REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

1. AGENCY USE ONLY (Leave Blank)	2. REPORT DATE 26 September, 2006	3. REPORT TYPE AND DATES COVERED Final Progress Report July 1, 2003 – June 30, 2006

4. TITLE AND SUBTITLE Study of nonlinear oscillations of elastic membrane	5. FUNDING NUMBERS DAAD 19-03-1-0204

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9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) U. S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211	10. SPONSORING / MONITORING AGENCY REPORT NUMBER 45276.5-MA-HSI

11. SUPPLEMENTARY NOTES
     The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12 a. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release; distribution unlimited.	12 b. DISTRIBUTION CODE .

13. ABSTRACT (Maximum 200 words)

     Nonlinear oscillations of a circular elastic membrane with a simply supported boundary are investigated. They are caused by a source of acoustic waves, so that the membrane serves as an acoustic receiver. Vertical deflections of the membrane and the acoustic pressure are to be found and, if possible, the direction to the source. The boundary value problem is solved by a nonlinear version of the method of eigenfunction expansions. Vertical deflections and the acoustic pressure are found in the form of an expansion into the series of eigenfunctions of the Laplace operator in a disc. Numerical simulations were conducted to establish the sensitivity of a nonlinear membrane. The nonlinear effect of frequency multiplication was established. Position of the vertical plane containing the source was found.

14. SUBJECT TERMS Circular elastic membrane, nonlinear oscillations		15. NUMBER OF PAGES 4
		16. PRICE CODE

STUDY OF NONLINEAR OSCILLATIONS OF ELASTIC MEMBRANES

1. Foreword

Membrane structures have been used since the earliest of times as a means of receiving acoustic signals. However, in the past their analysis relied mostly on trial and error while modern analysis provides powerful tools for creating mathematical theory of description of their motions. Although deformations of these structures are essentially nonlinear, the linear theory provides an important starting point for understanding this complicated behavior.

The necessity of creation of passive sensor technologies determines the importance of the study of elastic membrane oscillations. Unattended ground sensors can have as their basic element a circular membrane with fixed ends (clamped, simply supported or damped). Geometry of the membrane being chosen to be circular, the current investigation concentrates on the response analysis of the membrane.

2. Statement of the Problem, Goals and the Main Method

A thin circular elastic membrane with a simply supported boundary serves as a model of an acoustic receiver. An external source sends a flux of acoustic waves which fall on the membrane and incite it. This results in the propagation of elastic waves on the surface of the membrane. Vertical oscillations are of interest and horizontal deflections are neglected. Nonlinear oscillations of the surface of the membrane are governed by the damped Boussinesq equation. It is a second order in time and fourth order in space, dispersive, dissipative, nonlinear equation with a quadratic nonlinearity. Its linear part includes the fourth order classical elasticity operator and the third order term responsible for internal friction. Quadratic nonlinearity present in the equation includes second derivatives. It reflects dependence of the vertical deflection on the local curvature of the membrane.

Goals of investigation consist in developing an efficient algorithm of solving the direct nonlinear boundary-value problem in question, i.e. deriving the formulas for the vertical deflection of the membrane, proving the convergence of the corresponding series in a certain function space and finding the acoustic pressure. Conducting numerical simulations is necessary for the comparison of behavior of the linear and nonlinear membranes. In addition, a progress should be made in finding the direction to the acoustic source and exploiting the nonlinearity for making the membrane more sensitive. The latter goal stands out as a new type of an inverse nonlinear problem which has never been solved before.

Nonlinear version of the method of eigenfunction expansions is chosen as a main tool of constructing solutions. This method is well known for the classical linear models of wave propagation. Extending the method for solving nonlinear boundary value problems has

been a recent development made by the Principal Investigator of the present work. Solution is obtained in the form of the series of eigenfunctions of the Laplace operator in a disc with homogeneous boundary conditions and periodicity conditions in the angle. More precisely, it is a Fourier-Bessel series, where Bessel functions are the radial eigenfunctions and exponentials are the angular ones. Bessel function zeros depend on two indices which reflects the interaction of radial and angular eigenfunctions. Expanding the nonlinearity into the eigenfunction series and obtaining the estimates of the eigenfunction coefficients is the key issue in applying this method. In contrast to Galerkin's method which allows only to prove existence, our approach allows to construct solutions.

3. Results

Proving convergence of the series representing the solution and the nonlinearity forms the main difficulty and defines the function space where the solution exists. It was established that classical solutions of the boundary value problem can not be constructed in principle since convergence of the series is not enough to provide the necessary regularity. This is typical for many nonlinear initial-boundary-value problems. However, mild solutions can be constructed. They are defined as solutions of the integral equation obtained by integrating the original boundary-value problem with respect to time.

New function spaces were introduced, namely anisotropic Sobolev spaces. The norm in such a space is the sum of the L2-norms of the usual derivatives weighted by tangential derivatives. These spaces are sensitive to the energy transfer in the angular direction and reflect the interaction of angular waves and the radial ones.

A new family of special functions was introduced for improving the convergence of the series representing the solution. They owe their appearance to the nonlinearity and the circular geometry. These functions appear as a result of the series multiplication. They received the name of convolutions of Rayleigh functions with respect to the Bessel index. Rayleigh functions are well known in the classical theory of linear disc oscillations. They are defined as series of inverse powers of the Bessel function zeros and reflect the circular geometry for linear problems and the boundary conditions. Convolutions of these functions reflect circular geometry, boundary conditions and the nonlinearity. These functions allow to improve the convergence of the eigenfunction series representing the solutions and to improve greatly the computational effectiveness of the algorithm. Convolutions of Rayleigh functions were expressed in terms of the psi-function (logarithmic derivative of the gamma-function and its derivatives). Asymptotic expansions for large values of the angular index were obtained. These asymptotics are essential for the estimates of the Fourier-Bessel coefficients in the representations of solutions.

Eigenfunciton series representations were obtained for the vertical deflection of the circular membrane and the acoustic field in the plane of the membrane. These are *mild* solutions of the nonlinear boundary value problem in question. If a typical time-harmonic

signal is chosen as a source, then the final formulas show a typically nonlinear effect of frequency multiplication.

Numerical simulations were conducted in order to compare linear and nonlinear oscillations. Truncated eigenfunction series solutions were used for computations in the nonlinear case. Different angles for incident acoustic waves were tried. It was established that the created algorithm has excellent convergence properties. Just a few iterations were needed for securing enough precision. The conducted simulations clearly showed the picture of nonlinear waves traveling on the surface of the membrane and a much better sensitivity of the nonlinear membrane in comparison with the linear one.

As regards the most difficult problem of finding the angle to the acoustic source (inverse nonlinear problem which deserves the name of the problem of the 21st century), some progress has been made. Indeed, one can establish the plane where the source lies. On the border (perimeter) of the membrane there is a spot where the maximum of the wave energy is located. One can draw the diameter line through the center of the disc and the point of the maximum. The plane going through this line perpendicular to the membrane contains the source of acoustic waves. Rigorous mathematical proof of this fact is being developed.

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