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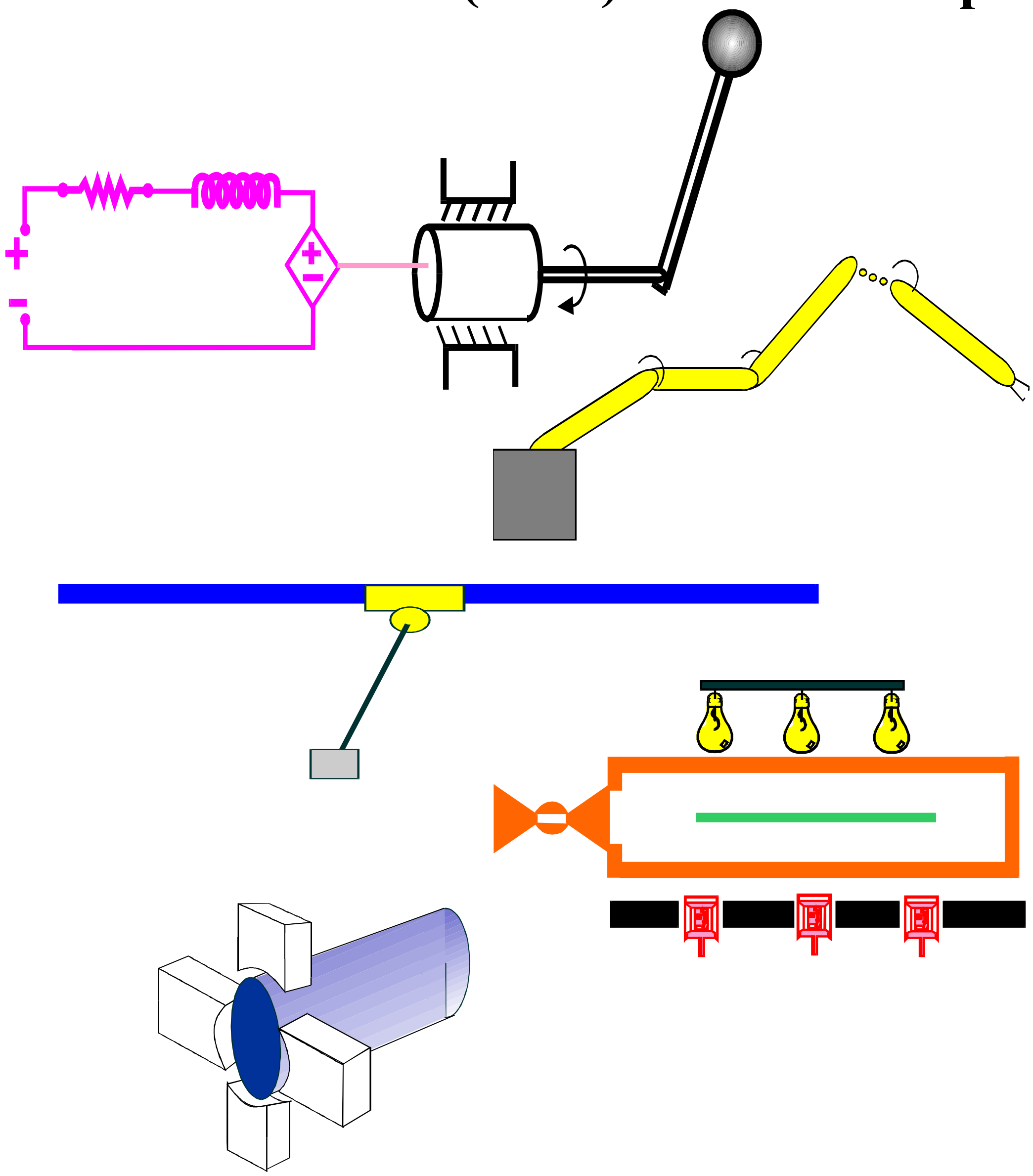

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Adaptive Control of Flat MIMO Nonlinear Systems With Additive Disturbance

Enver Tatlicioglu, Bin Xian, Darren M. Dawson, and Timothy Burg

Abstract: In this paper, two controllers are developed for flat multi-input/multi-output nonlinear systems. First, a robust adaptive controller is proposed and proven to yield semi-global asymptotic tracking in the presence of additive disturbances and parametric uncertainty. In addition to guaranteeing an asymptotic output tracking result, it is also proven that the parameter estimate vector is driven to a constant vector. In the second part of the paper, a learning controller is designed and proven to yield a semi-global asymptotic tracking result in the presence of additive disturbances where the desired trajectory is periodic. A continuous nonlinear integral feedback component is utilized in the design of both controllers and Lyapunov-based techniques are used to guarantee that the tracking error is asymptotically driven to zero. Numerical simulation results are presented for both controllers.

I. Introduction

Arguably, an interesting control problem is one that is both challenging from a theoretical perspective and applicable to real systems – the family of "flat" nonlinear systems appears to embody both of these properties. A flat system is characterized by a dynamic model where there exists a set of special outputs (equal to the number of inputs) such that the states and the inputs can be expressed in terms of outputs and a finite number of its derivatives [14]. A surprising number of practical machines match this form including mobile robots and cars, cars with multiple trailers, underwater vehicles, crane systems, induction motors, and planar satellite/manipulator systems [4], [7]. The reader is referred to [4] and [7] for a more detailed explanation of flatness and its applications to physical systems. It is the case of multi-input multi-output (MIMO) flat systems with parametric uncertainty and bounded disturbances that is considered here. Review of the basic control problem suggests and disqualifies certain solutions. It is probably wise at the outset to discard an exact model-based control approach for this problem given that any parameter estimation error and disturbances are not directly addressed, and hence, the system performance and stability cannot be predicted *a priori*. Given the parametric uncertainty in the proposed class of systems to be studied, an adaptive control solution may be warranted. However, an adaptive controller designed for a disturbance free system model may not compensate for the disturbances and may even go unstable under certain conditions. Enhancing the adaptive control approach with a robust component to form a robust adaptive controller can generally guarantee closed-loop signal boundedness in the presence of the additive disturbances. Unfortunately, while a robust adaptive controller can potentially guarantee the convergence of the tracking error to a bounded set (i.e., the tracking error can't necessarily be driven to zero) the asymptotic tracking result (where the tracking error is driven to zero) that would be shown for an adaptive controller applied to the disturbance free model will be lost. These trade-offs in performance and robustness have framed the last ten years of research in robust adaptive control.

Review of relevant work highlights some of the different tacks used to approach this problem. An adaptive backstepping controller was shown by Zhang and Ioannou in [18] for a class of single-input/single-output (SISO) linear systems with both input and output disturbances. The proposed controller demonstrates the use of a projection algorithm to bound the parameter estimates and guarantees an ultimately bounded tracking error. In an alternate approach, the work of Polycarpou and Ioannou [16] demonstrate a leakage-based adaptation law to compensate for parametric uncertainties. The proposed robust adaptive backstepping controller is applicable to a class of higher-order SISO systems with unknown nonlinearities. The suggested control law guarantees global uniform ultimate boundedness of the system state (with some restrictions on the bounding functions of the nonlinearities). Robust adaptive control laws were developed in [5], utilizing the modular design introduced in [12] and a tuning function design, for a class of systems similar to that studied in [16]. These authors show estimates on the effect of the bounded uncertainties and external disturbances on the tracking error. In [9], an adaptive backstepping controller for linear systems in the presence of output and multiplicative disturbances is designed. Ikhouane and Krstic, added a switching σ-modification to the tuning functions to obtain a tracking error proportional to the size of the perturbations. Marino and Tomei [13] proposed a robust adaptive tracking controller that achieves boundedness of all signals. The result is based on a class of SISO nonlinear systems that have ad-

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ditive disturbances but also unknown time-varying bounded parameters. It is significant that the result shows arbitrary disturbance attenuation. In [15], Pan and Basar proposed a robust adaptive controller for a similar class of systems in [13], where the tracking error is proven to be $\mathcal{L}_2-$bounded. In [6], Ge and Wang proposed a robust adaptive controller for SISO nonlinear systems with unknown parameters in the presence of disturbances, which ensure the global uniform boundedness of the tracking error.

Most of the research in adaptive control discussed above has focused on the convergence of the error signals and boundedness of the closed-loop system signals. As the sophistication in adaptive control techniques has evolved, additional questions about system performance have arisen. Notably, the final disposition of the parameters estimates in the closed-loop system has been examined. It is well established that without persistent excitation at the input, it is not typically possible to show the convergence of the parameter estimates to the corresponding system values (with an exception being a least-squares algorithm). In fact, for gradient and Lyapunov-type algorithms, convergence to a constant value, is typically not even guaranteed. Krstic summarizes this question well in [11] and also begins to provide some answers. In [11], it is shown that for the proposed adaptive controller; the parameter estimates will reach constant values after a sufficient amount of time. It is shown that the adaptation mechanism can be "turned off" after sufficient time and that the learned parameters can be used in a non-adaptive controller of the same structure to stabilize a restart of the system from new initial conditions. An important goal of the present work is to include a statement on parameter estimate limits for a controller proposed for the flat systems.

A recent paper by Cai *et al.* [2] presented a robust adaptive controller for MIMO nonlinear systems with parametric uncertainty and additive disturbances. With some restrictions placed on the disturbances, it was assumed that the disturbance is twice continuously differentiable and has bounded time derivatives up to second order, the proposed controller was proven to yield an asymptotic output tracking result. However, no mention of the convergence of the parameter estimates was made. Thinking out loud for a moment, it *might* stand to reason that if the robust part of the controller is compensating for the disturbances and an asymptotic tracking result is obtained then perhaps something special is happening to the parameter estimates. Exploring this vague notion with mathematical rigor, we will show that with a minor modification to the control in [2] and with some additional analysis of the stability result, we are able to formulate a new conclusion about the parameter estimates. What is shown is that this robust adaptive controller will yield constant parameter estimates even in the presence of the disturbance. The stability analysis parallels that presented in [2] but with the extended analysis the convergence of the parameter estimates is demonstrated. The main contribution of this paper is to add to the small number of results where parameter convergence has been shown. In the second part of the paper, a learning controller for the same class of flat systems is designed under the assumption that the reference trajectory is periodic (for past research related to the design of learning controllers, reader is referred to [1], [3], [8] and the references therein). This controller is proven to yield a semi-global asymptotic result in the presence of additive disturbances. In the design of both controllers, a continuous nonlinear integral feedback controller (see [17]) is utilized and Lyapunov-based techniques are used to guarantee that the tracking error is asymptotically driven to zero. Numerical simulation results are presented for both controllers to demonstrate their viability.

II. Adaptive Control

A. Problem Statement

A system model for the flat nonlinear systems is considered to be of the following form

$$x^{(n)} = f + G\left(u + d_1\right) + d_2 \tag{1}$$

where $x^{(i)}\left(t\right) \in \mathbb{R}^m$, $i = 0, ..., (n-1)$, are the system states, $f\left(x, \dot{x}, ..., x^{(n-1)}, \theta\right) \in \mathbb{R}^m$ and $G\left(x, \dot{x}, ..., x^{(n-1)}, \theta\right) \in \mathbb{R}^{m\times m}$ are nonlinear functions, $\theta \in \mathbb{R}^p$ is an unknown constant parameter vector, $d_1\left(t\right), d_2\left(t\right) \in \mathbb{R}^m$ are unknown additive nonlinear disturbances, and $u\left(t\right) \in \mathbb{R}^m$ is the control input. The system model is assumed to satisfy the following assumptions.

Assumption 1: The nonlinear function $G\left(\cdot\right)$ is symmetric, positive definite and satisfies the following inequalities

$$\underline{m}\left\|\xi\right\|^2 \leq \xi^T M\left(\cdot\right)\xi \leq \bar{m}\left(\cdot\right)\left\|\xi\right\|^2 \quad \forall\xi \in \mathbb{R}^m \tag{2}$$

where $M\left(x, \dot{x}, ..., x^{(n-1)}, \theta\right) \in \mathbb{R}^{m\times m}$ is defined as

$$M \triangleq G^{-1} \tag{3}$$

and $\underline{m} \in \mathbb{R}$ is a positive bounding constant, $\bar{m}\left(x, \dot{x}, ..., x^{(n-1)}\right) \in \mathbb{R}$ is a positive, globally invertible, nondecreasing function of each variable, and $\left\|\cdot\right\|$ denotes the Euclidean norm.

Assumption 2: The nonlinear functions, $f\left(\cdot\right)$ and $G\left(\cdot\right)$, are continuously differentiable up to their second derivatives (*i.e.,* $f\left(\cdot\right), G\left(\cdot\right) \in \mathcal{C}^2$).

Assumption 3: The nonlinear functions, $f\left(\cdot\right)$ and $M\left(\cdot\right)$, are affine in θ.

Assumption 4: The additive disturbances, $d_1\left(t\right)$ and $d_2\left(t\right)$, are assumed to be continuously differentiable and bounded up to their second derivatives (*i.e.,* $d_i\left(t\right) \in \mathcal{C}^2$ and $d_i\left(t\right), \dot{d}_i\left(t\right), \ddot{d}_i\left(t\right) \in \mathcal{L}_\infty$, $i = 1, 2$).

The output tracking error $e_1\left(t\right) \in \mathbb{R}^m$ is defined as follows

$$e_1 \triangleq x_r - x \tag{4}$$

where $x_r\left(t\right) \in \mathbb{R}^m$ is the reference trajectory satisfying the following property

$$x_r\left(t\right) \in \mathcal{C}^n\ ,\ x_r^{(i)}\left(t\right) \in \mathcal{L}_\infty\ ,\ i = 0, 1, ..., (n+2)\,. \tag{5}$$

The control design objective is to develop an adaptive control law that ensures $\left\|e_1^{(i)}\left(t\right)\right\| \to 0$ as $t \to \infty$, $i = 1, ..., n$, and that all signals remain bounded within the closed-loop system. To achieve the control objectives, the subsequent development is derived based on the assumption that the system states $x^{(i)}(t)$, $i = 0, ..., (n-1)$ are measurable.

B. Development of Robust Adaptive Control Law

The filtered tracking error signals, $e_i(t) \in \mathbb{R}^m$, $i = 2, 3, ..., n$ are defined as follows

$$e_2 \triangleq \dot{e}_1 + e_1 \tag{6a}$$
$$e_3 \triangleq \dot{e}_2 + e_2 + e_1 \tag{6b}$$
$$\vdots$$
$$e_n \triangleq \dot{e}_{n-1} + e_{n-1} + e_{n-2}. \tag{6c}$$

A general expression for e_i, $i = 2, 3, ..., n$ in terms of e_1 and its time derivatives is given as follows [17]

$$e_i = \sum_{j=0}^{i-1} a_{i,j} e_1^{(j)} \tag{7}$$

where the constants $a_{i,j}$ are defined as follows

$$\begin{aligned} a_{i,0} &= B(i) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i\right] \\ i &= 2, 3, ..., n \end{aligned} \tag{8}$$
$$\begin{aligned} a_{i,j} &= \sum_{k=1}^{i-1} B(i-k-j+1)\, a_{k+j-1,j-1} \\ i &= 3, 4, ..., n\ ,\ j = 1, 2, ..., (i-2) \end{aligned} \tag{9}$$
$$a_{i,i-1} = 1\ ,\ i = 1, 2, ..., n. \tag{10}$$

After utilizing (3), the system model can be rewritten as follows

$$Mx^{(n)} = h + u + d_1 + Md_2 \tag{11}$$

where $h(t) \in \mathbb{R}^m$ is defined as follows

$$h \triangleq Mf. \tag{12}$$

To facilitate the control development, the filtered tracking error signal, denoted by $r(t) \in \mathbb{R}^m$, is defined as follows

$$r \triangleq \dot{e}_n + \Lambda e_n \tag{13}$$

where $\Lambda \in \mathbb{R}^{m\times m}$ is a constant, diagonal, positive definite, gain matrix. After differentiating (13) and premultiplying by $M(\cdot)$, the following expression can be derived

$$\begin{aligned} M\dot{r} &= M\left(x_r^{(n+1)} + \sum_{j=0}^{n-2} a_{nj} e_1^{(j+2)} + \Lambda \dot{e}_n\right) \\ &\quad + \dot{M}x^{(n)} - \dot{h} - \dot{u} - \dot{d}_1 - M\dot{d}_2 - \dot{M}d_2 \end{aligned} \tag{14}$$

where (4), (7) and the first time derivative of (11) were utilized. The dynamics of $\dot{r}(t)$ in (14) can be arranged as follows

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_n - \dot{u} + N - \dot{d}_1 - M\dot{d}_2 - \dot{M}d_2 \tag{15}$$

where the auxiliary function $N\left(x, \dot{x}, ..., x^{(n)}, t\right) \in \mathbb{R}^m$ is defined as follows

$$\begin{aligned} N &\triangleq M\left(x_r^{(n+1)} + \sum_{j=0}^{n-2} a_{nj} e_1^{(j+2)} + \Lambda \dot{e}_n\right) \\ &\quad + \dot{M}\left(x^{(n)} + \frac{1}{2}r\right) + e_n - \dot{h}. \end{aligned} \tag{16}$$

To facilitate the subsequent analysis, (15) can be rearranged as follows

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_n - \dot{u} + \tilde{N} + N_r + \psi \tag{17}$$

where $\tilde{N}\left(x, \dot{x}, ..., x^{(n)}, t\right), N_r(t), \psi(t) \in \mathbb{R}^m$ are defined as follows

$$\begin{aligned} \tilde{N} &\triangleq \left(N - M\dot{d}_2 - \dot{M}d_2\right) \\ &\quad - \left(N_r - M_r\dot{d}_2 - \dot{M}_r d_2\right) \end{aligned} \tag{18}$$
$$N_r \triangleq N|_{x=x_r,\ \dot{x}=\dot{x}_r, ...,\ x^{(n)}=x_r^{(n)}} \tag{19}$$
$$\psi \triangleq -\dot{d}_1 - M_r\dot{d}_2 - \dot{M}_r d_2 \tag{20}$$

and $M_r(t) \in \mathbb{R}^{m\times m}$ is defined as follows

$$M_r \triangleq M|_{x=x_r,\ \dot{x}=\dot{x}_r, ...,\ x^{(n-1)}=x_r^{(n-1)}}\ . \tag{21}$$

Remark 1: By utilizing the Mean Value Theorem along with Assumptions 2 and 4, the following upper bound can be developed

$$\left\|\tilde{N}(\cdot)\right\| \leq \rho(\|z\|)\|z\| \tag{22}$$

where $z(t) \in \mathbb{R}^{(n+1)m\times 1}$ is defined as follows

$$z \triangleq \begin{bmatrix} e_1^T & e_2^T & ... & e_n^T & r^T \end{bmatrix}^T \tag{23}$$

and $\rho(\cdot) \in \mathbb{R}_{\geq 0}$ is some globally invertible, nondecreasing function.

Remark 2: After utilizing (5) and Assumption 4 along with (20) and its time derivative, then it is clear that $\psi(t)$, $\dot{\psi}(t) \in \mathcal{L}_\infty$.

Remark 3: After utilizing (5) and (16) along with (19) and its time derivative, then it is clear that $N_r(t)$, $\dot{N}_r(t) \in \mathcal{L}_\infty$.

Remark 4: In view of Assumption 3, $N_r(\cdot)$ defined in (19), can be linearly parameterized in the sense that

$$N_r \triangleq W_r\theta \tag{24}$$

where $W_r(t) \in \mathbb{R}^{m\times p}$ is the known regressor matrix and is a function of only $x_r(t)$ and its time derivatives.

Based on (17) and (24), the control input is designed as follows

$$\begin{aligned} u &\triangleq (K + I_m)\, e_n(t) - (K + I_m)\, e_n(t_0) \\ &\quad + \int_{t_0}^{t} \left[(K + I_m)\Lambda e_n(\tau) + W_r(\tau)\hat{\theta}(\tau)\right. \\ &\quad \left. + (C_1 + C_2)\operatorname{Sgn}(e_n(\tau))\right] d\tau \end{aligned} \tag{25}$$

where $\hat{\theta}(t) \in \mathbb{R}^p$ is generated via

$$\begin{aligned} \hat{\theta} &\triangleq \Gamma\int_{t_0}^{t} W_r^T(\tau)\Lambda e_n(\tau)\, d\tau - \Gamma\int_{t_0}^{t} \dot{W}_r^T(\tau)\, e_n(\tau)\, d\tau \\ &\quad + \Gamma W_r^T(t)\, e_n(t) - \Gamma W_r^T(t_0)\, e_n(t_0) \end{aligned} \tag{26}$$

with $K, C_1, C_2 \in \mathbb{R}^{m\times m}$ and $\Gamma \in \mathbb{R}^{p\times p}$ being constant, diagonal, positive definite, gain matrices, $I_m \in \mathbb{R}^{m\times m}$ being the standard identity matrix, and $\mathrm{Sgn}(\cdot)$ being the vector signum function defined as follows

$$\begin{aligned}\mathrm{Sgn}(\xi) &\triangleq \begin{bmatrix} \mathrm{sgn}(\xi_1) & \mathrm{sgn}(\xi_2) & \dots & \mathrm{sgn}(\xi_m)\end{bmatrix}^T \\ \forall \xi &= \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_m \end{bmatrix}^T. \end{aligned} \tag{27}$$

It should be noted that $\hat{\theta}(t_0) = 0_{p\times 1}$ and $u(t_0) = 0_{m\times 1}$ where $0_{p\times 1} \in \mathbb{R}^p$ and $0_{m\times 1} \in \mathbb{R}^m$ are vectors of zeros. Based on the structure of (25) and (26), the following are obtained

$$\dot{u} \triangleq (K + I_m) r + (C_1 + C_2) \mathrm{Sgn}(e_n) + W_r \hat{\theta} \tag{28}$$

$$\dot{\hat{\theta}} \triangleq \Gamma W_r^T r. \tag{29}$$

Finally, after substituting (28) into (17), the following closed-loop error system for $r(t)$ is obtained

$$\begin{aligned} M\dot{r} &= -\frac{1}{2}\dot{M}r - e_n - (K + I_m) r + W_r \tilde{\theta} \\ &\quad - (C_1 + C_2)\mathrm{Sgn}(e_n) + \tilde{N} + \psi \end{aligned} \tag{30}$$

where the parameter estimation error signal $\tilde{\theta}(t) \in \mathbb{R}^p$ is defined as follows

$$\tilde{\theta} \triangleq \theta - \hat{\theta}. \tag{31}$$

C. Stability Analysis

Theorem 1: The control law (25) and the update law (26) ensure the boundedness of all closed-loop system signals and $\left\| e_1^{(i)}(t) \right\| \to 0$ as $t \to \infty$, $i = 0, ..., n$, provided

$$\lambda_{\min}(\Lambda) > \frac{1}{2}, \tag{32}$$

$$C_{1i} > \left\| \psi_i(t) \right\|_{\mathcal{L}_\infty} + \frac{1}{\Lambda_i} \left\| \dot{\psi}_i(t) \right\|_{\mathcal{L}_\infty} \tag{33}$$

where the subscript $i = 1, ..., m$ denotes the ith element of the vector or diagonal matrix and the elements of K are selected sufficiently large relative to the system initial conditions.

Proof: See Appendix I.

Theorem 2: There exists a constant vector $\hat{\theta}_\infty \in \mathbb{R}^p$ such that

$$\lim_{t\to\infty} \hat{\theta}(t) = \hat{\theta}_\infty. \tag{34}$$

Proof: See Appendix II.

D. Numerical Simulation Results

A numerical simulation was performed to demonstrate the performance of the adaptive controller given in (25) and (26). A first-order flat system with following modelling functions is utilized [2]

$$\begin{aligned} f &= \begin{bmatrix} x_1 x_2 \\ x_2^2 \end{bmatrix}, \quad G = \begin{bmatrix} \dfrac{2+\cos x_1}{\theta_1} & 0 \\ 0 & \dfrac{3+\sin x_2}{\theta_2} \end{bmatrix}, \\ \theta &= \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \end{aligned} \tag{35}$$

$$\begin{aligned} d_1 &= \begin{bmatrix} \cos(2t) + \exp(-0.5t) \\ \sin(3t) + \exp(-0.5t) \end{bmatrix}, \\ d_2 &= \begin{bmatrix} \sin(2t) + \exp(-0.5t) \\ \cos(3t) + \exp(-0.5t) \end{bmatrix} \end{aligned} \tag{36}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$. The nonlinear disturbances defined in (36), are chosen to show the validity of the proposed controller for nonrepeating disturbances. The reference trajectory was selected as

$$x_r = \begin{bmatrix} x_{r1} \\ x_{r2} \end{bmatrix} = \begin{bmatrix} \sin t \left(1 - \exp\left(-\dfrac{t^3}{5}\right)\right) \\ 2\sin t \left(1 - \exp\left(-\dfrac{t^3}{2}\right)\right) \end{bmatrix}. \tag{37}$$

The initial conditions of the system were set to $x(t_0) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}^T$ and $\hat{\theta}(t_0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, while the controller parameters were chosen as $\Lambda = I_2$, $K = 20I_2$, $C_1 = 10I_2$, $C_2 = 5I_2$, and $\Gamma = 20I_2$ where $I_2 \in \mathbb{R}^{2\times 2}$ is the standard identity matrix. In Figures[1] 1 and 2, the state $x(t)$ and the reference trajectory $x_r(t)$ are presented, respectively. The tracking error $e_1(t)$ is presented in Figure 3. From Figure 3, it is clear that the tracking objective is satisfied. In Figures 4 and 5, the parameter estimate $\hat{\theta}(t)$ and the control input $u(t)$ are presented, respectively. From Figure 4, it is clear that the parameter estimate vector is driven to a constant vector. In Figures 6 and 7, the additive disturbances $d_1(t)$ and $d_2(t)$ are presented, respectively.

III. Learning Control

A. Problem Statement

A system model for the flat nonlinear systems is considered to be of the following form

$$x^{(n)} = f + G(u + d_1) + d_2 \tag{38}$$

where $x^{(i)}(t) \in \mathbb{R}^m$, $i = 0, ..., (n-1)$ are the system states, $f(x, \dot{x}, ..., x^{(n-1)}) \in \mathbb{R}^m$ and $G(x, \dot{x}, ..., x^{(n-1)}) \in \mathbb{R}^{m\times m}$ are nonlinear functions, $d_1(t), d_2(t) \in \mathbb{R}^m$ are unknown additive disturbances, and $u(t) \in \mathbb{R}^m$ is the control input. The system model is assumed to satisfy Assumptions 1, 2, and 4.

The output tracking error $e_1(t)$ is defined in (4) and in this case the reference trajectory satisfies the following property

$$\begin{aligned} x_r^{(i)}(t+T) &= x_r^{(i)}(t), \; x_r^{(i)}(t) \in \mathcal{L}_\infty \\ i &= 0, 1, ..., (n+2) \end{aligned} \tag{39}$$

where $T \in \mathbb{R}^+$ is the period of the reference trajectory.

The control objective is to develop a nonlinear control that ensures $\|e_1(t)\| \to 0$ as $t \to \infty$. To achieve the control objective, the subsequent development is derived based on the assumption that the system states $x^{(i)}(t)$, $i = 0, ..., (n-1)$ are measurable.

[1]The results of this simulation section are presented in Appendix IV

B. Development of Learning Control Law

The open-loop error system development for the learning control law is exactly the same as the open-loop error system development for the adaptive control law. The control design is assumed to continue after Remark 1 of Section II-B.

Remark 5: After utilizing (39) and Assumption 4 along with (20) and its time derivative, then it is clear that $\psi(t)$, $\dot{\psi}(t) \in \mathcal{L}_\infty$.

Remark 6: After utilizing (16) and (39) along with (19) and its time derivative, then it is clear that $N_r(t)$, $\dot{N}_r(t) \in \mathcal{L}_\infty$.

Remark 7: After utilizing (39), it is clear that $N_r(t)$ satisfies the following equation

$$N_r(t+T) = N_r(t). \tag{40}$$

Based on (17), the control input is designed as follows

$$\begin{aligned} u(t) \triangleq\; & (K+I_m)e_n(t) - (K+I_m)e_n(t_0) \\ & + \int_{t_0}^{t} \left[(K+I_m)\Lambda e_n(\tau) + C_1 \mathrm{Sgn}(e_n(\tau))\right] d\tau \\ & + \hat{W}_r(t) \end{aligned} \tag{41}$$

where $K, C_1, \Lambda \in \mathbb{R}^{m\times m}$ are constant, diagonal, positive definite, gain matrices, $\mathrm{Sgn}(\cdot)$ is defined in (27), and $\hat{W}_r(t) \in \mathbb{R}^m$ is defined as follows

$$\begin{aligned} \hat{W}_r(t) \triangleq\; & \hat{W}_r(t-T) + k_L \Lambda \int_{t_0}^{t} e_n(\tau)\, d\tau \\ & + k_L e_n(t) - k_L e_n(t_0) \end{aligned} \tag{42}$$

where $k_L \in \mathbb{R}$ is a positive gain. It should be noted that since $\hat{W}_r(t_0) = 0_{m\times 1}$ it follows that $u(t_0) = 0_{m\times 1}$. The auxiliary function $\hat{N}_r(t) \in \mathbb{R}^m$ is defined as

$$\hat{N}_r \triangleq \dot{\hat{W}}_r\,. \tag{43}$$

By utilizing (43) along with (42), the following can be obtained

$$\hat{N}_r(t) = \hat{N}_r(t-T) + k_L r(t). \tag{44}$$

Taking the time derivative of (41) along (42) and (43) generates

$$\dot{u} = (K+I_m)r + C_1 \mathrm{Sgn}(e_n) + \hat{N}_r(t). \tag{45}$$

Finally, after substituting (45) into (17), the closed-loop error system for $r(t)$ is obtained as follows

$$\begin{aligned} M\dot{r} =\; & -\frac{1}{2}\dot{M}r - e_n - (K+I_m)r \\ & - C_1 \mathrm{Sgn}(e_n) + \tilde{N} + \tilde{N}_r + \psi \end{aligned} \tag{46}$$

where $\tilde{N}_r(t) \in \mathbb{R}^m$ is defined as follows

$$\tilde{N}_r \triangleq N_r - \hat{N}_r. \tag{47}$$

By utilizing (40) and (44), $\tilde{N}_r(t)$ can be rewritten as follows

$$\tilde{N}_r(t) = \tilde{N}_r(t-T) - k_L r. \tag{48}$$

C. Stability Analysis

Theorem 3: The control law (41) and (42) ensures that $\|e_1(t)\| \to 0$ as $t \to \infty$, provided that (32) and (33) are satisfied and the elements of K are selected sufficiently large relative to the system initial conditions.

Proof: See Appendix III.

D. Numerical Simulation Results

A numerical simulation was performed to demonstrate the performance of the learning controller given in (41) and (42). The flat system model in (35), (36) with the following reference trajectory is utilized

$$x_r = \begin{bmatrix} x_{r1} \\ x_{r2} \end{bmatrix} = \begin{bmatrix} \sin(\pi t) \\ \cos(\pi t) \end{bmatrix}. \tag{49}$$

The initial conditions of the system were set to $x(t_0) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}^T$, while the controller parameters were chosen as $\Lambda = 20I_2$, $K = 20I_2$, $C_1 = 10I_2$, and $k_L = 1$. In Figures[2] 8 and 9, the state $x(t)$ and the reference trajectory $x_r(t)$ are presented, respectively. The tracking error $e_1(t)$ is presented in Figure 10. From Figure 10, it is clear that the tracking objective is satisfied. In Figures 11, 12, 13, and 14, the control input $u(t)$, $\hat{W}_r(t)$, the additive disturbances $d_1(t)$ and $d_2(t)$ are presented, respectively.

IV. Conclusion

Two controllers were developed for flat MIMO nonlinear systems in the presence of additive disturbances. The robust adaptive controller was proven to yield a semi-global asymptotic tracking result in the presence of parametric uncertainty along with additive disturbances. The adaptive controller and the adaptation law were designed such that, the parameter estimate vector is proven to go to a constant vector. In the second part of the paper, the learning controller was proven to yield a semi-global asymptotic result in the presence of additive disturbances and when the desired trajectory is periodic. In the development of both controllers, the bounded additive disturbances were assumed to be twice continuously differentiable and have bounded time derivatives up to second order. Since no assumptions were made regarding the periodicity of the disturbances, it is clear that the suggested controllers compensated for both repeating and nonrepeating disturbances. For each controller, Lyapunov-based techniques were used to guarantee that the tracking error is asymptotically driven to zero. Numerical simulation results were presented for both controllers where nonrepeating disurbances were utilized.

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APPENDIX I
PROOF OF THEOREM 1

Lemma 1: Let the auxiliary functions $L_1(t)$, $L_2(t) \in \mathbb{R}$ be defined as follows

$$L_1 \triangleq r^T(\psi - C_1\mathrm{Sgn}(e_n)) \ , \ L_2 \triangleq -\dot{e}_n^T C_2 \mathrm{Sgn}(e_n). \tag{50}$$

If C_1 is selected to satisfy the sufficient condition (33), then

$$\int_{t_0}^t L_1(\tau)\,d\tau \le \zeta_{b1} \quad , \quad \int_{t_0}^t L_2(\tau)\,d\tau \le \zeta_{b2} \tag{51}$$

where $\zeta_{b1}, \zeta_{b2} \in \mathbb{R}$ are positive constants defined as

$$\begin{aligned} \zeta_{b1} &\triangleq \sum_{i=1}^m C_{1i}\left|e_{ni}(t_0)\right| - e_n^T(t_0)\,\psi(t_0) \\ \zeta_{b2} &\triangleq \sum_{i=1}^m C_{2i}\left|e_{ni}(t_0)\right|. \end{aligned} \tag{52}$$

Proof: After substituting (13) into (50) and then integrating $L_1(t)$ in time, results in the following expression

$$\begin{aligned} \int_{t_0}^t L_1(\tau)\,d\tau = & \int_{t_0}^t e_n^T(\tau)\,\Lambda^T\left[\psi(\tau)\right. \\ & \left.-C_1\mathrm{Sgn}(e_n(\tau))\right]d\tau \\ & +\int_{t_0}^t \frac{de_n^T(\tau)}{d\tau}\psi(\tau)\,d\tau \\ & -\int_{t_0}^t \frac{de_n^T(\tau)}{d\tau}C_1\mathrm{Sgn}(e_n(\tau))\,d\tau. \end{aligned} \tag{53}$$

After integrating the second integral on the right-hand side of (53) by parts, the following expression is obtained

$$\begin{aligned} \int_{t_0}^t L_1(\tau)\,d\tau = & \int_{t_0}^t e_n^T(\tau)\,\Lambda^T\left[\psi(\tau)\right. \\ & \left.-C_1\mathrm{Sgn}(e_n(\tau))\right]d\tau + e_n^T(\tau)\,\psi(\tau)\Big|_{t_0}^t \\ & -\int_{t_0}^t e_n^T(\tau)\,\frac{d\psi(\tau)}{d\tau}d\tau \\ & -\sum_{i=1}^m C_{1i}\left.\left|e_{ni}(\tau)\right|\right|_{t_0}^t \\ = & \int_{t_0}^t e_n^T(\tau)\,\Lambda^T\left[\psi(\tau)\right. \\ & \left.-\Lambda^{-1}\frac{d\psi(\tau)}{d\tau} - C_1\mathrm{Sgn}(e_n(\tau))\right]d\tau \\ & +e_n^T(t)\,\psi(t) - e_n^T(t_0)\,\psi(t_0) \\ & -\sum_{i=1}^m C_{1i}\left(\left|e_{ni}(t)\right| - \left|e_{ni}(t_0)\right|\right). \end{aligned} \tag{54}$$

The right-hand side of (54) can be upper-bounded as follows

$$\begin{aligned} \int_{t_0}^t L_1(\tau)\,d\tau \le & \int_{t_0}^t \sum_{i=1}^m \left|e_{ni}(\tau)\right|\Lambda_i\left[\left|\psi_i(\tau)\right|\right. \\ & \left.+\frac{1}{\Lambda_i}\left|\frac{d\psi_i(\tau)}{d\tau}\right| - C_{1i}\right]d\tau \\ & +\sum_{i=1}^m \left|e_{ni}(t)\right|\left(\left|\psi_i(t)\right| - C_{1i}\right) + \zeta_{b1}. \end{aligned} \tag{55}$$

If C_1 is chosen according to satisfy (33), then the first inequality in (51) can be proven from (55). The second inequality in (51) can be obtained by integrating $L_2(t)$ defined in (50) as follows

$$\begin{aligned}\int_{t_0}^{t} L_2(\tau)\,d\tau &= -\int_{t_0}^{t} \dot{e}_n^T(\tau)\,C_2\mathrm{Sgn}(e_n(\tau))\,d\tau \quad (56)\\ &= \zeta_{b2} - \sum_{i=1}^{m} C_{2i}\,|e_{ni}(t)| \leq \zeta_{b2}.\end{aligned}$$

■

The following is the proof of Theorem 1.

Proof: Let the auxiliary functions $P_1(t)$, $P_2(t) \in \mathbb{R}$ be defined as follows

$$P_1 \triangleq \zeta_{b1} - \int_{t_0}^{t} L_1(\tau)\,d\tau \quad (57)$$

$$P_2 \triangleq \zeta_{b2} - \int_{t_0}^{t} L_2(\tau)\,d\tau \quad (58)$$

where $L_1(t)$, $L_2(t)$, ζ_{b1} and ζ_{b2} were defined in Lemma 1. The proof of Lemma 1 ensures that $P_1(t)$ and $P_2(t)$ are non-negative. The non-negative function $V(s(t),t) \in \mathbb{R}$ is defined as follows

$$V \triangleq \frac{1}{2}\sum_{i=1}^{n} e_i^T e_i + \frac{1}{2}r^T M r + P_1 + P_2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \quad (59)$$

where $s(t) \in \mathbb{R}^{[(n+1)m+2+p]\times 1}$ is defined as follows

$$s = \begin{bmatrix} z^T & \sqrt{P_1} & \sqrt{P_2} & \tilde{\theta}^T \end{bmatrix}^T. \quad (60)$$

After utilizing (2), (59) can be bounded as follows

$$W_1(s) \leq V(s,t) \leq W_2(s) \quad (61)$$

where $W_1(s)$, $W_2(s) \in \mathbb{R}$ are defined as follows

$$W_1(s) \triangleq \lambda_1 \|s\|^2 \quad , \quad W_2(s) \triangleq \lambda_2(\|s\|)\,\|s\|^2 \quad (62)$$

and[3] $\lambda_1, \lambda_2(\cdot) \in \mathbb{R}$ are defined as follows

$$\begin{aligned}\lambda_1 &= \frac{1}{2}\min\left\{1, \underline{m}, \lambda_{\min}\left(\Gamma^{-1}\right)\right\},\\ \lambda_2 &= \max\left\{1, \frac{1}{2}\bar{m}(\|s\|), \frac{1}{2}\lambda_{\max}\left(\Gamma^{-1}\right)\right\}. \quad (63)\end{aligned}$$

By differentiating (59), the following expression can be obtained

$$\begin{aligned}\dot{V} &= -\sum_{i=1}^{n-1} e_i^T e_i - e_n^T\Lambda e_n + e_{n-1}^T e_n - r^T r \quad (64)\\ &\quad + r^T\tilde{N} - r^T K r - e_n^T \Lambda C_2 \mathrm{Sgn}(e_n)\end{aligned}$$

where (6a)-(6c), (13), (29), (30) and (50) were utilized. By using (22), (32), and the triangle inequality, an upper-bound on (64) can be obtained as follows

$$\begin{aligned}\dot{V} &\leq -\lambda_3\|z\|^2 + \|r\|\,\rho(\|z\|)\,\|z\|\\ &\quad -\lambda_{\min}(K)\,\|r\|^2 - \sum_{i=1}^{m}\Lambda_i C_{2i}\,|e_{ni}(t)|\\ &\leq -\left(\lambda_3 - \frac{\rho^2(\|z\|)}{4\lambda_{\min}(K)}\right)\|z\|^2\\ &\quad - \sum_{i=1}^{m}\Lambda_i C_{2i}\,|e_{ni}(t)| \quad (65)\end{aligned}$$

where $\lambda_3 \triangleq \min\left\{\frac{1}{2}, \lambda_{\min}(\Lambda) - \frac{1}{2}\right\}$. The following inequality can be developed

$$\dot{V} \leq W(s) - \sum_{i=1}^{m}\Lambda_i C_{2i}\,|e_{ni}(t)| \quad (66)$$

where $W(s) \in \mathbb{R}$ denotes the following non-positive function

$$W(s) \triangleq -\beta_0\|z\|^2 \quad (67)$$

with $\beta_0 \in \mathbb{R}$ being a positive constant, and provided that $\lambda_{\min}(K)$ is selected according to the following sufficient condition

$$\begin{aligned}\lambda_{\min}(K) &\geq \frac{\rho^2(\|z\|)}{4\lambda_3} \quad (68)\\ \text{or } \|z\| &\leq \rho^{-1}\left(2\sqrt{\lambda_3\lambda_{\min}(K)}\right).\end{aligned}$$

Based on (59)-(63) and (65)-(67) the regions D and S can be defined as follows

$$\begin{aligned}\mathcal{D} &= \left\{s : \|s\| < \rho^{-1}\left(2\sqrt{\lambda_3\lambda_{\min}(K)}\right)\right\} \quad (69)\\ \mathcal{S} &= \left\{s \in \mathcal{D} : \right. \quad (70)\\ &\qquad \left. W_2(s) < \lambda_1\left(\rho^{-1}\left(2\sqrt{\lambda_3\lambda_{\min}(K)}\right)\right)^2\right\}\end{aligned}$$

Note that the region of attraction in (70) can be made arbitrarily large to include any initial conditions by increasing $\lambda_{\min}(K)$ (i.e., a semi-global stability result). Specifically, (62) and (70) can be used to calculate the region of attraction as follows

$$\begin{aligned}W_2(s(t_0)) &< \lambda_1\left(\rho^{-1}\left(2\sqrt{\lambda_3\lambda_{\min}(K)}\right)\right)^2 \quad (71)\\ &\Longrightarrow \|s(t_0)\| < \sqrt{\frac{\lambda_1}{\lambda_2(\|s(t_0)\|)}}\\ &\qquad \rho^{-1}\left(2\sqrt{\lambda_3\lambda_{\min}(K)}\right)\end{aligned}$$

which can be rearranged as

$$\lambda_{\min}(K) \geq \frac{1}{4\lambda_3}\rho^2\left(\sqrt{\frac{\lambda_2(\|s(t_0)\|)}{\lambda_1}}\,\|s(t_0)\|\right). \quad (72)$$

By utilizing (23), (52) and (60) the following explicit expression for $\|s(t_0)\|$ can be derived as follows

$$\begin{aligned}\|s(t_0)\|^2 &= \sum_{i=1}^{n}\|e_i(t_0)\|^2 + \|r(t_0)\|^2 \quad (73)\\ &\quad + \zeta_{b1} + \zeta_{b2} + \|\theta\|^2.\end{aligned}$$

[3] Using (4) and (6a)-(6c) it can be shown that $\left\|(x,\dot{x},...,x^{(n-1)})\right\| \leq \vartheta(\|\mathbf{s}\|)$ where $\vartheta(\cdot)$ is some positive function. Thus, $\bar{m}(x,\dot{x},...,x^{(n-1)}) \leq \bar{m}(\|s\|)$.

From (59), (66), (70)-(72), it is clear that $V(s,t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$; hence $s(t)$, $z(t)$, $\tilde{\theta}(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. From (66) it is easy to prove that $e_n(t) \in \mathcal{L}_1$ $\forall s(t_0) \in \mathcal{S}$. From (13), it is clear that $\dot{e}_n(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. By using (4), (5) and (7), it can be proved that $x^{(i)}(t) \in \mathcal{L}_\infty$, $i = 0, 1, ..., n$, $\forall s(t_0) \in \mathcal{S}$. Then, it is clear that $M(t)$, $\dot{M}(t)$, $f(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. The facts that $r(t)$, $\tilde{\theta}(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$ can be used along with (31) and (29) to prove that $\hat{\theta}(t)$, $\dot{\hat{\theta}}(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. After using these boundedness statements along with (11) and (28), it is clear that $u(t)$, $\dot{u}(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. The previous boundedness statements and Remarks 1, 2, 3 can be used along with (17), to prove that $\dot{r}(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. These boundedness statements can be used along with the time derivative of (67) to prove that $\dot{W}(s(t)) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$; hence $W(s(t))$ is uniformly continuous. Standard signal chasing algorithms can be used to prove that all remaining signals are bounded. A direct application of Theorem 8.4 in [10] can be used to prove that $\|z(t)\| \to 0$ as $t \to \infty$ $\forall s(t_0) \in \mathcal{S}$. Based on the definition of $z(t)$, it is easy to show that $\|e_i(t)\|, \|r(t)\| \to 0$ as $t \to \infty$ $\forall s(t_0) \in \mathcal{S}$, $i = 1, 2, ..., n$. From (13), it is clear that $\|\dot{e}_n(t)\| \to 0$ as $t \to \infty$ $\forall s(t_0) \in \mathcal{S}$. By utilizing (7) recursively it can be proven that $\left\|e_1^{(i)}(t)\right\| \to 0$ as $t \to \infty$, $i = 1, 2, ..., n$ $\forall s(t_0) \in \mathcal{S}$. ■

Appendix II
Proof of Theorem 2

Proof: The fact that $W_r(t)$ is a function of only $x_r(t)$ and its time derivatives, can be used along with the boundedness expression in (5), to show that $W_r(t), \dot{W}_r(t) \in \mathcal{L}_\infty$. After considering the fact that $e_n(t) \in \mathcal{L}_1$ (see the proof of Theorem 1), it is clear that $W_r^T(t)\Lambda e_n(t)$, $\dot{W}_r^T(t) e_n(t) \in \mathcal{L}_1$. This assures the existence of the limits for the first and second terms in (26), i.e., $\lim_{t\to\infty} \int_{t_0}^t W_r^T(\tau)\Lambda e_n(\tau)d\tau$ and $\lim_{t\to\infty} \int_{t_0}^t \dot{W}_r^T(\tau) e_n(\tau)d\tau$ exist (see Theorem 3.1 of [11]). Based on the fact that $e_n(t) \to 0$ as $t \to \infty$ $\forall s(t_0) \in \mathcal{S}$ (see the proof of Theorem 1) then it is clear that $\lim_{t\to\infty} W_r^T(t) e_n(t) = 0$. Utilizing the above facts along with the fact that $W_r^T(t_0) e_n(t_0)$ is constant, it follows that $\lim_{t\to\infty} \hat{\theta}(t) = \hat{\theta}_\infty$. ■

Appendix III
Proof of Theorem 3

Proof: Let $V(s,t) \in \mathbb{R}$ denotes the following non-negative function

$$V \triangleq \frac{1}{2}\sum_{i=1}^{n} e_i^T e_i + \frac{1}{2} r^T M r + P_1 + V_g \tag{74}$$

where $P_1(t)$ was defined in Lemma 1 and $V_g(t) \in \mathbb{R}$ is a non-negative function defined as follows

$$V_g \triangleq \frac{1}{2k_L}\int_{t-T}^{t} \tilde{N}_r^T(\tau)\tilde{N}_r(\tau)d\tau \tag{75}$$

where $s(t)$ is defined as follows

$$s \triangleq \left[\begin{array}{ccc} z^T & \sqrt{P_1} & \sqrt{V_g} \end{array}\right]^T. \tag{76}$$

After utilizing (2), (74) can be bounded as follows

$$W_1(s) \leq V(s,t) \leq W_2(s) \tag{77}$$

where $W_1(s)$, $W_2(s) \in \mathbb{R}$ are defined as follows

$$W_1(s) \triangleq \lambda_1 \|s\|^2 \quad , \quad W_2(s) \triangleq \lambda_2(\|s\|)\|s\|^2 \tag{78}$$

and $\lambda_1, \lambda_2(\cdot) \in \mathbb{R}$ are defined as follows

$$\lambda_1 \triangleq \frac{1}{2}\min\{1, \underline{m}\} \;,\; \lambda_2 \triangleq \max\left\{1, \frac{1}{2}\bar{m}(\|s\|)\right\}. \tag{79}$$

After taking the time derivative of (74), the following expression can be obtained

$$\begin{aligned} \dot{V} \;=\; & -\sum_{i=1}^{n-1} e_i^T e_i - e_n^T \Lambda e_n + e_{n-1}^T e_n \\ & -r^T r + r^T \tilde{N} - r^T K r - \frac{k_L}{2} r^T r \end{aligned} \tag{80}$$

where (6a)-(6c), (13), (46), (48) and (50) were utilized. By (22), (32) and the triangle inequality, an upper-bound on (80) can be obtained as follows

$$\begin{aligned} \dot{V} \;\leq\; & -\lambda_3\|z\|^2 + \|r\|\rho(\|z\|)\|z\| \\ & -\left(\lambda_{\min}(K) + \frac{k_L}{2}\right)\|r\|^2 \\ \leq\; & -\left(\lambda_4 - \frac{\rho^2(\|z\|)}{4\lambda_{\min}(K)}\right)\|z\|^2 \end{aligned} \tag{81}$$

where $\lambda_3 \triangleq \min\left\{\frac{1}{2}, \lambda_{\min}(\Lambda) - \frac{1}{2}\right\}$ and $\lambda_4 \triangleq \min\left\{\lambda_3, \frac{k_L}{2}\right\}$. The following inequality can be developed

$$\dot{V} \leq W(s) \leq \bar{W}(s) \tag{82}$$

where $W(s), \bar{W}(s) \in \mathbb{R}$ denote the following non-positive functions

$$W(s) \triangleq -\beta_0\|z\|^2 \quad , \quad \bar{W}(s) \triangleq -\beta_0\|e_1\|^2 \tag{83}$$

with $\beta_0 \in \mathbb{R}$ being a positive constant, and provided that $\lambda_{\min}(K)$ is selected according to the following sufficient condition

$$\begin{aligned} \lambda_{\min}(K) \;&\geq\; \frac{\rho^2(\|z\|)}{4\lambda_4} \\ \text{or } \|z\| \;&\leq\; \rho^{-1}\left(2\sqrt{\lambda_4\lambda_{\min}(K)}\right). \end{aligned} \tag{84}$$

Based on (74)-(79) and (81)-(83), the regions D and S can be defined as follows

$$\mathcal{D} = \left\{s : \|s\| < \rho^{-1}\left(2\sqrt{\lambda_4\lambda_{\min}(K)}\right)\right\} \tag{85}$$

$$\mathcal{S} = \left\{s \in \mathcal{D} : W_2(s) < \lambda_1\left(\rho^{-1}\left(2\sqrt{\lambda_4\lambda_{\min}(K)}\right)\right)^2\right\}. \tag{86}$$

Note that the region of attraction in (86) can be made arbitrarily large to include any initial conditions by increasing $\lambda_{\min}(K)$ (i.e., a semi-global stability result). Specifically,

(78) and (86) can be used to calculate the region of attraction as follows

$$W_2\left(s\left(t_0\right)\right) \quad < \quad \lambda_1\left(\rho^{-1}\left(2\sqrt{\lambda_4\lambda_{\min}\left(K\right)}\right)\right)^2 \tag{87}$$

$$\implies \quad \left\|s\left(t_0\right)\right\| < \sqrt{\frac{\lambda_1}{\lambda_2\left(\left\|s\left(t_0\right)\right\|\right)}} \rho^{-1}\left(2\sqrt{\lambda_4\lambda_{\min}\left(K\right)}\right),$$

which can be rearranged as

$$\lambda_{\min}\left(K\right) \geq \frac{1}{4\lambda_4}\rho^2\left(\sqrt{\frac{\lambda_2\left(\left\|s\left(t_0\right)\right\|\right)}{\lambda_1}}\left\|s\left(t_0\right)\right\|\right). \tag{88}$$

By utilizing (23), (52) and (76) the following explicit expression for $\left\|s\left(t_0\right)\right\|$ can be derived as follows

$$\left\|s\left(t_0\right)\right\|^2 = \sum_{i=1}^{n}\left\|e_i\left(t_0\right)\right\|^2 + \left\|r\left(t_0\right)\right\|^2 + \zeta_{b1}. \tag{89}$$

From (74), (82), (86)-(88), it is clear that $V\left(s,t\right) \in \mathcal{L}_\infty$ $\forall s\left(t_0\right) \in \mathcal{S}$; hence $s\left(t\right), z\left(t\right) \in \mathcal{L}_\infty$ $\forall s\left(t_0\right) \in \mathcal{S}$. From (13), it is clear that $\dot{e}_n\left(t\right) \in \mathcal{L}_\infty$ $\forall s\left(t_0\right) \in \mathcal{S}$. Using (4) and (39), it can be proved that $x^{(i)}\left(t\right) \in \mathcal{L}_\infty$, $i = 0, 1, ..., n$, $\forall s\left(t_0\right) \in \mathcal{S}$. Then, it is clear that $M\left(t\right), \dot{M}\left(t\right), f\left(t\right) \in \mathcal{L}_\infty$ $\forall s\left(t_0\right) \in \mathcal{S}$. By using these boundedness statements along with (11) it is clear that $u\left(t\right) \in \mathcal{L}_\infty$ $\forall s\left(t_0\right) \in \mathcal{S}$. These boundedness statements can be used along with the time derivative of (83) to prove that $\dot{\bar{W}}\left(s\left(t\right)\right) \in \mathcal{L}_\infty$ $\forall s\left(t_0\right) \in \mathcal{S}$; hence $\bar{W}\left(s\left(t\right)\right)$ is uniformly continuous. A direct application of Theorem 8.4 in [10] can be used to prove that $\left\|e_1\left(t\right)\right\| \to 0$ as $t \to \infty$ $\forall s\left(t_0\right) \in \mathcal{S}$. It should be noted that for finite time the subsequent analysis can be easily extended to prove that $\hat{N}_r\left(t\right)$, $\dot{u}\left(t\right)$, $\dot{r}\left(t\right)$, $\tilde{N}_r\left(t\right)$ are bounded. ■

Remark 8: It should be noted that when $\hat{W}_r\left(t\right)$ is designed as follows

$$\hat{W}_r\left(t\right) \quad \triangleq \quad \int_{t_0}^{t}\left[\mathrm{Sat}_\beta\left(\hat{N}_r\left(\tau - T\right)\right) + k_L\Lambda e_n\left(\tau\right)\right]d\tau + k_L e_n\left(t\right) - k_L e_n\left(t_0\right) \tag{90}$$

where $\hat{N}_r\left(t\right)$ was introduced in (43) and $\mathrm{Sat}_\beta\left(\cdot\right) \in \mathbb{R}^m$ is a saturation function vector, then the previous analysis can be modified to prove that $\hat{N}_r\left(t\right)$, $\dot{u}\left(t\right)$, $\dot{r}\left(t\right)$, $\tilde{N}_r\left(t\right)$ are bounded for all time and thus $\left\|e_1^{(i)}\left(t\right)\right\|$ converge to zero for $i = 1, ..., n$.

Appendix IV
Numerical Simulation Results

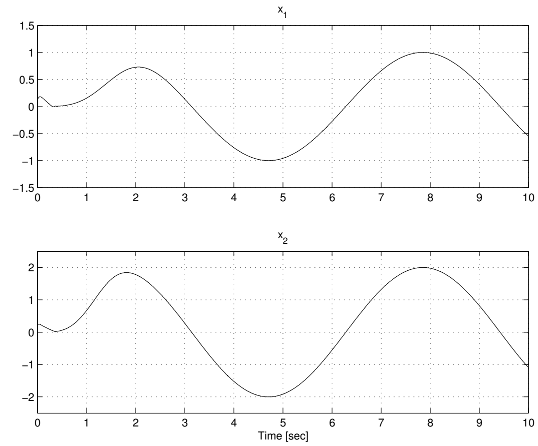

Fig. 1. (Adaptive Controller) State $x\left(t\right)$

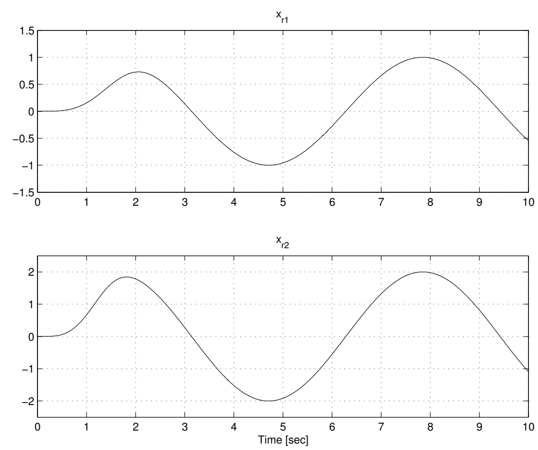

Fig. 2. (Adaptive Controller) Reference Trajectory $x_r\left(t\right)$

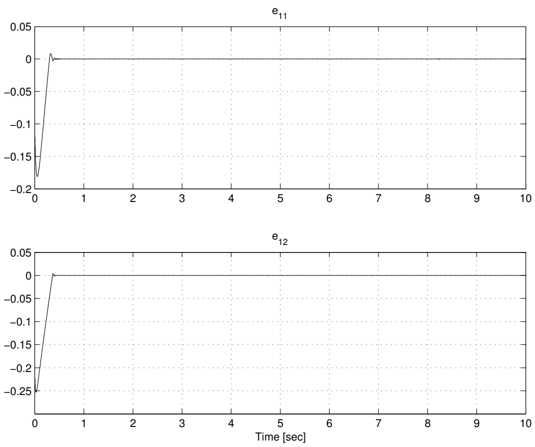

Fig. 3. (Adaptive Controller) Tracking Error $e_1(t)$

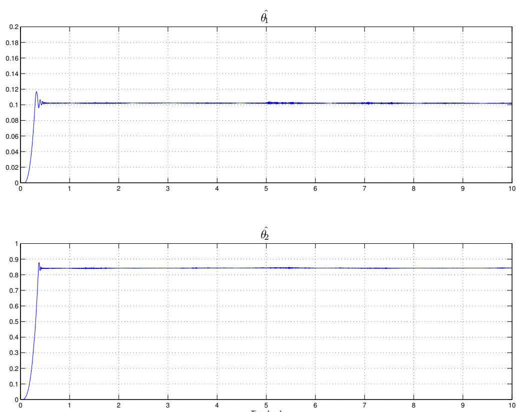

Fig. 4. (Adaptive Controller) Parameter Estimate $\hat{\theta}(t)$

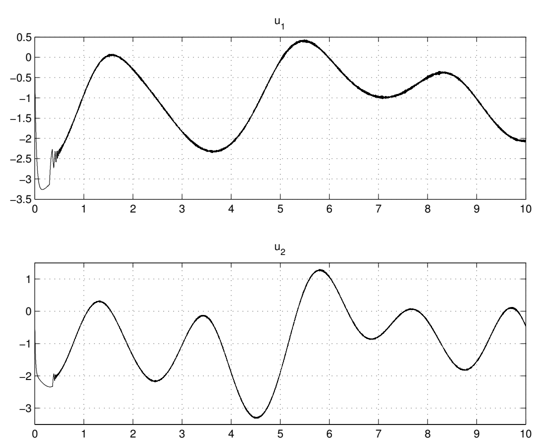

Fig. 5. (Adaptive Controller) Control Input $u(t)$

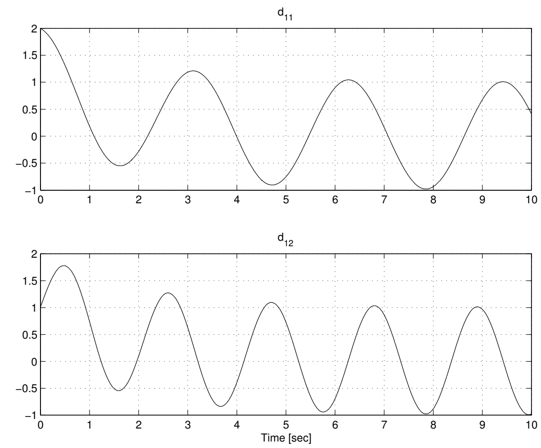

Fig. 6. (Adaptive Controller) Additive Disturbance $d_1(t)$

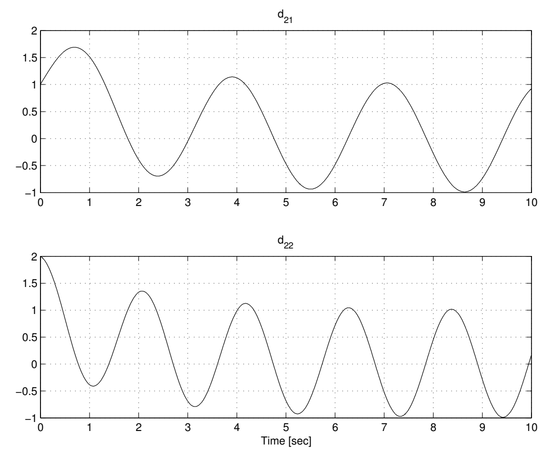

Fig. 7. (Adaptive Controller) Additive Disturbance $d_2(t)$

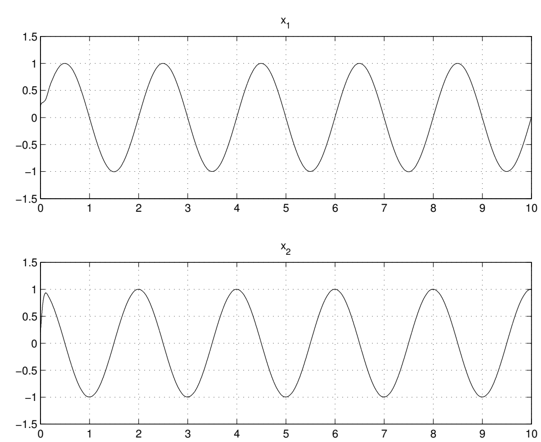

Fig. 8. (Learning Controller) State $x(t)$

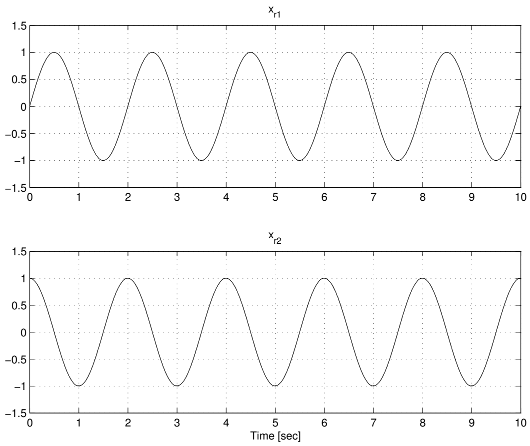

Fig. 9. (Learning Controller) Reference Trajectory $x_r(t)$

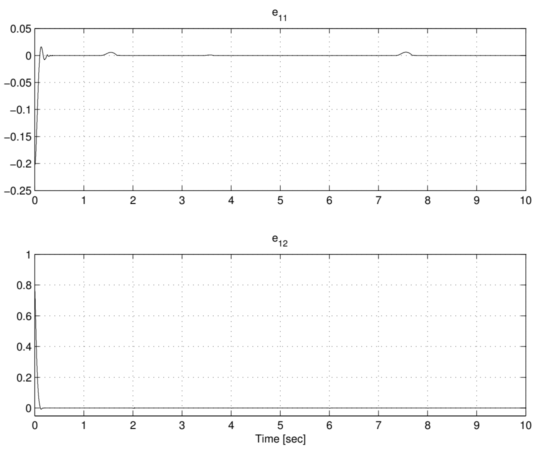

Fig. 10. (Learning Controller) Tracking Error $e_1(t)$

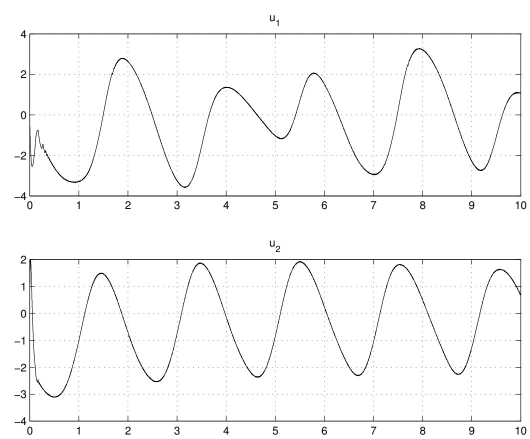

Fig. 11. (Learning Controller) Control Input $u(t)$

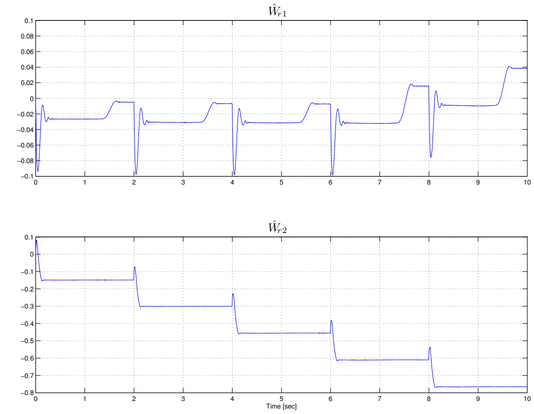

Fig. 12. (Learning Controller) $\hat{W}_r(t)$

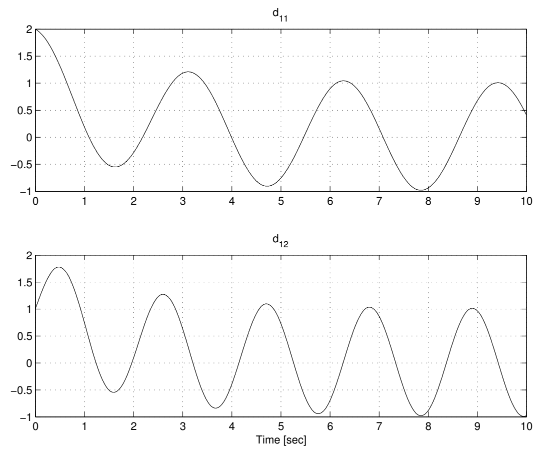

Fig. 13. (Learning Controller) Additive Disturbance $d_1(t)$

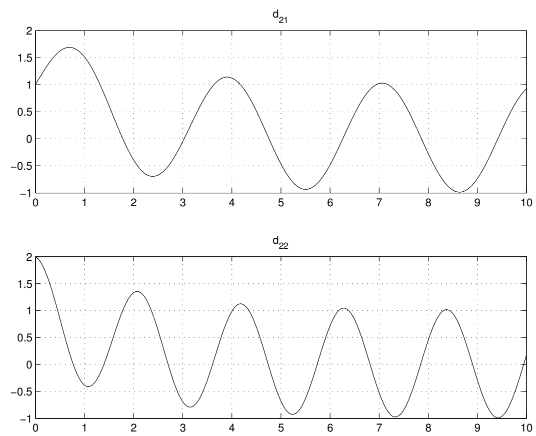

Fig. 14. (Learning Controller) Additive Disturbance $d_2(t)$