The research described in this paper was supported in part by the Air Force Office of Scientific Research under Grant AFOSR-82-0258 and in part by the National Science Foundation under Grant ECS-8700903.
### Generalized Riccati Equations for Two-Point Boundary-Value Descriptor Systems

**Authors:**
Massachusetts Institute of Technology, Laboratory for Information and Decision Systems, 77 Massachusetts Avenue, Cambridge, MA, 02139-4307

**DISTRIBUTION/AVAILABILITY STATEMENT**
Approved for public release; distribution unlimited

**ABSTRACT**

**SUBJECT TERMS**

**SECURITY CLASSIFICATION OF:**
- a. REPORT: unclassified
- b. ABSTRACT: unclassified
- c. THIS PAGE: unclassified

**LIMITATION OF ABSTRACT**

**NUMBER OF PAGES**
3

**NAME OF RESPONSIBLE PERSON**
The research described in this paper was supported in part by the Air Force Office of Scientific Research under Grant AFOSR-82-0258 and in part by the National Science Foundation under Grant ECS-8700903.
In the case of the optimal smoother, it is shown in [3] that if the following generalized Riccati equations

\[ \theta = A'(E\theta^{-1}E^* + BB^*)^{-1}A + C'R^{-1}C \]  
(14)

\[ \psi = A'(E\psi^{-1}E^* + C'R^{-1}C)A' + BB' \]  
(15)

have positive definite solutions \( \psi \) and \( \theta \) then there exist invertible matrices \( M \) and \( N \) such that

\[ MN^{-1} = \begin{bmatrix} I & 0 \\ 0 & A'S^{-1}E_\theta^{-1} \end{bmatrix} \]  
(16)

\[ MN^{-1} = \begin{bmatrix} A'T^{-1}E^* & 0 \\ 0 & I \end{bmatrix} \]  
(17)

Moreover, the eigenvalues of \( AT^{-1}E^* \) and \( A'S^{-1}E_\theta^{-1} \) are inside or on the unit circle. Equation (3.5) is called the descriptor Hamiltonian equation and the above decomposition is the descriptor Hamiltonian diagonalization. Of course, we would like \( AT^{-1}E^* \) and \( A'S^{-1}E_\theta^{-1} \) to be strictly stable. This occurs only when the descriptor Hamiltonian has no eigenmodes on the unit circle i.e. it is forward-backward stable.

Theorem 3:
If the system is forward-backward detectable and stabilizable (i.e. the modes on the unit circle are strongly reachable and strongly observable) then the corresponding smoother is forward-backward stable.

IV. Generalized Riccati Equations
In this section we study the generalized algebraic Riccati equation.

\[ \varphi = A(E\varphi^{-1}E + C'R^{-1}C)^{-1}A' + BB' \]  
(18)

Theorem 4:
If \((E,A,B)\) and \((C,E,A)\) are strongly reachable and observable respectively then (18) has a unique positive definite solution.

The approach used to prove this theorem is similar to that in [6] for the standard Riccati equation. Details will be presented in a future paper. Existence proceeds as follows. From Theorem 3 and the fact that eigenmodes of the smoother occur in reciprocal pairs, we know that we can write

\[ \begin{bmatrix} E & -BB^* \\ 0 & -A' \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E^* \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} \]  
(19)

The proof then proceeds by first showing that \( F \) is invertible, then that \( E'GF + C'R^{-1}C > 0 \) and finally that

\[ \varphi = (A(E'GF - C'R^{-1}C)^{-1}A' + BB') \]  
(20)

satisfies (18). To prove uniqueness, let \( \varphi_1 \) and \( \varphi_2 \) be two positive definite solutions of (18), let \( \Delta \varphi = \varphi_1 - \varphi_2 \), and

\[ T_i = E_i\varphi_i^{-1}E + C'R^{-1}C \]  
(21)

for \( i = 1, 2 \).

Some algebra then yields

\[ \Delta \varphi = AT^{-1}_1E_i\varphi_1^{-1}A' + BB' \]  
(22)

But \( AT^{-1}_1E_i\varphi_1^{-1} \) and \( BB' \) are strictly stable (see [3]); thus \( \Delta \varphi = 0 \).

References


