at $\Omega = \dot{\Omega} = 0$; and Ω_{RP}, Ω_{RD}, α, m, c, and the system time delays are known quantities. The phase plane for this condition is plotted as follows:

Step 1. Plot the drop-out and pull-in lines:

$$\Omega_R = \Omega + m\dot{\Omega} \pm mc\ddot{\Omega} . \tag{36}$$

(a) For the pull-in line A, $\ddot{\Omega}$ is dependent on the acceleration of the disturbance:

$$\ddot{\Omega}_{\text{pull-in}} = \frac{\alpha_D}{\alpha} . \tag{37}$$

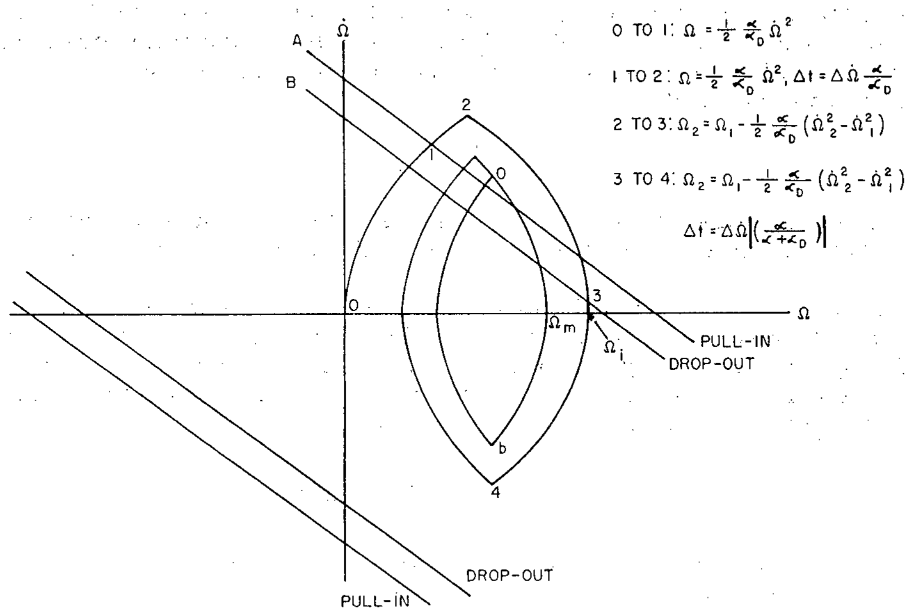

Figure 4 - Phase plane with continuous disturbance

Therefore

$$\pm\,\Omega_{RP} = \Omega + m\dot{\Omega} - mc\left(\frac{\alpha_D}{\alpha}\right). \tag{38}$$

Both pull-in lines should be plotted to determine the possibility of the vehicle trajectory intersecting the negative pull-in line.

(b) For the drop-out line B, $\ddot{\Omega}$ depends on the acceleration due to the disturbance plus the correcting moment acceleration:

$$\ddot{\Omega}_{drop\text{-}out} = \frac{\alpha_D + \alpha}{\alpha}, \tag{39}$$

where $\alpha_D > 0$ and the sign of α is such as to oppose vehicle motion, i.e., opposite to the sign of $\dot{\Omega}_{RP}$. Therefore

$$\pm\,\Omega_{RD} = \Omega + m\dot{\Omega} + mc\left|\frac{\alpha_D + \alpha}{\alpha}\right|. \tag{40}$$

Step 2. Determine the initial trajectory due to the disturbing acceleration.

(a) The initial trajectory will be the parabola

$$\Omega = \frac{1}{2}\frac{\alpha}{\alpha_D}\dot{\Omega}^2. \tag{41}$$

(b) This trajectory will continue after intersection with the pull-in line, point 1, until point 2. This length is determined by

$$\Delta\dot{\Omega} = \Delta t\,\frac{\alpha_D}{\alpha}, \tag{42}$$

where Δt is the time delay of the system resulting from relay action, solenoid action, and thrust build-up.

Step 3. At point 2 the vehicle motion will be the result of the disturbance and an opposing control acceleration:

$$\Omega_2 = \Omega_1 + \frac{1}{2}\frac{\alpha}{\alpha_D}(\dot{\Omega}_2^2 - \dot{\Omega}_1^2). \qquad (43)$$

This trajectory will continue after intersection with the drop-out line at point 3 until point 4, because of the time delay of the system:

$$\Delta\dot{\Omega} = \Delta t \left|\frac{\alpha_D + \alpha}{\alpha}\right|. \qquad (44)$$

Step 4. At point 4 the vehicle motion is again the result of the disturbance. The cycle is now repeated, using the same parabolic equations, until a limit cycle a, b is reached. The maximum excursion Ω_m will occur at the extreme intersection of the Ω- axis and the parabola representing the disturbance plus the correction. The efficiency number is the same as that defined in Example 1:

$$\text{eff. no.} = \frac{\text{jet-on time per cycle}}{\text{period of cycle}}. \qquad (45)$$

In the present case

$$\text{jet-on time} = 2\dot{\Omega}_{\text{limit cycle}}\left|\frac{\alpha}{\alpha + \alpha_D}\right|; \qquad (46)$$

$$\text{jet-off time} = 2\dot{\Omega}_{\text{limit cycle}}\left|\frac{\alpha}{\alpha_D}\right|. \qquad (47)$$

ANALYTIC SOLUTIONS FOR PHASE PLANE PROBLEMS

It is possible to determine most of the necessary design criteria for a jet-relay control system analytically. An analytical approach to the solution of phase plane diagrams makes it possible to use digital computer facilities and reduces the time required to evaluate system design variations.

In the following pages, equations which may be used to solve for the necessary design parameters of a jet relay control system will be developed and the limitations of this type of solution will be explained.

SYSTEM WITH NO DISTURBANCE

The phase plane diagram for a system with no disturbances will be of the form shown in Fig. 5. This is a diagram showing the system limit cycle. At point 0, jet control thrust becomes effective, resulting in a parabolic vehicle trajectory on the phase plane which intersects the relay drop-out line at point 1. From 1 to 2, the vehicle continues on the parabolic path for a distance corresponding to the drop-out time delay. From point 2 to point (-3) the vehicle has a constant velocity since no acceleration exists. At point (-3) the vehicle trajectory intersects the relay pull-in line. Vehicle motion continues along the constant velocity line from point (-3) to point (-0), a distance corresponding to the pull-in time delay. From point (-0) where control thrust again becomes effective, the vehicle motion will continue in a manner similar to that previously explained and return to point 0. It may be shown analytically that the phase plane diagram of the limit cycle for a system with no disturbances will be symmetric about the Ω and $\dot{\Omega}$ axes.

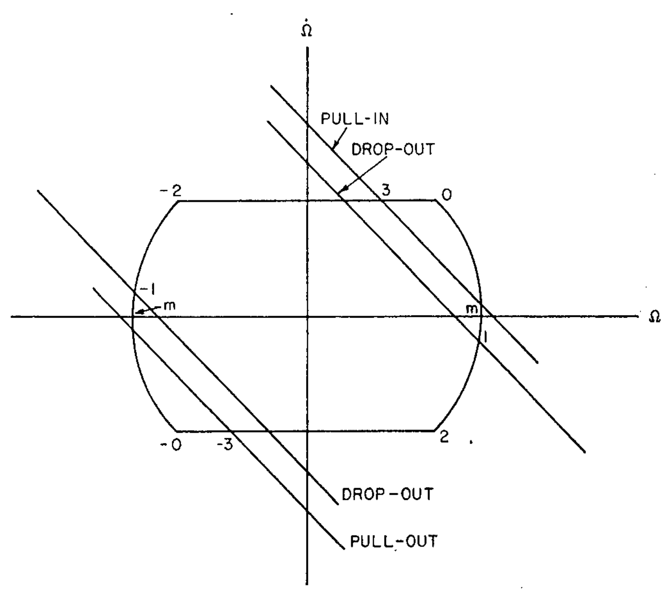

Figure 5 - Phase plane with no disturbance

Alteration of the pattern of vehicle motion on the phase plane may be accomplished by varying the lead circuit parameters and thus changing the pull-in and drop-out line positions, by changing the drop-out or pull-in time delays, or by changing the control system dead spot. It may be possible by varying these parameters to over-compensate the system, in which case the system will become semicontinuous.

The case of an over-compensated system is illustrated in Fig. 6. At point 0, jet control thrust becomes effective and results in a parabolic vehicle trajectory on the phase plane which intersects the relay drop-out line at point 1. The distance from point 1 to point 3 on the parabola corresponds to the drop-out time delay. In this case the vehicle trajectory intersects the relay pull-in line, point 2, before completion of the relay drop-out time delay. Under this condition the pull-in time delay begins at point 2 and extends to point (-0). In other words, part of the pull-in time delay is occurring during the drop-out delay. At point 3 the drop-out time delay ends and the vehicle moves with a constant velocity from point 3 to point (-0), a distance corresponding to the remainder of the pull-in time delay. The pattern from point (-0) to point 0 is symmetric with the previous pattern. In the development of an analytic solution for the limit cycle of a system with no disturbances the over-compensated system is not considered, since this type of system is inefficient.

Since the phase plane pattern for a limit cycle (Fig. 5) is symmetric with respect to the Ω and $\dot{\Omega}$ axes, it is possible to completely describe the properties of the system by knowing the coordinates of only one of the points where a slope discontinuity exists. Therefore, the problem of solving analytically for the properties of a system in the limit cycle, reduces to the problem of solving for the coordinates Ω_o, $\dot{\Omega}_o$ at point 0. It is known from the symmetry of the pattern that

$$\Omega_o = \Omega_2 = -\Omega_{-o} = -\Omega_{-2}, \tag{48}$$

$$\dot{\Omega}_o = -\dot{\Omega}_2 = -\dot{\Omega}_{-o} = \dot{\Omega}_{-2}. \tag{49}$$

The first step is to solve for the intersection of the relay drop-out line and the parabolic trajectory. The equation for the relay drop-out line will be

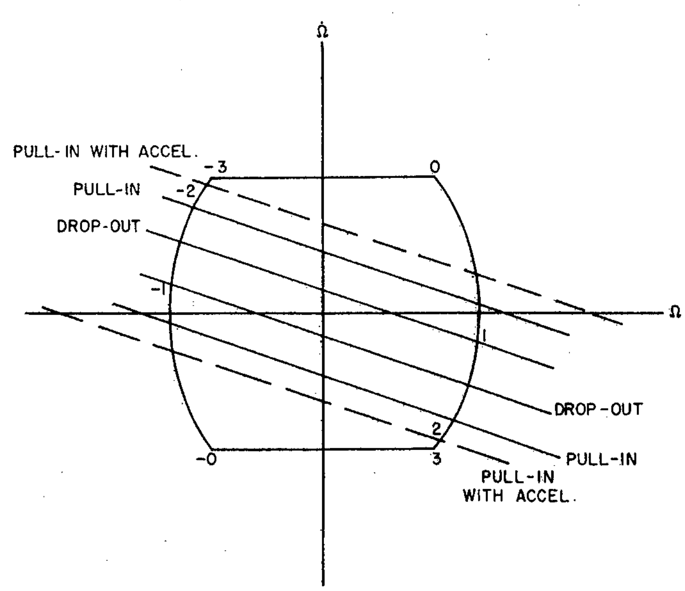

Figure 6 - Over-compensated system

$$\Omega_{RD} = \Omega_1 + m\dot{\Omega}_1 + mc \,, \tag{50}$$

and the equation for the parabolic trajectory will be

$$\Omega_1 = \Omega_o - \frac{1}{2}(\dot{\Omega}_1^2 - \dot{\Omega}_o^2) \,. \tag{51}$$

Solving these equations simultaneously yields

$$\dot{\Omega}_1 = m \pm \sqrt{m^2 - 2\left[(\Omega_{RD} - mc) - \left(\Omega_o + \frac{\dot{\Omega}_o^2}{2}\right)\right]} \,, \tag{52}$$

where the minus sign applies in the present case. Now,

$$+\,|\dot{\Omega}_2 - \dot{\Omega}_1| = \Delta t_D \,, \tag{53}$$

where Δt_D is the time delay at drop-out; since by Eq. (49) $\dot{\Omega}_2 = -\dot{\Omega}_o$,

$$\dot{\Omega}_o - \Delta t_D = -\dot{\Omega}_1 \,, \tag{54}$$

and

$$\dot{\Omega}_o - \Delta t_D + m = \sqrt{m^2 - 2\left[(\Omega_{RD} - mc) - \left(\Omega_o + \frac{\dot{\Omega}_o^2}{2}\right)\right]} \,. \tag{55}$$

Squaring both sides of this equation and combining terms yields

$$\Omega_o = \frac{\Delta t_D^2}{2} + \dot{\Omega}_o(m - \Delta t_D) - \Delta t_D m + \Omega_{RD} - mc \,. \tag{56}$$

The time delay at pull-in is

$$\Delta t_P = \frac{\Omega_o - \Omega_3}{\dot{\Omega}_o} \tag{57}$$

and, at the intersection of the constant-velocity trajectory and the pull-in line,

$$\Omega_3 = \Omega_{RP} - m\dot{\Omega}_o \,; \tag{58}$$

combining these two equations yields

$$\Omega_o = \Omega_{RP} + \dot{\Omega}_o(\Delta t_P - m) \,. \tag{59}$$

This expression for Ω_o may be equated to Eq. (56) to yield a solution for $\dot{\Omega}_o$:

$$\dot{\Omega}_o[2m - (\Delta t_D + \Delta t_P)] = -\frac{\Delta t_D^2}{2} + \Delta t_D m + (\Omega_{RP} - \Omega_{RD}) + mc \,. \tag{60}$$

For the case where the time delays are equal, i.e., $\Delta t_D = \Delta t_P$,

$$\dot{\Omega}_o 2(m - \Delta t) = -\frac{\Delta t^2}{2} + \Delta t\; m + (\Omega_{RP} - \Omega_{RD}) + mc \tag{61}$$

and

$$\Omega_o = (\Omega_{RP} - m\dot{\Omega}_o + \Delta t_P \dot{\Omega}_o) \,. \tag{62}$$

When the coordinates Ω_o, $\dot{\Omega}_o$ of point 0 are known, the maximum excursion in the limit cycle may be determined from the parabolic relationship

$$\Omega_m = \Omega_o + \frac{1}{2}\dot{\Omega}_o^2 . \tag{63}$$

The efficiency number of the system may be determined from Ω_o and $\dot{\Omega}_o$ as follows:

$$\text{jet-on time} = 2(2\dot{\Omega}_o) , \tag{64}$$

$$\text{jet-off time} = 2\left(\frac{2\Omega_o}{\dot{\Omega}_o}\right), \tag{65}$$

$$\text{period} = 4\left(\dot{\Omega}_o + \frac{\Omega_o}{\dot{\Omega}_o}\right), \tag{66}$$

$$\text{eff. no.} = \frac{1}{1 + \dfrac{\Omega_o}{\dot{\Omega}_o^2}} . \tag{67}$$

It should be reiterated that the above equations apply to a control system with no disturbances present, and that the equations are not valid for an over-compensated system. One indication from the analytic solution that a system has been over-compensated is the appearance of a negative Ω_o. It is suggested that whenever the analytic solution is in doubt a quick graphical check be made. A quick means for a graphic check is to plot Ω_o and Ω_o at all four points, connect the trajectories and then measure the delay times. The measured times should be equivalent to the known times.

SYSTEM WITH A CONTINUOUS DISTURBANCE

The phase plane diagram for a system with a continuous disturbance will be of the form shown in Figure 7. This is a diagram showing the system limit cycle. At point 0 the control thrust becomes effective and the result is a parabolic trajectory for which the resultant acceleration is due to the difference between the disturbance and the control force. The vehicle trajectory intersects the drop-out line at point 1. From there to point 2 the vehicle motion continues on the same parabolic path for a distance corresponding to the drop-out time delay. From point 2 to point 3, the vehicle trajectory in the phase plane is again parabolic with an acceleration due to the disturbing force only. At point 3, the trajectory intersects the relay pull-in line. The vehicle motion continues on the parabolic trajectory due to the disturbance from point 3 to point 0, a distance which corresponds to the relay pull-in time delay. The pattern is then repeated.

In the case of a continuous disturbance the phase plane pattern in the limit cycle is symmetric with respect to the Ω axis. Because of this symmetry it is possible to describe the properties of the system completely when the coordinates at the intersection of the two parabolas are known.

Because of the symmetry of the pattern (Fig. 7)

$$\dot{\Omega}_o = -\dot{\Omega}_2 , \tag{68}$$

$$\Omega_o = \Omega_2 . \tag{69}$$

The first step is to solve for the intersection of the trajectory with no correcting force applied, and the pull-in line. The equation for the pull-in line will be

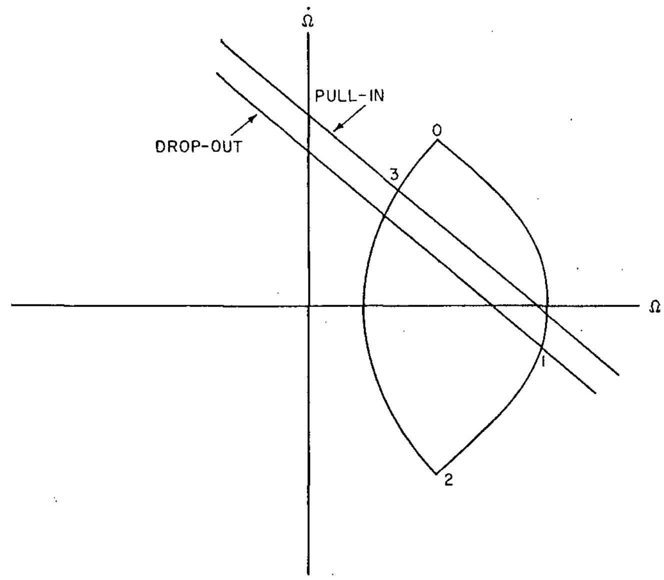

Figure 7 - Resultant phase plane for a system with a continuous disturbance

$$\Omega_{RP} = \Omega_3 + m\dot{\Omega}_3 - \frac{|\alpha_D|}{|\alpha|} mc , \tag{70}$$

and the equation for the parabolic trajectory will be

$$\Omega_3 = \Omega_2 + \frac{|\alpha|}{2|\alpha_D|}(\dot{\Omega}_3^2 - \dot{\Omega}_2^2) . \tag{71}$$

Solving these equations simultaneously gives

$$\dot{\Omega}_3 = -m\frac{|\alpha_D|}{|\alpha|} \pm \sqrt{\left(\frac{|\alpha_D|}{|\alpha|} m\right)^2 - \frac{2|\alpha_D|}{|\alpha|}\left(\Omega_2 - \frac{\alpha}{2|\alpha_D|}\dot{\Omega}_2^2\right) + \frac{2|\alpha_D|}{|\alpha|}\left(\Omega_{RP} + mc\frac{|\alpha_D|}{|\alpha|}\right)} . \tag{72}$$

The next step is to solve for the intersection of the trajectory with correcting force applied and the drop-out line. The equation for the relay drop-out line will be

$$\Omega_{RD} = \Omega_1 + m\dot{\Omega}_1 + mc\frac{(\alpha - \alpha_D)}{\alpha} , \tag{73}$$

and the equation for the parabolic trajectory in this case will be

$$\Omega_1 = \left(\Omega_o + \frac{|\alpha|}{2|\alpha + \alpha_D|}\dot{\Omega}_o^2\right) - \frac{|\alpha|}{2|\alpha + \alpha_D|}\dot{\Omega}_1^2 . \tag{74}$$

Solving these equations simultaneously gives

$$\dot{\Omega}_1 = \frac{|\alpha + \alpha_D| m}{|\alpha|} - \sqrt{\left(\frac{|\alpha + \alpha_D| m}{|\alpha|}\right)^2 - \frac{2|\alpha + \alpha_D|}{\alpha}\left[\left(\Omega_{RD} - mc\frac{|\alpha + \alpha_D|}{|\alpha|}\right) - \left(\Omega_o + \frac{|\alpha|}{2|\alpha + \alpha_D|}\dot{\Omega}_o^2\right)\right]} . \tag{75}$$

Next the time delays are considered. The drop-out time delay will be

$$\dot{\Omega}_2 - \dot{\Omega}_1 = -\Delta t_D \frac{|\alpha + \alpha_D|}{|\alpha|} . \tag{75}$$

By substitution from Eq. (68) this becomes

$$\dot{\Omega}_o - \Delta t_D \frac{|\alpha + \alpha_D|}{|\alpha|} = -\dot{\Omega}_1 . \tag{77}$$

The pull-in time delay will be

$$\dot{\Omega}_o - \Delta t_P \frac{|\alpha_D|}{|\alpha|} = \dot{\Omega}_3 . \tag{78}$$

Now considering Eqs. (68) and (69), Eqs. (72) and (78) may be combined and rewritten

$$\Omega_o = -\frac{\Delta t_P^2}{2}\left|\frac{\alpha_D}{\alpha}\right| + m\Delta t_P\left|\frac{\alpha_D}{\alpha}\right| + \left(\Omega_{RP} + mc\left|\frac{\alpha_D}{\alpha}\right|\right) - \dot{\Omega}_o(m - \Delta t_P) ; \tag{79}$$

and similarly Eqs. (75) and (77):

$$\Omega_o = \frac{\Delta t_D^2}{2}\frac{|\alpha + \alpha_D|}{|\alpha|} - m\Delta t_D \frac{|\alpha + \alpha_D|}{|\alpha|} + \left(\Omega_{RD} - mc\frac{|\alpha + \alpha_D|}{|\alpha|}\right) + \dot{\Omega}_o(m - \Delta t_D) . \tag{80}$$

Finally, Eqs. (79) and (80) may be combined to yield a solution for $\dot{\Omega}_o$:

$$\dot{\Omega}_o[2m - (\Delta t_D + \Delta t_P)] = -\frac{\Delta t_D^2}{2}\frac{|\alpha + \alpha_D|}{|\alpha|} - \frac{\Delta t_P^2}{2}\left|\frac{\alpha_D}{\alpha}\right| + m\left(\Delta t_D \frac{|\alpha + \alpha_D|}{|\alpha|} + \Delta t_P\left|\frac{\alpha_D}{\alpha}\right|\right) + (\Omega_{RP} - \Omega_{RD}) + mc . \tag{81}$$

For the case where the time delays are equal, i.e., $\Delta_D = \Delta t_P$, the value of $\dot{\Omega}_o$ will be identical to the value obtained for a system with no disturbance,

$$\dot{\Omega}_o[2(m - \Delta t)] = -\frac{\Delta t^2}{2} + m\Delta t + (\Omega_{RP} - \Omega_{RD}) + mc . \tag{82}$$

With $\dot{\Omega}_o$ known, Ω_o may be obtained from Eq. (79):

$$\Omega_o = -\frac{\Delta t_P^2}{2}\left|\frac{\alpha_D}{\alpha}\right| + m\Delta t_P\left|\frac{\alpha_D}{\alpha}\right| + \left(\Omega_{RP} + mc\left|\frac{\alpha_D}{\alpha}\right|\right) - \dot{\Omega}_o(m - \Delta t_P) . \tag{83}$$

The maximum excursion in the limit cycle may be determined by using the parabolic relationship

$$\Omega_m = \Omega_o + \frac{|\alpha|}{2|\alpha + \alpha_D|}\dot{\Omega}_o^2 . \tag{84}$$

The efficiency number as defined earlier, for a system with a continuous disturbance, depends only on the disturbing and control accelerations. This may be shown as follows:

$$\text{eff. no.} = \frac{\text{jet-on time per cycle}}{\text{period of cycle}} ; \tag{85}$$

$$\begin{aligned}\text{period of cycle} &= \text{jet-on time} + \text{jet-off time} \\ &= 2\dot{\Omega}_o\left(\frac{|\alpha|}{|\alpha + \alpha_D|} + \frac{\alpha}{\alpha_D}\right) .\end{aligned} \tag{86}$$

Since α and α_D will always be opposite in sign the equation may be rewritten without the absolute value signs as follows:

$$\begin{aligned} \text{period of cycle} &= 2\dot{\Omega}_o \left(\frac{\alpha}{\alpha - \alpha_D} + \frac{\alpha}{\alpha_D} \right) \\ &= 2\dot{\Omega}_o \left(\frac{\alpha^2}{\alpha_D(\alpha - \alpha_D)} \right); \end{aligned} \tag{87}$$

$$\text{eff. no.} = \frac{2\dot{\Omega}_o \dfrac{\alpha}{(\alpha - \alpha_D)}}{2\dot{\Omega}_o \left(\dfrac{\alpha^2}{\alpha_D(\alpha - \alpha_D)} \right)} = \frac{\alpha_D}{\alpha}. \tag{88}$$

SUMMARY OF EQUATIONS FOR THE ANALYTIC SOLUTION OF PHASE PLANE PROBLEMS

SYSTEM WITH NO DISTURBANCE

For a system with no disturbance the characteristic coordinates, derived originally as Eqs. (59) and (60), are

$$\dot{\Omega}_o[2m - (\Delta t_D + \Delta t_P)] = -\frac{\Delta t_D^2}{2} + \Delta t_D m + (\Omega_{RP} - \Omega_{RD}) + mc\,, \tag{89}$$

$$\Omega_o = \Omega_{RP} + \dot{\Omega}_o(\Delta t_P - m)\,; \tag{90}$$

the efficiency number of the system, derived originally as Eq. (67), is

$$\text{eff. no.} = \frac{1}{1 + \frac{\Omega_o}{\dot{\Omega}_o^2}}\,; \tag{91}$$

and the maximum excursion in the limit cycle is

$$\Omega_m = \Omega_o + \frac{1}{2}\dot{\Omega}_o^2\,. \tag{92}$$

SYSTEM WITH A CONTINUOUS DISTURBANCE

For a system with a continuous disturbance the characteristic coordinates are

$$\begin{aligned}\dot{\Omega}_o[2m - (\Delta t_D + \Delta t_P)] = &-\frac{\Delta t_D^2}{2}\frac{|\alpha + \alpha_D|}{|\alpha|} - \frac{\Delta t_P^2}{2}\left|\frac{\alpha_D}{\alpha}\right| \\ &+ m\left(\Delta t_D\frac{|\alpha + \alpha_D|}{|\alpha|} + \Delta t_P\left|\frac{\alpha_D}{\alpha}\right|\right) + \Omega_{RP} - \Omega_{RD} + mc\,,\end{aligned} \tag{93}$$

$$\Omega_o = -\frac{\Delta t_P^2}{2}\left|\frac{\alpha_D}{\alpha}\right| + m\Delta t_P\left|\frac{\alpha_D}{\alpha}\right| + \Omega_{RP} + mc\left|\frac{\alpha_D}{\alpha}\right| - \dot{\Omega}_o(m - \Delta t_P)\,; \tag{94}$$

the efficiency number of the system is

$$\text{eff. no.} = \frac{\alpha_D}{\alpha}\,; \tag{95}$$

and the maximum excursion in the limit cycle is

$$\Omega_m = \Omega_o + \frac{|\alpha|}{2|\alpha + \alpha_D|}\dot{\Omega}_o^2\,. \tag{96}$$

GENERAL PRINCIPLES FOR OPTIMIZATION IN SYSTEM DESIGN

The design of a relay jet control system consists of determination of the control thrust and amplifier dead spot necessary to satisfy the system limitations and determination of the lead network parameters which will result in the greatest system efficiency (the lowest efficiency number). Since every system will have some limiting angular excursion, the choice of the amplifier dead spot and control acceleration will be based on this limitation.

Consider a very simple system with a disturbance, no lead network, no time delays, and no relay hysteresis (Fig. 8). In this case the vehicle trajectory will immediately be in a limit cycle and by solution of the equations for the trajectory, the system maximum excursion may be expressed as follows:

$$\Omega_m = \Omega_D \left(1 + \frac{\alpha_D}{(\alpha - \alpha_D)}\right), \tag{97}$$

where

Ω_D = amplifier dead spot,

α_D = disturbing acceleration,

α = control acceleration, and

Ω_m = maximum excursion.

The consideration of time delays and relay hysteresis would increase the maximum excursion, while the use of a lead network would reduce it. Since these effects tend to

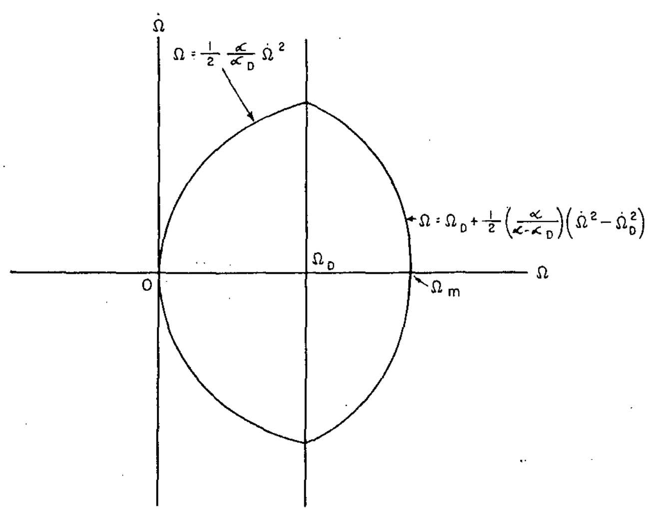

Figure 8 - Phase plane without lead circuit or hysteresis

compensate they are neglected in the initial design choice of control acceleration and dead spot. The relationship between maximum excursion, control acceleration, and dead spot for the simple system described above is shown in Fig. 9.

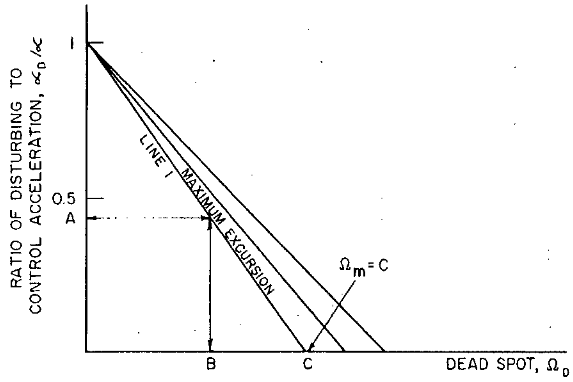

Figure 9 - Dead spot vs ratio of disturbing to control acceleration for different maximum excursions

For a system with an initial disturbance, which eventually oscillates in a limit cycle with no disturbance, the efficiency number as defined earlier becomes less (the efficiency becomes greater) with increasing α and also with larger dead spots.

For a system with a continuous disturbance the most efficient system has the largest possible control acceleration, since in this case efficiency depends only on control acceleration. The optimum lead network will depend on the time delay and the relay hysteresis as well as the control acceleration and dead spot.

After selection of a control acceleration and amplifier dead spot which satisfy the system limitations, it is necessary to determine the optimum lead network.

For a system with only an initial disturbance, the lead network is optimized on the basis of maximum system efficiency in the limit cycle. Figure 10 shows the relationship between γ and T, the lead circuit parameters, and efficiency number in the limit. From consideration of this figure the value of γ chosen for the lead network should be the largest value which can be attained and still provide an acceptable system noise level. After the determination of γ, a series of calculations employing the equations developed earlier, may be used to determine the optimum value of T.

For a system with a continuous disturbance, the lead network is optimized on the basis of the smallest maximum excursion in the limit cycle. The relationship between γ and T, and the maximum excursion in the limit cycle are shown in Fig. 11. After the largest γ consistent with system noise considerations has been chosen, T may be determined by a series of calculations employing the equations developed earlier. An interesting side light on this type of system is illustrated in Fig. 12. This figure compares the initial system excursion (the first excursion experienced by the system if it is initially at rest and the disturbance appears as a step input) with the maximum excursion in the limit. It may be seen that the optimum value of T on the basis of the maximum limit cycle excursion results in an initial excursion which has approximately the same value, indicating that the system is nearly critically damped for this value of T.

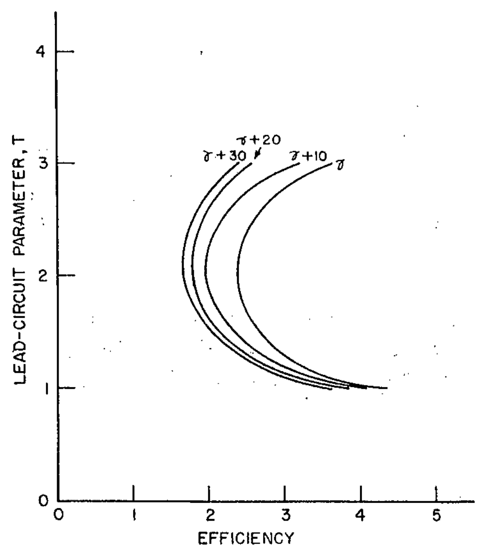

Figure 10 - Lead circuit parameter T vs efficiency number for different values of γ

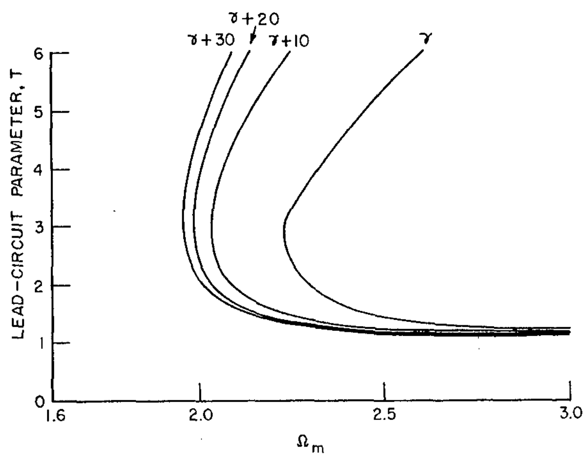

Figure 11 - Effect of different values of γ on the relation between the lead circuit parameter T and the maximum excursion in the limit cycle, for a system with a constant disturbance

The system time delay has not yet been mentioned. Figure 13 shows the relationship between the system efficiency, or maximum excursion in the limit, and the system time delay. It is obvious that time delays should be minimized as much as possible.

Relay hysteresis generally has a relatively small effect on the system efficiency.

ACKNOWLEDGMENTS

The assistance of Donald C. Green in the preparation of this report is gratefully acknowledged. The idea for the general application of phase plane

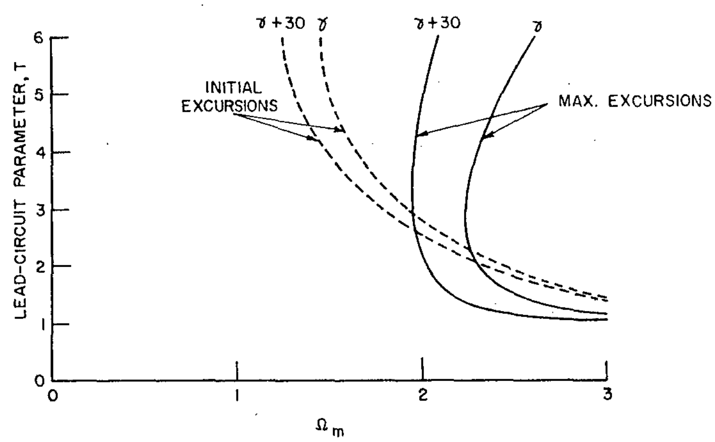

Figure 12 - Relation between the lead circuit parameter T and the initial and maximum excursions in the limit cycle for different values of γ, in a system with a constant disturbance

analysis to relay jet control systems was obtained from unpublished work done by J. S. Pistner of The Martin Company, Baltimore, Maryland. The guidance and continued interest of Mr. Frank H. Ferguson is greatly appreciated.

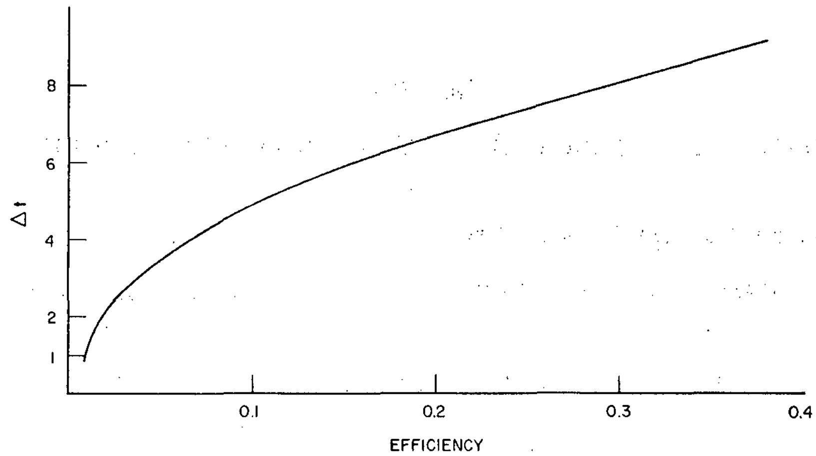

Figure 13 - Relation between efficiency number and system time delay

Appendix A

TRANSIENT CORRECTION FOR RELAY PULL-IN AND DROP-OUT LINES*

Correction when a Constant Velocity Exists

The actual relay pull-in boundary on each side of the zero displacement axis is

$$\pm \Omega_{RP} = \Omega + m\dot{\Omega} - m\dot{\Omega}\, e^{-\frac{\tau}{c}} . \tag{A1}$$

Since pull-in occurs along a constant $\dot{\Omega}$ axis, $\ddot{\Omega} = 0$.

Assume that it is required to find a relay pull-in point in the lower half of the phase plane. Here, $\dot{\Omega}$ is known and a negative quantity. Along a constant $\dot{\Omega}$ path

$$\tau = \frac{\Delta\Omega}{|\dot{\Omega}|} \quad \frac{\Omega_o - \Omega_P}{|\dot{\Omega}|} , \tag{A2}$$

where

Ω_o = the initial point on constant-velocity path, and

Ω_P = the actual relay pull-in point.

The relay will pull in when the following conditions obtain:

$$\begin{aligned} \Omega_P - \Omega_{RP} &= m\dot{\Omega} - m\dot{\Omega}\, e^{-\frac{\tau}{c}} \\ &= m\dot{\Omega}\left(1 - e^{-\frac{\Omega_o - \Omega_P}{|\dot{\Omega}|c}}\right); \\ 1 - \frac{\Omega_P - \Omega_{RP}}{m\dot{\Omega}} &= e^{-\frac{\Omega_o - \Omega_P}{|\dot{\Omega}|c}} , \\ \therefore \ln\left(1 - \frac{\Omega_P - \Omega_{RP}}{m\dot{\Omega}}\right) &= -\frac{\Omega_o - \Omega_P}{|\dot{\Omega}|c} . \end{aligned} \tag{A3}$$

Adding and subtracting $\Omega_{RP}/|\dot{\Omega}|c$ from the right-hand side of this equation and multiplying through by c/m yields

$$\frac{c}{m} \ln\left(1 - \frac{\Omega_P - \Omega_{RP}}{m\dot{\Omega}}\right) = \frac{\Omega_P - \Omega_{RP}}{m|\dot{\Omega}|} - \left(\frac{\Omega_o - \Omega_{RP}}{m|\dot{\Omega}|}\right) . \tag{A4}$$

Let

$$\frac{\Omega_P - \Omega_{RP}}{m\dot{\Omega}} = y , \tag{A5}$$

$$\frac{\Omega_o - \Omega_{RP}}{m|\dot{\Omega}|} = x ; \tag{A6}$$

---

*Adapted from unpublished work done by J. S. Pistner of The Martin Co.

then

$$\frac{c}{m}\ \ln(1 - y) = y - x\,. \tag{A7}$$

For any value of the lead circuit parameter $c/m = (1/\gamma - 1)$, x will approach infinity as y approaches unity. Hence, if a curve of $y - x = \ln(1 - y)$, in other words $c/m = 1$ or $\gamma = 2$, is plotted in the xy plane (Fig. A1) there will be a point somewhere on this curve with the same value of y as would exist on a curve where c/m has a value other than unity. The value $x_o = \Omega_o - \Omega_{RP}/m\dot{\Omega}$ is known in each particular case.

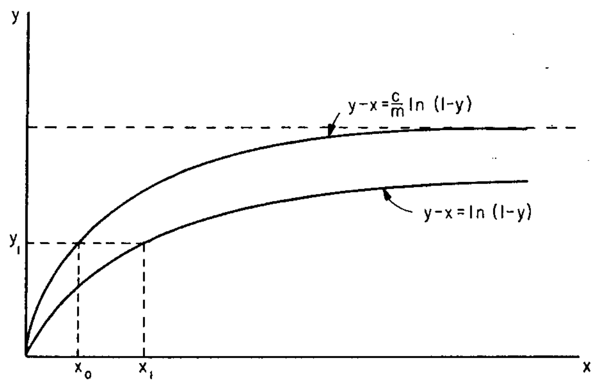

Figure A1 - Comparison of transient correction curves (no acceleration) for c/m = 1

It is required to find y_1 for the case when c/m is not unity. But y_1 will also be on the curve of $y - x = \ln(1 - y)$ with a corresponding abscissa x_1. Substituting the coordinates of these points in the equations of the two curves yields

$$y_1 - x_o = \frac{c}{m}\ln(1 - y_1)\,, \tag{A8}$$

$$y_1 - x_1 = \ln(1 - y_1)\,; \tag{A9}$$

or

$$y\left(1 - \frac{m}{c}\right) = x - \frac{m}{c}\,x_o\,, \tag{A10}$$

$$y = \frac{x}{1 - \frac{m}{c}} - \frac{\frac{m}{c}\,x_o}{1 - \frac{m}{c}}\,; \tag{A11}$$

or

$$y = \frac{1}{\gamma - 2}\,x + \frac{\gamma - 1}{\gamma - 2}\,x_o\,. \tag{A12}$$

This is the equation of a straight line with the following properties:

$$\text{slope} = -\frac{1}{\gamma - 2}\,, \tag{A13}$$

$$\text{y-intercept} = \frac{\gamma - 1}{\gamma - 2}\,x_o\,, \tag{A14}$$

$$\text{x-intercept} = (\gamma - 1)x_o\,. \tag{A15}$$

This straight line may be drawn, since x_o and γ are known in each particular case. The intersection of this line with the curve $y - x = \ln(1 - y)$ then gives the required value of y.

Since the slope of the straight line only depends on the value of γ, for a particular lead circuit all lines corresponding to different values of x_o will be parallel to each other. The procedure for finding the actual relay pull-in point when Ω_{RP}, γ, and $x_o = \Omega_o - \Omega_{RP}/m|\dot{\Omega}|$ are given is therefore as follows: On Fig. A2, draw the straight line of Eq. (A12). Read off the value of y where the straight line intersects the curve $y - x = \ln(1 - y)$. The actual relay pull-in point is then given by

$$y = \frac{\Omega_P - \Omega_{RP}}{m|\dot{\Omega}|}, \tag{A15}$$

or

$$\Omega_P = m|\dot{\Omega}|y + \Omega_{RP}. \tag{A16}$$

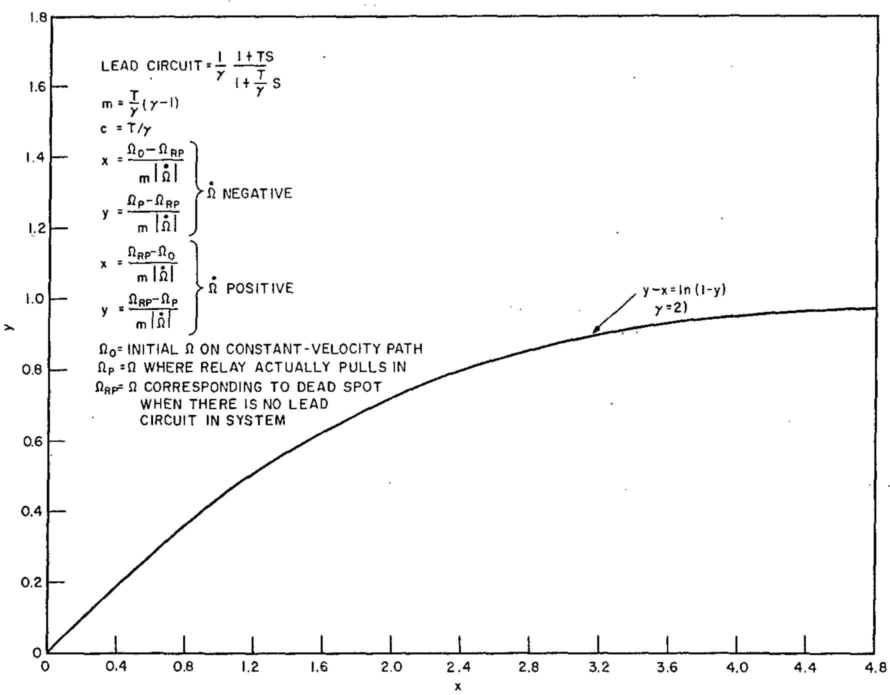

Figure A2 - Transient correction curve (no acceleration) for $c/m = 1$

Operations in the upper half of the phase plane, where $\dot{\Omega}$ is positive, can be analyzed in a similar manner. Here it can be shown that if

$$x = \frac{\Omega_{RP} - \Omega_o}{m|\dot{\Omega}|}, \tag{A17}$$

$$y = \frac{\Omega_{RP} - \Omega_P}{m|\dot{\Omega}|}, \tag{A18}$$

the foregoing procedure for finding Ω_P may be used except that now

$$\Omega_P = \Omega_{RP} - m|\dot{\Omega}|y. \tag{A19}$$

For a lead network where $\gamma = 2$, it is only required to calculate x_o; then y is directly read off the curve $y - x = \ln(1 - y)$.

Correction when a Constant Acceleration Exists

To determine the actual relay drop-out or pull-in point when an acceleration exists consider the general equation for relay pull-in or drop-out

$$\Omega_R = \Omega + m\dot{\Omega} - mc\,\ddot{\Omega}\left(1 - e^{-\frac{\tau}{c}}\right). \tag{A20}$$

The steady-state drop-out line will be

$$\Omega_R = \Omega + m\dot{\Omega} - mc\,\ddot{\Omega}\ . \tag{A21}$$

Let Ω_A be the approximate relay drop-out point (steady-state), and Ω_T the actual relay drop-out point; then

$$\Omega_R - \Omega_A = m\dot{\Omega}_A - mc\,\ddot{\Omega}\ , \tag{A22}$$

$$\Omega_R - \Omega_T = m\dot{\Omega}_T - mc\,\ddot{\Omega}\left(1 - e^{-\frac{\tau}{c}}\right). \tag{A23}$$

Dividing both equations through by $mc\ddot{\Omega}$ gives

$$\frac{\Omega_R - \Omega_T}{mc\ddot{\Omega}} = \frac{\dot{\Omega}_T}{c\ddot{\Omega}} - \left(1 - e^{-\frac{\tau}{c}}\right), \tag{A24}$$

$$\frac{\Omega_R - \Omega_A}{mc\ddot{\Omega}} = \frac{\dot{\Omega}_A}{c\ddot{\Omega}} - 1\ . \tag{A25}$$

Now τ', the nondimensional time to go from the beginning of the acceleration $\ddot{\theta}$ to the drop-out line will be

$$\tau' = \frac{\Delta\dot{\Omega}\,\alpha}{\ddot{\theta}} = \frac{\Delta\Omega}{\ddot{\Omega}}\ ; \tag{A26}$$

therefore

$$\frac{\Omega_R - \Omega_T}{mc\ddot{\Omega}} = \frac{\dot{\Omega}_T}{c\ddot{\Omega}} - \left(1 - e^{-\frac{\Delta\dot{\Omega}}{c\ddot{\Omega}}}\right). \tag{A27}$$

In order to solve the equations, consider the phase plane axis to have been transformed so that $\dot{\Omega}_1 = 0$. Then

$$\Delta\dot{\Omega} = \dot{\Omega}_2\ . \tag{A28}$$

With this transformation, substitute

$$\frac{\Omega}{\ddot{\Omega}c} = y\ , \tag{A29}$$

$$\frac{\Omega}{\ddot{\Omega}mc} = x\ . \tag{A30}$$

Then Eqs. (A25) and (A27) become

$$(C + 1) - x = y, \tag{A31}$$

$$(C + 1) - x = y + e^{-y}, \tag{A32}$$

where

$$C = \frac{\Omega_F}{mc\ddot{\Omega}}. \tag{A33}$$

The general equation for the trajectory on the phase plane with an acceleration $\ddot{\Omega}$ will be

$$\Omega_2 - \Omega_1 = \frac{1}{2}\frac{\dot{\Omega}_2^2}{\ddot{\Omega}}. \tag{A34}$$

Dividing this equation through by $mc\ddot{\Omega}$ gives

$$\frac{\Omega_2 - \Omega_1}{mc\ddot{\Omega}} = \frac{1}{2}\frac{c}{m}\frac{\dot{\Omega}_2^2}{c^2\ddot{\Omega}^2}. \tag{A35}$$

Now substitute

$$x = \frac{\Omega_2}{mc\ddot{\Omega}}, \tag{A36}$$

$$y = \frac{\dot{\Omega}_2}{c\ddot{\Omega}}, \tag{A37}$$

and the equation becomes

$$x - D = \frac{1}{2}\frac{c}{m}y^2, \tag{A38}$$

where

$$D = \frac{\Omega_1}{mc\ddot{\Omega}}. \tag{A39}$$

If these curves are plotted it may be seen that actual relay pull-in or drop-out occurs before the trajectory intersects the steady-state pull-in or drop-out curve (Fig. A3).

Examination of the equation for the trajectory, Eq. (A38), indicates that, plotted on log-log paper, it is the equation of a family of straight lines of constant slope. Hence if Eqs. (A31) and (A32) are also plotted on log-log paper, the actual relay drop-out or pull-in point will be given by the intersection of the straight line (representing the trajectory) drawn through the approximate relay drop-out or pull-in point determined from the phase plane plot, and the line representing the actual relay drop-out or pull-in point.

In summary, the procedure for finding the actual relay pull-in or drop-out point when an acceleration exists is as follows:

1. On the phase plane plot draw the steady-state relay pull-in and drop-out lines.

2. Find the nondimensional change in velocity, $\Delta\dot{\Omega}$, from the initial point on the parabola to the approximate relay pull-in or drop-out point as given by the intersection of the parabola and the steady-state relay pull-in or drop-out line.

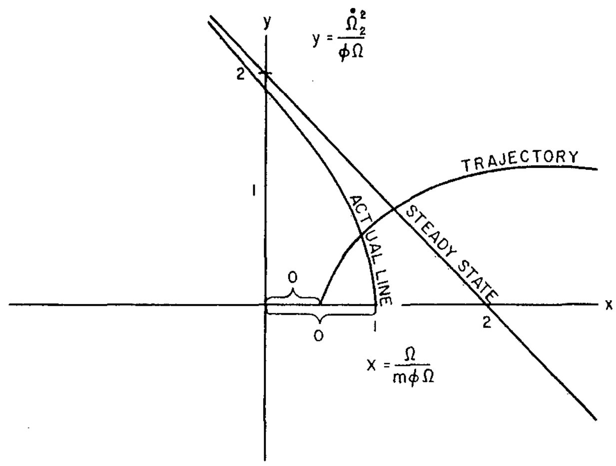

Figure A3 - Relation between actual relay pull-in and drop-out and phase plane trajectory

3. Divide $|\Delta\dot{\Omega}|$ by $c\ddot{\Omega}$.

4. Enter this value $|\Delta\dot{\Omega}|/c\ddot{\Omega}$ on the transient correction plot (Fig. A4) and find the corresponding point on the steady-state line.

5. Through this point, draw the straight line representing the parabola. This is a straight line parallel to the straight line labelled as parabola (reference) on the plot.

6. The intersection of this line with the actual drop-out line determines the actual drop-out point. Read off this value of $|\Delta\dot{\Omega}/c\ddot{\Omega}|$.

7. Multiply this value by $c\ddot{\Omega}$ and adjust the value of $|\Delta\dot{\Omega}|$ on the phase plane plot to obtain the actual relay pull-in or drop-out point.

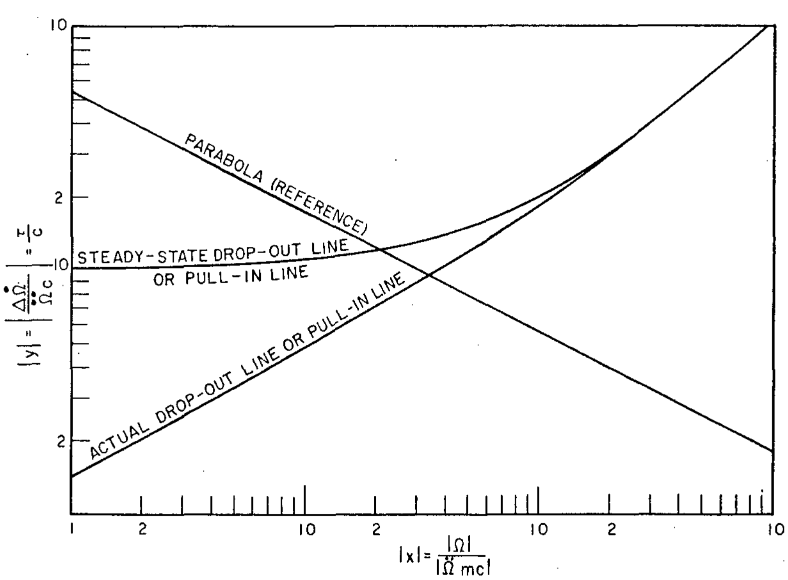

Figure A4 - Transient correction for relay drop-out or pull-in line when an acceleration exists

Appendix B

LIST OF SYMBOLS

α = angular acceleration due to a control moment
α_D = angular acceleration due to a disturbance
α_g = system angular acceleration
γ = lead circuit parameter
θ = vehicle angular displacement
$\dot{\theta}$ = vehicle angular velocity
$\ddot{\theta}$ = vehicle angular acceleration
θ_o = initial angular displacement
$\dot{\theta}_o$ = initial angular velocity
θ_R = value of θ corresponding to the relay drop-out or pull-in point when there is no lead circuit in the system
$\tau = t/t_b$ = nondimensional time scale
Ω = nondimensional angular displacement
$\dot{\Omega}$ = nondimensional angular velocity
$\ddot{\Omega}$ = nondimensional angular acceleration
Ω_D = amplifier dead spot
Ω_m = maximum excursion in limit cycle
Ω_{RP} = nondimensional pull-in point determined by dead spot
Ω_{RD} = nondimensional drop-out point determined by dead spot

$c = T/\gamma$
$E_o(s)$ = output voltage
$E_i(s)$ = input voltage
$m = T(\gamma - 1)/\gamma$
t = time
t_b = 1 second (time)
Δt = system time delay
Δt_D = system time delay at drop-out
Δt_P = system time delay at pull-in
T = lead circuit parameter

* * *