

Generalized Finite Element Methods: Main Ideas, Results, and Perspective

Ivo Babuška * Uday Banerjee † John E. Osborn ‡

Abstract

This paper is an overview of the main ideas of the Generalized Finite Element Method (GFEM). We present the basic results, experiences with, and potentials of this method. The GFEM is a generalization of the classical Finite Element Method — in its h , p , and h - p versions — as well as of the various forms of meshless methods used in engineering.

AMS(MOS) subject classifications. 65N30, 65N15, 41A10, 42A10, 41A30

1 Introduction

A numerical method to approximate the solution of a boundary value problem (BVP) for partial differential equations (PDE) has two major components:

- (a) The selection of a family $\{\omega_j\}_{j=1}^N$ of small sets that form a cover of the domain of the BVP, and, for each j , a finite dimensional *local approximation* space V_j of functions with the property that functions in V_j can accurately approximate the solution of the BVP on ω_j , *i.e.*, locally. The approximate solution of the BVP is then sought from the space S of global functions, obtained by “pasting together” the functions in V_j , in such a way that good local approximability of the V_j s ensure good global approximability of S . The functions in S are often of the form $\sum_j \phi_j v_j$, with $v_j \in V_j$, and where $\{\phi_j\}$ is a partition of unity with respect to $\{\omega_j\}$. We note that each $v_j \in V_j$ can be viewed as a vector of real numbers (the coefficients with respect to some basis for V_j). Consequently, a functions in the space S , which has the form $\sum_j \phi_j v_j$, can also be viewed as a vector \mathbf{c} of real numbers.

*Institute for Computational Engineering and Sciences, ACE 6.412, University of Texas at Austin, Austin, TX 78712. This research was partially supported by NSF Grant # DMS-0341982 and ONR Grant # N00014-99-1-0724.

†Department of Mathematics, 215 Carnegie, Syracuse University, Syracuse, NY 13244. E-mail address: banerjee@syr.edu. WWW home page URL: <http://bhaskara.syr.edu>. This research was partially supported by NSF Grant # DMS-0341899.

‡Department of Mathematics, University of Maryland, College Park, MD 20742. E-mail address: jeo@math.umd.edu. WWW home page URL: <http://www.math.umd.edu/~jeo>. This research was supported by NSF Grant # DMS-0341982.

Report Documentation Page

Form Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE 2005		2. REPORT TYPE		3. DATES COVERED -	
4. TITLE AND SUBTITLE Generalized Finite Element Methods: Main Ideas, Results, and Perspective				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Office of Naval Research, One Liberty Center, 875 North Randolph Street Suite 1425, Arlington, VA, 22203-1995				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT see report					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES 38	19a. NAME OF RESPONSIBLE PERSON
a. REPORT unclassified	b. ABSTRACT unclassified	c. THIS PAGE unclassified			

- (b) A discretization principle that selects an approximate solution of the BVP from the space S ; in other words, the discretization principle associates a specific vector \mathbf{c} , *i.e.*, a specific element of S , to the exact solution of the BVP. This element of S is then viewed as an approximate solution of the BVP.

Given the local spaces V_j , and the derived global space S , a discretization principle determines the approximate solution in S by approximating the partial differential operator, and thereby reduces the BVP to a system of linear or non-linear algebraic equations for the vector \mathbf{c} . When the system is linear, the associated matrix is often sparse. The accuracy of the approximate solution depends on the stability of the discretized partial differential operator and on the approximation properties of the space S , which in turn depends on the approximation properties of the spaces V_j .

We first discuss briefly the choice of the space S , as indicated in (a), for different numerical methods. In a large family of methods, classical interpolation theory provides guidance in the choice of the spaces V_j , and thus S . Specifically, let $\{x_j\}$ be a set of given distinct points, called *nodes*, in the domain of definition of the BVP, and suppose that g is a function whose values g_j at nodes x_j , *i.e.*, $g_j = g(x_j)$, are given. Then the space S is such that there exists a unique interpolating function $f \in S$ such that $f(x_j) = g_j$. The approximation property of the space S is related to the interpolation error, *i.e.*, $g - f$, and this error depends on the distribution of the nodes $\{x_j\}$, which could be regular or irregular (scattered nodes), and on the bounds of higher derivatives of the function g . The space of polynomials, piecewise polynomials, and the combination of radial basis functions are examples of the space S with this interpolation property. We mention that the uniqueness of the interpolating function $f \in S$, with respect to the given data $\{g_j\}$, depends strongly on the distribution of nodes, as well as on the space S (and thus on spaces V_j). For a given distribution of the nodes, the space S may not have unique solvability of the interpolation problem. To resolve this problem in certain situations, various stabilization techniques have been reported in the literature; *e.g.*, see [10] in the context of thin-plate spline radial functions. We also refer to [11] for a detailed discussion on radial basis functions. The interpolation problem for the space of polynomials or piecewise polynomials, and its sensitivity on the distribution of nodes, is well studied in the literature.

But there are other methods, *e.g.*, certain meshless methods, where the choice of V_j , and thus S , is not dictated by the idea of interpolation. In these methods, the local spaces V_j are constructed from particle shape functions, *e.g.*, RKP shape functions, and the elements of the space S are of the form $\sum_j v_j$, where $v_j \in V_j$ (*i.e.*, $\phi_j = 1$ in $\sum_j \phi_j v_j$). The approximability of the spaces V_j and S , is ensured by so called “reproducing property” of the particle shape functions. For a detailed discussion of these spaces, we refer to [4, 26].

We will now briefly discuss the discretization principle indicated in (b). Different discretization principles, together with given global approximating spaces S , give rise to different methods for the approximation of the solution of a BVP;

e.g., finite difference methods (FDM), finite volume methods (FVM), collocation methods, and methods based on weighted residuals. We note that the FDM and collocation methods can be viewed as obtained from the discretization by the Petrov-Galerkin method (in the most general setting) with Dirac functions used as test functions. Establishing stability and obtaining error estimates for these methods is subtle and difficult, even when the spaces V_j , and consequently S , have good approximation properties. For example, though the convergence analysis of FDM with regularly distributed nodes is well-established, not much is known when the nodes are irregularly distributed ([14]). The convergence of the collocation method using radial functions was analyzed in [18]. For a survey of application of these methods, we refer to [24].

A variant of the collocation method, obtained from the discretization by the Petrov-Galerkin method using test functions with small supports (instead of Dirac functions as mentioned in the last paragraph), have also been reported in the engineering literature, but without rigorous mathematical analysis ([1, 25, 43, 32]). These methods, which are also used to approximate solutions of non-linear equations, CFD, and other engineering problems, often lack robustness. Various ad-hoc stabilization techniques are used in the implementation of these methods, without rigorous mathematical examination.

There is yet another class of methods that is based on a discretization principle where the trial and test functions belong to the same Hilbert space, say the Sobolev space $H^1(\Omega)$ (for second order elliptic BVP). This principle is referred to as the Galerkin method or Bubnov-Galerkin method ([30]). Typical representatives of this class are Finite Element Method (FEM) – with its h , p , and $h-p$ versions and mixed FEM. In these methods, the functions in the local spaces V_j have to be “pasted together” so that the space S is a subspace of $H^1(\Omega)$. This is achieved by considering V_j ’s consisting of piecewise polynomials (or pull-back polynomials) of special form, defined with respect to an appropriate mesh. Certain classes of meshless methods, *e.g.*, RKP method, fall in this category. In these meshless methods, the spaces V_j are subspaces of the energy space, and consequently the elements of S , which are linear combinations of elements in V_j (mentioned before), are automatically in the energy space.

In this paper, we present the main ideas of Generalized Finite Element Method (GFEM), which is a Galerkin (or Bubnov-Galerkin) method. The local spaces V_j consists of functions, not necessarily polynomials, that reflect the available information on the unknown solution and thus ensure good local approximation. Then a partition of unity $\{\phi_j\}$ is used to “paste” these spaces together to form S , which is a subspace of the energy space and has good global approximability. The GFEM has been extensively discussed in a series of papers ([16, 37, 38, 40, 39]), where its effectiveness was shown when applied to problems with domains having complicated boundaries, problems with micro-scales, and problems with boundary layers. We will present the theoretical basis of the GFEM, proving major results. In addition, we will discuss various procedures for the selection of local approximating functions and comment on certain issues in implementation.

The partition of unity approach was first used in [5] to obtain an accurate approximation to the solution (which is non-smooth) of BVP for PDEs with rough coefficients; the method in [5] was referred to as the Special Finite Element Method. The importance of such an approach was seen in [7], which showed that standard FE approximations converge arbitrarily slowly when approximating solutions to problems with rough coefficients. Based on the ideas in [5], the GFEM was elaborated on in [6, 27, 28], where it was referred to as the Partition of Unity Method (PUM). Later in [37, 38], the method was referred to as GFEM, since the classical FEM is a special case of this method. Currently, the partition of unity approach is used in various directions under various names — Method of Clouds, XFEM (extended FEM), and Method of Spheres ([15, 41, 36, 13]). These methods differ primarily in the form of partition of unity functions used and in the use of different local spaces.

2 The Galerkin Method

Suppose we are interested in solving the stationary heat conduction problem on the domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. Specifically, we consider the problem

$$\begin{cases} -\operatorname{div}(a(x, y) \operatorname{grad} u) = f, & \text{for } (x, y) \in \Omega \\ u = 0 & \text{on } \Gamma_1 \\ a \frac{\partial u}{\partial n} = g & \text{on } \Gamma_2. \end{cases} \quad (2.1)$$

Here $f = f(x, y)$ is the heat gain from internal sources per unit volume, $a = a(x, y)$ is the coefficient of thermal conductivity, $g = g(x, y)$ is the heat flow per unit length across Γ_2 . We consider f , a , and g to be given, we specify the temperature to be 0 on Γ_1 , and specify the heat flow per unit length across Γ_2 to be g , and seek the steady state temperature $u = u(x, y)$ throughout the domain Ω . The function $a(x, y)$ could be rough, *i.e.*, fail to have continuous derivatives, but is assumed to satisfy

$$0 < \alpha \leq a(x, y) \leq \beta < \infty.$$

As usual, we give our problem a weak, or variational, formulation. Let

$$\mathcal{E}(\Omega) = \mathcal{E} = \left\{ v : \|v\|_{\mathcal{E}(\Omega)}^2 < \infty \right\}, \quad (2.2)$$

where

$$\|v\|_{\mathcal{E}(\Omega)}^2 = \|v\|_{\mathcal{E}}^2 = \int_{\Omega} a(x, y) \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] dx dy. \quad (2.3)$$

We note that under mild restrictions on Γ , $\|v\|_{\mathcal{E}(\Omega)} < \infty$ implies

$$\|v\|_{L_a^2(\Omega)}^2 = \|v\|_{L_a^2}^2 = \int_{\Omega} a|u|^2 dx dy < \infty, \quad (2.4)$$

i.e., $v \in L^1_a(\Omega)$. We then let

$$\mathcal{E}_{\Gamma_1}(\Omega) = \mathcal{E}_{\Gamma_1} = \{v : v \in \mathcal{E}(\Omega), u = 0 \text{ on } \Gamma_1\}, \quad (2.5)$$

where the Dirichlet boundary condition is imposed in the sense of trace. If $\Gamma_1 = \emptyset$, then $\mathcal{E}_{\Gamma_1}(\Omega) = \mathcal{E}(\Omega)$. The space \mathcal{E}_{Γ_1} is the *energy space* for our problem and $\|v\|_{\mathcal{E}}$ is the *energy norm* of v . (Strictly speaking, $\|v\|_{\mathcal{E}(\Omega)}$ is not a norm on $\mathcal{E}(\Omega)$; it is, however, a norm up to *rigid body motions*, which in this situation are the constants.)

On $\mathcal{E}_{\Gamma_1} \times \mathcal{E}_{\Gamma_1}$ define

$$B(u, v) = \int_{\Omega} a(x, y) \left[\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right] dx dy$$

and

$$F(v) = \int_{\Omega} f v dx dy + \int_{\Gamma_2} g v ds,$$

where we assume $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_2)$. Then the weak formulation reads,

$$\begin{cases} \text{Find } u \in \mathcal{E}_{\Gamma_1} \text{ satisfying} \\ B(u, v) = F(v) \text{ for all } v \in \mathcal{E}_{\Gamma_1}. \end{cases} \quad (2.6)$$

Remark 2.1. If $\Gamma_2 = \emptyset$, then (2.1) is Dirichlet Problem. If the lengths of both Γ_1 and Γ_2 are positive, then (2.1) is a mixed Dirichlet-Neumann Problem. In either of these cases, Problem (2.1) ((2.6)) is uniquely solvable. If $\Gamma_1 = \emptyset$, then (2.1) is a Neumann Problem. In this case (2.1) ((2.6)) will be solvable provided $\int_{\Omega} f dx dy + \int_{\Gamma_2} g ds = 0$. To ensure uniqueness, one needs an auxiliary condition: say $\int_{\Omega} u dx dy = 0$.

We next consider the approximation of the solution u of (2.1) ((2.6)) by the Galerkin Method (Bubnov-Galerkin method). Toward this end we suppose we have a finite dimensional space $S \subset \mathcal{E}_{\Gamma_1}$, and consider the problem

$$\begin{cases} \text{Find } u_S \in S \text{ satisfying} \\ B(u_S, v) = F(v) \text{ for all } v \in S. \end{cases} \quad (2.7)$$

This problem, like Problem (2.6), has a unique solution, and is equivalent to a system of linear algebraic equations. Specifically, if ϕ_1, \dots, ϕ_m spans the space S and we write $u_S = \sum_{j=1}^m c_j \phi_j$, Problem (2.7) becomes

$$\sum_{j=1}^m B(\phi_i, \phi_j) c_j = F(\phi_i), \quad i = 1, \dots, m. \quad (2.8)$$

If $\{\phi_j\}_{j=1}^m$ is a basis for S , then the linear system (2.8) is nonsingular and is uniquely solvable. If $\{\phi_j\}_{j=1}^m$ is not a basis, *i.e.*, it fails to be linearly independent, the system (2.8) is solvable since (2.7) is solvable, but solutions of (2.8) are not unique. (The family $\{\phi_j\}_{j=1}^m$ is said to span S if any $v \in S$ can be written

as $v = \sum_{j=1}^N c_j \phi_j$ for some coefficients c_j ; it is said to be a *basis* if, in addition, it is linearly independent, *i.e.*, $\sum_{j=1}^m c_j \phi_j = 0$ implies $c_j = 0, j = 1, \dots, m$). We note, however, that if $\{c_j^{(1)}\}_{j=1}^m$ and $\{c_j^{(2)}\}_{j=1}^m$ are solutions of (2.8), then

$$u_S = \sum_{j=1}^m c_j^{(1)} \phi_j = \sum_{j=1}^m c_j^{(2)} \phi_j.$$

Whenever we have a spanning set ϕ_1, \dots, ϕ_m , we refer to the functions ϕ_j as *shape functions*. If the shape function have small supports, the matrix of the system (2.8) is sparse. The Finite Element Method (FEM) is of this type, with piecewise polynomial shape functions defined on a mesh.

We will measure the accuracy of u_S in the energy norm. Letting $e_S = u - u_S$ be the *error*, and consider the *energy norm of the error*:

$$\|e_S\|_{\mathcal{E}} = (B(e_S, e_S))^{1/2}. \quad (2.9)$$

The main feature of the Galerkin Method is that

$$\|u - u_S\|_{\mathcal{E}} = \|e_S\|_{\mathcal{E}} \leq \|u - \xi\|_{\mathcal{E}}, \quad \text{for any } \xi \in S. \quad (2.10)$$

We thus need to construct S so that

$$S \subset \mathcal{E}_{\Gamma_1}(\Omega) \quad (2.11)$$

and so that

$$\text{there exists } \xi = \xi^u \in S \text{ so that } \|u - \xi^u\|_{\mathcal{E}(\Omega)} \text{ is small.} \quad (2.12)$$

Of course, it is also important that the approximating space S lead to a reasonably solvable linear system (2.8). Constructing S so that (2.11) and (2.12) are satisfied are our major goals.

In many important problems the character (smoothness) of the solution changes from one part of the domain to another, so it is natural to attempt to approximate u separately on these parts of Ω . There is often a natural division of Ω into subdomains, ω_j , so that, for each j , we can find a function ξ_j^u that approximates u well on ω_j . More precisely, we have open sets $\omega_1, \dots, \omega_N$, called *patches* satisfying

$$\omega_j \subset \Omega \text{ and } \Omega = \bigcup_{j=1}^N \omega_j \text{ (they form an open cover of } \Omega), \quad (2.13)$$

and function $\xi_j^u \in \mathcal{E}(\omega_j)$ satisfying

$$\|u - \xi_j^u\|_{\mathcal{E}(\omega_j)} \text{ is small,} \quad (2.14)$$

where $\mathcal{E}(\omega_j)$ and $\|u - \xi_j^u\|_{\mathcal{E}(\omega_j)}$ are defined by (2.2) and (2.3), with Ω replaced by ω_j . We will speak of $\{\omega_j\}$ as a partition of Ω . We then need to “paste” these

approximating functions together to obtain a function $\xi^u \in S$ satisfying (2.12). These two aspects of our development — the existence of local approximations and the process of pasting them together — are largely independent.

For each j we wish first to construct ξ_j^u on ω_j so that (2.14) is satisfied. Then we wish to construct $\xi^u \in S$ using the ξ_j^u — pasting them together — so that

$$K_1 \sum_{j=1}^N \|u - \xi_j^u\|_{\mathcal{E}(\omega_j)}^2 \leq \|u - \xi^u\|_{\mathcal{E}(\Omega)}^2 \leq K_2 \sum_{j=1}^N \|u - \xi_j^u\|_{\mathcal{E}(\omega_j)}^2, \quad (2.15)$$

where K_1, K_2 are independent of u and the number of patches (N), but do depend on the form (character) of the patches. Our main focus in Section 3 will be to prove the upper bound in (2.15). The lower bound will be true in some situations, but not in others.

These issues will be discussed in detail in the next section. We end this section by noting that to find a suitable ξ_j^u and to show that $\|u - \xi_j^u\|_{\mathcal{E}(\omega_j)}$ is small, we need to use the available information on the (unknown) solution. For example, if $a(x, y), f(x, y), g(x, y)$ are smooth functions and $\Gamma_1 = \Gamma$ is also smooth, then $u(x, y)$ will be a smooth function. From standard polynomial approximation theory we thus know that there is a quadratic polynomial ξ_j^u that approximates u well on ω_j :

$$\|u - \xi_j^u\|_{\mathcal{E}(\omega_j)} \leq h_j^3 K_j C_j,$$

where K_j is a bound on the third derivatives of u on ω_j ($|D^3 u| \leq K_j$) and h_j is the diameter of ω_j and C_j depends on the form of ω_j .

3 Local and Global Approximation

In this section we show how to accomplish the goals stated in Section 2 — namely (2.11) and (2.12) — by means of local approximation and the pasting process, which are largely separate. As indicated in Section 2, let $\{\omega_j\}_{j=1}^N$ be open sets (patches) satisfying

$$\omega_j \subset \Omega \text{ and } \Omega = \bigcup_{j=1}^N \omega_j.$$

We assume in addition that any $x \in \Omega$ belongs to at most κ of the subdomains ω_j . Then let $\{\phi_j\}_{j=1}^N$ be a family of functions defined on Ω , having piecewise continuous first derivatives, and satisfying the following properties:

$$\phi_j(x, y) = 0, \text{ for } (x, y) \in \Omega \setminus \omega_j, j = 1, \dots, N; \quad (3.1)$$

$$\sum_{j=1}^N \phi_j(x, y) = 1, \text{ for } (x, y) \in \Omega; \quad (3.2)$$

$$\max_{(x,y) \in \Omega} |\phi_j(x,y)| \leq C_1, \quad j = 1, \dots, N; \quad (3.3)$$

$$\max_{(x,y) \in \Omega} |\nabla \phi_j(x,y)| \leq \frac{C_2}{\text{diam}(\omega_j)}, \quad j = 1, \dots, N; \quad (3.4)$$

where $0 < C_1, C_2 < \infty$. Here $\text{diam}(\omega_j)$ denotes the diameter of ω_j . Property (3.2) states that $\{\phi_j\}$ is a *partition of unity* on Ω .

As an example, consider the classical FEM with triangular elements satisfying the minimal angle condition, with nodal points A_j . Let ω_j be the patch or finite element star associated with the node A_j , *i.e.*, the union of triangles with A_j as one of their vertices. It is easy to see that the family ω_j creates a partition of Ω . Further, let ϕ_j be the piecewise linear functions with

$$\phi_i(A_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Then it is easily seen that the family $\{\phi_j\}$ satisfies (3.1)-(3.4) with $C_1 = 1$ and C_2 depending on the minimal angle condition.

We next mention another example. Let

$$\Omega = \{(x,y) : 0 < x < 1, 0 < y < 1\}$$

and let $A_{\underline{k}} = A_{i,j} = (ih, jh)$, $h = \frac{1}{m}$, $i, j = 0, 1, \dots, m$. Let $\omega_{\underline{k}}^h$ be the intersection of Ω and the open disk centered at $A_{\underline{k}}$ with radius Rh , where R is such that $\{\omega_{\underline{k}}^h\}$ is a cover of Ω . Letting $\phi(r)$, $0 \leq r \leq \infty$, be a function with bounded first derivative and with $\phi(r) > 0$, for $0 \leq r < R$, and $\phi(r) = 0$ for $r \geq R$, define

$$\tilde{\phi}_{\underline{k}}^{(h)}(x,y) = \phi \left(\left(\left(\frac{x-ih}{h} \right)^2 + \left(\frac{y-jh}{h} \right)^2 \right)^{1/2} \right).$$

The family $\{\tilde{\phi}_{\underline{k}}^{(h)}\}$ satisfies (3.1), (3.3), and (3.4), but not, in general, (3.2). If we define

$$\phi_{\underline{k}}^h(x,y) = \frac{\tilde{\phi}_{\underline{k}}^h(x,y)}{\sum_{\underline{l}} \tilde{\phi}_{\underline{l}}^h(x,y)},$$

then the family $\{\phi_{\underline{k}}^h\}$ satisfies all the conditions (3.1)-(3.4). To prove (3.3) and (3.4) for this family, we use the fact that

$$\sum_{\underline{l}} \tilde{\phi}_{\underline{l}}^{(h)}(x,y) \geq \tau > 0, \quad \text{for } (x,y) \in \Omega.$$

The functions in the family $\{\phi_{\underline{k}}^h\}$ are called *Shepard functions* ([35]).

To every ω_j of the partition $\{\omega_j\}$ we associate an $m(j)$ -dimensional space of functions defined on ω_j :

$$V_j = \{\xi_j : \xi_j = \sum_{i=1}^{m(j)} b_{ji} \xi_{ji}, \quad b_{ji} \in \mathbb{R}, \quad \xi_{ji} \in \mathcal{E}(\omega_j) \text{ and } \xi_{ji} = 0 \text{ on } \bar{\omega}_j \cap \Gamma_1\}. \quad (3.5)$$

The space V_j is called a *local approximation space*. Note that the (essential) Dirichlet boundary condition is built into V_j . Then we let

$$\begin{aligned} S^{GFEM} &= \left\{ \psi = \sum_{j=1}^N \phi_j \xi_j : \text{where } \xi_j \in V_j \right\} \\ &= \text{span of } \{ \eta_{ji}, i = 1, \dots, m(j), j = 1, \dots, N \}, \end{aligned} \quad (3.6)$$

where

$$\eta_{ji} = \phi_j \xi_{ji} \quad (3.7)$$

are the shape functions for the S^{GFEM} . The space S^{GFEM} is called the Generalized Finite Element *global approximation space*.

Theorem 3.1 *We have*

$$S^{GFEM} \subset \mathcal{E}_{\Gamma_1}(\Omega). \quad (3.8)$$

Proof. Using (3.1) we see that $(\phi_j \xi_{ji})(x, y) = 0$ for $(x, y) \in \partial\omega_j \cap \Omega$. Hence $\phi_j \xi_{ji}$ can be extended as zero to all of Ω , and $\phi_j \xi_{ji}$, so extended, will be in $\mathcal{E}(\Omega)$. Furthermore, since $\xi_{ji} = 0$ on $\bar{\omega}_j \cap \Gamma_1$, we see that $\phi_j \xi_{ji}|_{\Gamma_1} = 0$. So, for all j and i , $\phi_j \xi_{ji} \in \mathcal{E}_{\Gamma_1}(\Omega)$, and hence the span of these functions is in $\mathcal{E}(\Omega)$. This is the desired result. \square

Remark 3.1. Theorem 3.1 establishes (2.11), one of the goals discussed in Section 2.

The Generalized Finite Element Method (GFEM) is now defined to be the Galerkin Method (2.7) with

$$S = S^{GFEM}.$$

We denote the approximate solution by $u_S = u_{GFEM}$. If we can now construct a $\xi^u \in S^{GFEM}$ so that (2.12) is satisfied, then from (2.10) we know that $\|u - u^{GFEM}\|_{\mathcal{E}}$ is small. We now turn to the construction of such a ξ^u .

For each j , we assume the exact solution u of Problem (2.1), more generally any $u \in \mathcal{E}_{\Gamma_1}$, can be accurately approximated on ω_j by a function $\xi_j^u \in V_j$; specifically that

$$\|u - \xi_j^u\|_{L_a^2(\omega_j)}^2 = \int_{\omega_j} a |u - \xi_j^u|^2 dx dy \leq \epsilon_1^2(j) \quad (3.9)$$

and

$$\|u - \xi_j^u\|_{\mathcal{E}(\omega_j)}^2 = \int_{\omega_j} a |\nabla(u - \xi_j^u)|^2 dx dy \leq \epsilon_2^2(j). \quad (3.10)$$

Then define the *global approximation*

$$\xi^u = \sum_{j=1}^N \phi_j \xi_j^u \in S^{GFEM}. \quad (3.11)$$

We see that the local approximation is ensured by the appropriate selection of the spaces V_j ; and the pasting together is handled by multiplication by the partition of unity functions, ϕ_j . We now estimate $\|u - \xi_j^u\|_{L^2(\Omega)}$ and $\|u - \xi_j^u\|_{\mathcal{E}(\Omega)}$.

Theorem 3.2 *Suppose $u \in \mathcal{E}_{\Gamma_1}(\Omega)$. Then*

$$\|u - \xi^u\|_{L_a^2(\Omega)} \leq \kappa^{1/2} C_1 \left(\sum_{j=1}^N \epsilon_1^2(j) \right)^{1/2} \quad (3.12)$$

and

$$\|u - \xi^u\|_{\mathcal{E}(\Omega)} \leq (2\kappa)^{1/2} \left(C_2^2 \sum_{j=1}^N \frac{\epsilon_1^2(j)}{\text{diam}^2(\omega_j)} + C_1^2 \sum_{j=1}^N \epsilon_2^2(j) \right)^{1/2}, \quad (3.13)$$

where C_1 and C_2 are as in (3.3) and (3.4), respectively.

Proof. We will first prove (3.12). Recalling the definition of ξ^u in (3.11) and using the fact that $\{\phi_j\}$ is a partition of unity on Ω , we have

$$\|u - \xi^u\|_{L_a^2(\Omega)}^2 = \int_{\Omega} a |u - \xi^u|^2 dx dy = \int_{\Omega} a \left| \sum_{j=1}^N \phi_j (u - \phi_j) \right|^2 dx dy. \quad (3.14)$$

Using the fact that any $x \in \Omega$ is in at most κ subdomains ω_j we see that the sum $\sum_{j=1}^N \phi_j (u - \xi_j^u)$ has at most κ terms for any $(x, y) \in \Omega$. Hence, using the Schwartz inequality, we have

$$\left| \sum_{j=1}^N \phi_j (u - \xi_j^u) \right|^2 \leq \kappa \sum_{j=1}^N |\phi_j (u - \xi_j^u)|^2.$$

Thus, using (3.3) and (3.9) in (3.14), we have

$$\begin{aligned} \|u - \xi^u\|_{L_a^2(\Omega)}^2 &\leq \kappa \int_{\Omega} a \sum_{j=1}^N |\phi_j (u - \xi_j^u)|^2 dx dy \\ &\leq \kappa C_1^2 \sum_{j=1}^N \int_{\omega_j} a |(u - \xi_j^u)|^2 dx dy \\ &= \kappa C_1^2 \sum_{j=1}^N \epsilon_1^2(j), \end{aligned} \quad (3.15)$$

which is (3.12).

Now we turn to the proof of (3.13), which is similar. Proceeding as above, we have

$$\begin{aligned}
\|u - \xi^u\|_{\mathcal{E}}^2 &= \int_{\Omega} a |\nabla(u - \xi^u)|^2 dx dy \\
&= \int_{\Omega} a \left| \nabla \sum_{j=1}^N \phi_j (u - \xi_j^u) \right|^2 dx dy \\
&= \int_{\Omega} a \left| \sum_{j=1}^N [(u - \xi_j^u) \nabla \phi_j + \phi_j \nabla(u - \xi_j^u)] \right|^2 dx dy \\
&\leq 2 \int_{\Omega} a \left(\sum_{j=1}^N (u - \xi_j^u) \nabla \phi_j \right)^2 dx dy + 2 \int_{\Omega} a \left(\sum_{j=1}^N \phi_j \nabla(u - \xi_j^u) \right)^2 dx dy \\
&\leq 2\kappa \int_{\omega_j} a \sum_{j=1}^N |(u - \xi_j^u) \nabla \phi_j|^2 dx dy + 2\kappa \int_{\omega_j} a \sum_{j=1}^N |\phi_j \nabla(u - \xi_j^u)|^2 dx dy.
\end{aligned}$$

Hence, using (3.9) and (3.10), we obtain

$$\|u - \xi^u\|_{\mathcal{E}} \leq 2\kappa \left(C_2^2 \sum_{j=1}^N \frac{\epsilon_1^2(j)}{\text{diam}^2(\omega_j)} + C_1^2 \sum_{j=1}^N \epsilon_2^2(j) \right), \quad (3.16)$$

which is (3.13). \square

Since $\epsilon_2(j)$ is usually proportional to $\epsilon_1(j)/\text{diam}(\omega_j)$, the terms in (3.13) are in some sense balanced. The next theorem gives sufficient conditions to ensure this balance.

Theorem 3.3 *Suppose $u \in \mathcal{E}_{\Gamma_1}$. Suppose the patches $\{\omega_j\}$ and the local approximation spaces $\{V_j\}$ satisfy the following assumptions:*

(a) *For all j for which $\bar{\omega}_j \cap \Gamma_1 = \emptyset$, V_j contains the constant functions, and*

$$\|v\|_{L_a^2(\omega_j)} \leq C_3 \text{diam}(\omega_j) \|v\|_{\mathcal{E}(\omega_j)}, \text{ for all } v \in \mathcal{E}(\omega_j) \text{ satisfying } \int_{\omega_j} a v dx dy = 0, \quad (3.17)$$

i.e., for all v with weighted a -average over ω_j equal to 0;

(b) *For all j for which $|\bar{\omega}_j \cap \Gamma_1| > 0$,*

$$\|v\|_{L_a^2(\omega_j)} \leq C_4 \text{diam}(\omega_j) \|v\|_{\mathcal{E}(\omega_j)}, \text{ for all } v \in \mathcal{E}(\omega_j) \text{ with } v|_{\bar{\omega}_j \cap \Gamma_1} = 0. \quad (3.18)$$

(Note that we require C_3 and C_4 to be independent of j). Then there exists $\xi_j^u \in V_j$ so that the corresponding global approximation,

$$\tilde{\xi}^u = \sum_{j=1}^N \phi_j \tilde{\xi}_j^u, \quad (3.19)$$

satisfies

$$\|u - \tilde{\xi}^u\|_{L_a^2(\Omega)} \leq C_5 \left(\sum_{j=1}^N \text{diam}^2(\omega_j) \epsilon_2^2(j) \right)^{1/2}, \quad (3.20)$$

where $C_5 = \sqrt{\kappa} C_1 (C_3^2 + C_4^2)^{1/2}$, and

$$\|u - \tilde{\xi}^u\|_{\mathcal{E}} \leq C_6 \left(\sum_{j=1}^N \epsilon_2^2(j) \right)^{1/2}, \quad (3.21)$$

where $C_6 = \{2\kappa (C_1^2 + C_2^2 (C_3^2 + C_4^2))\}^{1/2}$.

Remark 3.2. Estimates (3.17) and (3.18) are Poincaré inequalities. In Remarks 3.4 and 3.5, we give simple geometric conditions on the patches ω_j that imply (3.17) and (3.18) hold uniformly in j . Specifically, we bound C_3 and C_4 in term of simple geometric data.

Proof. Let ξ_j^u satisfy (3.9) and (3.10). We divide the index set $A = \{1, \dots, N\}$ into two disjoint sets:

$$A_{int} = \{j : 1 \leq j \leq N, \bar{\omega}_j \cap \Gamma_1 = \emptyset\}$$

and

$$A_{bd} = \{j : 1 \leq j \leq N, \bar{\omega}_j \cap \Gamma_1 \neq \emptyset\}.$$

For $j \in A_{int}$, let $\tilde{\xi}_j^u = \xi_j^u + r_j$, where r_j is a constant chosen so that $u - \tilde{\xi}_j^u$ has zero a -average on ω_j . By assumption (a), $\tilde{\xi}_j^u \in V_j$. Then, using (3.17) with $v = u - \tilde{\xi}_j^u$ and noting that $\nabla(u - \tilde{\xi}_j^u) = \nabla(u - \xi_j^u)$, from (3.10) we have

$$\begin{aligned} \|u - \tilde{\xi}_j^u\|_{L_a^2(\omega_j)}^2 &\leq C_3^2 \text{diam}^2(\omega_j) \int_{\omega_j} a |\nabla(u - \tilde{\xi}_j^u)|^2 dx dy \\ &= C_3^2 \text{diam}^2(\omega_j) \int_{\omega_j} a |\nabla(u - \xi_j^u)|^2 dx dy \\ &\leq C_3^2 \text{diam}^2(\omega_j) \epsilon_2^2(j). \end{aligned} \quad (3.22)$$

We also have

$$\|u - \tilde{\xi}_j^u\|_{\mathcal{E}(\omega_j)}^2 = \int_{\omega_j} a |\nabla(u - \xi_j^u)|^2 dx dy \leq \epsilon_2^2(j). \quad (3.23)$$

For $j \in A_{bd}$, let $\tilde{\xi}_j^u = \xi_j^u$. Now $u|_{\bar{\omega}_j \cap \Gamma_1} = 0$, and we know that $\tilde{\xi}_j^u|_{\bar{\omega}_j \cap \Gamma_1} = 0$. Thus, using (3.18), with $v = u - \tilde{\xi}_j^u$, and (3.10), we have

$$\|u - \tilde{\xi}_j^u\|_{L_a^2(\omega_j)} = \|u - \xi_j^u\|_{L_a^2(\omega_j)} \leq C_4 \text{diam}(\omega_j) \|u - \xi_j^u\|_{\mathcal{E}(\omega_j)} \leq C_4 \text{diam}(\omega_j) \epsilon_2(j). \quad (3.24)$$

Also,

$$\|u - \tilde{\xi}_j^u\|_{\mathcal{E}(\omega_j)}^2 \leq \epsilon_2^2(j). \quad (3.25)$$

Following the steps leading to (3.15) in the proof of Theorem 3.2, and using (3.22) and (3.24), we get

$$\begin{aligned}
\|u - \tilde{\xi}^u\|_{L_a^2(\Omega)}^2 &\leq \kappa C_1^2 \sum_{j \in A} \|u - \tilde{\xi}_j^u\|_{L_a^2(\omega_j)}^2 \\
&= \kappa C_1^2 \left\{ \sum_{j \in A_{int}} \|u - \tilde{\xi}_j^u\|_{L_a^2(\omega_j)}^2 + \sum_{j \in A_{bd}} \|u - \tilde{\xi}_j^u\|_{L_a^2(\omega_j)}^2 \right\} \\
&\leq \kappa C_1^2 (C_3^2 + C_4^2) \sum_{j \in A} \text{diam}^2(\omega_j) \epsilon_2^2(j), \tag{3.26}
\end{aligned}$$

which is (3.20) with $C_5 = \sqrt{\kappa} C_1 (C_3^2 + C_4^2)^{1/2}$. Similarly, following the steps leading to (3.16) in the proof of Theorem 3.2, and using (3.22)–(3.25) we obtain

$$\|u - \tilde{\xi}^u\|_{\mathcal{E}}^2 \leq 2\kappa (C_1^2 + C_2^2 (C_3^2 + C_4^2)) \sum_{j \in A} \epsilon_2^2(j), \tag{3.27}$$

which is (3.21) with $C_6 = \sqrt{2\kappa} (C_1^2 + C_2^2 (C_3^2 + C_4^2))^{1/2}$. \square

The idea of GFEM, in particular the use of a partition of unity and local shape functions, was first introduced in [5]. A result similar to Theorems 3.2 and 3.3 was proved in that paper. The GFEM was further developed in [6, 28]. Our presentation of Theorems 3.2 and 3.3 closely follows [6, 28].

Remark 3.3. Theorem 3.3 establishes (2.12), the second goal discussed in Section 2.

Remark 3.4. Suppose each ω_j is convex, $d_j = \text{diam}(\omega_j)$, and ω_j contains a ball of diameter $\tilde{d}_j \geq \frac{d_j}{\kappa_1}$, with κ_1 independent of j . Then

$$C_3 \leq 2\kappa_1 \left(\frac{\beta}{\alpha} \right)^{3/2}, \tag{3.28}$$

where C_3 is the Poincaré constant in (3.17). This follows directly from Theorem 8.1 in the Appendix (Section 8).

Remark 3.5. Suppose $\bar{\omega}_j \cap \Gamma_1$ is an arc. Let $S_{\bar{\omega}_j \cap \Gamma_1}(x)$ be the sector subtending this arc, and let $\gamma_{\bar{\omega}_j \cap \Gamma_1}$ be the angle of this sector. Suppose each ω_j is convex, $d_j = \text{diam}(\omega_j)$, and suppose $\tilde{\omega}_j$ is a disk of diameter $\tilde{d}_j \geq \frac{d_j}{\kappa_2}$, whose closure lies in ω_j . Assume

$$\gamma_{\bar{\omega}_j \cap \Gamma_1}(x) \geq \gamma_0, \text{ for all } x \in \tilde{\omega}_j, j = 1, 2, \dots, N.$$

Then

$$C_4 \leq \left\{ \left(\frac{\beta}{\alpha} \right)^{3/2} 2\kappa_1 + \left(\frac{\beta}{\alpha} \right) \frac{\kappa_2 \pi}{\gamma_0} \right\}, \tag{3.29}$$

where C_4 is the Poincaré constant in (3.18). This follows directly from Theorem 8.2 in the Appendix (Section 8).

Remark 3.6. In Theorems 3.1 and 3.2 we have imposed only minimal conditions on the patch ω_j . In Theorems 3.3 we imposed additional conditions. We note, however, that the conditions on the ω_j can be considerably relaxed. The ω_j can, in particular, be multiply connected. The condition that $\bar{\omega}_j \cap \Gamma_1$ is an arc can be relaxed; in particular, it can be a disconnected set (see Remark 8.1).

We return now to the GFEM. Suppose the hypotheses of Theorem 3.3 are satisfied, and suppose u is the solution of (2.6). It follows from (2.10), with $\xi = \xi^u$, and (3.21) that

$$\|u - u_{GFEM}\|_{\mathcal{E}(\Omega)} \leq C \|u - \tilde{\xi}_j^u\|_{\mathcal{E}(\Omega)} \leq C \left(\sum \epsilon_2^2(j) \right)^{1/2}, \quad (3.30)$$

which is the main error estimate for the GFEM. It will be useful to state this estimate in the following alternate form:

$$\|u - u_{GFEM}\|_{\mathcal{E}(\Omega)} \leq C \inf_{\xi_j^u \in V_j} \left(\sum \|u - \xi_j^u\|_{\mathcal{E}(\omega_j)} \right)^{1/2}. \quad (3.31)$$

We can write u^{GFEM} as

$$u_{GFEM} = \sum_{j=1}^N \sum_{i=1}^{m(j)} c_{ji} \eta_{ji}, \quad (3.32)$$

where $\mathbf{c} = \{c_{ji}\}$ is the solution of the linear system (see (2.8))

$$\sum_{j=1}^N \sum_{i=1}^{m(j)} B(\eta_k, \eta_{ji}) c_{ji} = F(\eta_k), \quad 1 \leq k \leq m(l), 1 \leq l \leq N,$$

or

$$A\mathbf{c} = \mathbf{F}, \quad (3.33)$$

where A is the stiffness matrix, whose elements are

$$A(l, k; j, i) = B(\eta_k, \eta_{ji}) = \int_{\omega_j \cap \omega_l} \nabla \eta_k \cdot \nabla \eta_{ji} \, dx, \quad (3.34)$$

and \mathbf{F} is the load vector, whose components are

$$F(l; k) = \int_{\omega_l} f \eta_k \, dx + \int_{\Gamma_1 \cap \bar{\omega}_l} g \eta_k \, ds. \quad (3.35)$$

The GFEM is a very general method. We show in the next section that it is an umbrella covering many standard FEMs, hence the name *Generalized FEM*. Using polynomial functions together with other special functions we get the XFEM (see [36, 41]), which is a special case of the GFEM. The specific selections of ϕ_j and V_j lead to the methods referred to in the literature by different names.

Remark 3.6. We have addressed only second order boundary value problems. In an analogous way the GFEM can be used to approximate the solutions of $2m$ th order boundary value problems, where the bilinear form includes derivatives of orders up to m . Instead of (3.4) we would assume

$$\max_{(x,y) \in \Omega} |D^\alpha \phi_j(x,y)| \leq \frac{C_2}{(\text{diam } \omega_j)^m},$$

where $\alpha = (l,k), l,k \geq 0, k+l = m$. In addition, we have to assume that ϕ_j has piecewise continuous derivatives of orders up to m on Ω , and that ϕ_j and its normal derivatives of orders up to m are 0 on $\bar{\omega}_j \cap \Gamma_1$.

4 Relation Between GFEM and Classical FEM

The GFEM is based on the generalization of the idea of classical FEMs. We will illustrate this by showing that certain classical FEMs can be cast in the framework of a GFEM by appropriately choosing the partition of unity functions $\{\phi_j\}$ and the local approximation spaces $\{V_j\}$. We will also comment on the linear system obtained from the GFEM, and will examine Theorems 3.2 and 3.3 in the context of a classical FEM that can be viewed as a GFEM.

Example 1: The classical FEM in 1-d, based on continuous, piecewise polynomials of degree k , is same as the a suitably chosen GFEM. We show this here for $k = 2$ by proving that the finite dimensional approximating space used in this GFEM is same as the classical FEM space.

Suppose $\Omega = I = (0,1)$, and for a fixed positive integer N , let $x_j = jh$, $0 \leq j \leq N$, with $h = 1/N$, be uniformly distributed nodes in I . We consider the “triangulation” of I by the intervals $I_j = (x_j, x_{j+1})$. The standard FEM space, relative to this triangulation, is given by

$$S^{FEM} = \{v \in C(0,1) : v|_{I_j} \in \mathcal{P}^k(I_j), j = 0, 1, \dots, N-1\}. \quad (4.1)$$

We construct a GFEM space as follows: To each node x_j , we associate a function ϕ_j , which is the usual piecewise linear continuous hat functions centered at x_j such that $\phi_j(x_i) = \delta_{ji}$. We let $\bar{\omega}_j \equiv \text{supp } \phi_j = [x_{j-1}, x_{j+1}]$, $1 \leq j \leq N-1$. For $j = 0, N$, we define $\bar{\omega}_0 = \text{supp } \phi_0 = [x_0, x_1]$ and $\bar{\omega}_N = \text{supp } \phi_N = [x_{N-1}, x_N]$. We recall that the sets ω_j 's were introduced in Section 2. Clearly, $\{\phi_j\}_{j=0}^N$ form a partition of unity in I and satisfy (3.1)–(3.4). For the local approximation spaces V_j , $0 \leq j \leq N$, we consider

$$V_j = \text{span}\{1, x - x_j\}.$$

We then define the GFEM space as

$$S^{GFEM} = \{\psi : \psi = \sum_{j=0}^N \phi_j(x) l_j(x)\}, \quad (4.2)$$

where

$$l_j(x) \equiv \alpha_j + \beta_j(x - x_j) \in V_j, \quad \alpha_j, \beta_j \in \mathbb{R}.$$

The functions $l_j \in V_j$ are only defined in ω_j , but since $\phi_j(x) = 0$ at x_{j-1} and x_j , $\phi_j(x)l_j(x)$ has a natural continuous zero-extension to I . We will show that $S^{FEM} = S^{GFEM}$.

Since the functions $\phi_j(x)l_j(x)$ are continuous on I , it is clear that functions in S^{GFEM} are also continuous on I . Also, since ϕ_j and l_j are piecewise linear, it is clear that every $\psi \in S^{GFEM}$ is a C^0 , piecewise quadratic function. Thus $S^{GFEM} \subset S^{FEM}$. We next show that $S^{FEM} \subset S^{GFEM}$, *i.e.*, for a given $q(x) \in S^{FEM}$, we can find constants α_i, β_i , and hence $l_i(x)$ for $0 \leq i \leq N$ such that

$$q(x) = \sum_{i=0}^N \phi_i(x)l_i(x), \quad x \in I. \quad (4.3)$$

We first note that equality of $q(x)$ and $\sum_{i=0}^N \phi_i(x)l_i(x)$ at the nodes x_j , $0 \leq j \leq N$, implies

$$q(x_j) = \sum_{i=0}^N \phi_i(x_j)l_i(x_j) = \phi_j(x_j)l_j(x_j) = \alpha_j. \quad (4.4)$$

We now consider the function $\sum_{i=0}^N \phi_i(x)l_i(x)$ with these α_i 's. Since $q(x)$ and $\sum_{i=0}^N \phi_i(x)l_i(x)$ are both continuous, have same values at nodes x_j , $0 \leq j \leq N$, and their restrictions to the I_j s are quadratics, they will be equal for all $x \in I$ if they are equal at the mid points of I_j s, *i.e.*,

$$q(x_{j+1/2}) = \sum_{i=0}^N \phi_i(x_{j+1/2})l_i(x_{j+1/2}), \quad 0 \leq j \leq N-1,$$

where $x_{j+1/2} \equiv x_j + h/2$. Imposing these conditions yields

$$\begin{aligned} q(x_{j+1/2}) &= \sum_{i=0}^N \phi_i(x_{j+1/2})l_i(x_{j+1/2}) \\ &= \phi_j(x_{j+1/2})l_j(x_{j+1/2}) + \phi_{j+1}(x_{j+1/2})l_{j+1}(x_{j+1/2}) \\ &= \frac{1}{2} [\alpha_j + \beta_j(x_{j+1/2} - x_j) + \alpha_{j+1} + \beta_{j+1}(x_{j+1/2} - x_{j+1})] \\ &= \frac{1}{2} \left[\alpha_j + \alpha_{j+1} + \frac{h}{2}(\beta_j - \beta_{j+1}) \right], \end{aligned}$$

which can be written as

$$\beta_j - \beta_{j+1} = [2q(x_{j+1/2}) - (\alpha_j + \alpha_{j+1})] \frac{2}{h}, \quad 0 \leq j \leq N-1. \quad (4.5)$$

For an arbitrarily given value of β_0 , we can solve for β_i , $1 \leq i \leq N$ uniquely in terms of β_0 . Using these β_i 's and the α_i 's as given in (4.4), we have constructed

$l_i(x)$, $0 \leq i \leq N$ such that (4.3) is satisfied. Thus $S^{FEM} \subset S^{GFEM}$, and using the fact that $S^{GFEM} \subset S^{FEM}$ (shown above), we have $S^{GFEM} = S^{FEM}$.

It is well known that for $k = 2$, a basis of S^{FEM} consists of nodal hat functions $\phi_i(x)$, $0 \leq i \leq N$, and the *quadratic bubble functions*, $B_i(x)$, $0 \leq i \leq N - 1$, given by

$$B_i(x) = \begin{cases} \frac{1}{h^2} (x - x_i)(x_{i+1} - x), & x_i \leq x \leq x_{i+1}; \\ 0, & \text{otherwise.} \end{cases} \quad (4.6)$$

It will be useful later in this section to have an expression for $B_i(x)$ of the form (4.3). From (4.4) with $q(x) = B_i(x)$, it is clear that

$$\alpha_j = B_i(x_j) = 0, \quad 0 \leq j \leq N. \quad (4.7)$$

Also, since

$$B_i(x_{j+1/2}) = \begin{cases} \frac{1}{4}, & j = i \\ 0, & j \neq i, \end{cases}$$

from (4.5), with $q(x) = B_i(x)$, we have

$$\beta_j - \beta_{j+1} = \begin{cases} \frac{1}{h}, & j = i \\ 0, & j \neq i, \end{cases}$$

We can solve this system uniquely in terms of β_0 . If we take $\beta_0 = 1/h$, the solution of this system is

$$\beta_j = \begin{cases} \beta_0 = \frac{1}{h}, & 1 \leq j \leq i, \\ 0, & i + 1 \leq j \leq N. \end{cases} \quad (4.8)$$

Thus using (4.7) and (4.8) in (4.3), we get

$$B_i(x) = \frac{1}{h} \sum_{j=0}^i \phi_j(x) (x - x_j). \quad (4.9)$$

The above expression for $B_i(x)$ is of the form (4.3) and thus $B_i(x)$ is a linear combination of the shape functions η_{jk} of S^{GFEM} .

Remark 4.1. We recall from Section 3 that the functions in the local approximation space V_j , for j for which $\bar{\omega}_j \cap \Gamma_1 \neq \emptyset$, must satisfy the Dirichlet boundary condition on $\bar{\omega}_j \cap \Gamma_1$. In this 1-d setting, if the exact solution u of a BVP satisfies the boundary condition $u(0) = 0$ at $x = 0$, we take $\alpha_0 = 0$ and

$$V_0 = \text{span} \{x\};$$

the functions in V_0 satisfy the boundary condition at $x = 0$. Likewise, if $u(1) = 0$ is the specified boundary condition at $x = 1$, we take $\alpha_N = 0$ and

$$V_N = \text{span} \{x - 1\};$$

the functions in V_N satisfy the boundary condition at $x = 1$. A minor modification of the above analysis shows that $S^{GFEM} = S^{FEM}$ in this case also.

Example 2: Consider the domain $\Omega = (0, 1)^2$ and for a fixed positive integer N , let $x_i = ih$, $y_j = jh$, where $h = 1/N$ and $0 \leq i, j \leq N$. We consider a “triangulation” of Ω by the squares $\Omega_{i,j} \equiv (x_i, x_{i+1}) \times (y_j, y_{j+1})$, $0 \leq i, j \leq N-1$. The nodes of this triangulation are $A_{i,j} \equiv (x_i, y_j)$, $0 \leq i, j \leq N$. A standard FEM space with respect to this triangulation of Ω is

$$S^{FEM} = \{v \in C^0(\Omega) : v|_{\Omega_{i,j}} \in Q^k(\Omega_{i,j})\}, \quad (4.10)$$

where $Q^k(\Omega_{i,j}) = \text{span} \{x^l y^m\}_{l,m=0}^k$, *i.e.*, the space of polynomials of degree $\leq k$ in each variable. It is possible to find a GFEM space, S^{GFEM} , with suitably chosen partition of unity functions $\{\phi_{i,j}(x, y)\}$ and local approximation spaces $V_{i,j}$, so that $S^{GFEM} = S^{FEM}$. We again do this for $k = 2$. For $k = 2$, the functions in S^{FEM} are C^0 piecewise biquadratics. We construct a GFEM space as follows: To each node $A_{i,j}$, we associate a function

$$\phi_{i,j}(x, y) \equiv \phi_i(x)\phi_j(y), \quad (4.11)$$

where $\phi_i(x)$ and $\phi_j(y)$ are one dimensional hat functions centered at x_i and y_j respectively, as discussed in Example 1. $\phi_{i,j}$ is the standard piecewise bilinear hat function centered at $A_{i,j}$ satisfying $\phi_{i,j}(A_{i,j}) = 1$ and $\phi_{i,j}$ is zero at every other node. We let $\bar{\omega}_{i,j} \equiv \text{supp } \phi_{i,j} = [x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]$. We note that when $i = 0$ or $j = 0$, we replace x_{i-1} by x_i or y_{j-1} by y_i , accordingly, in the definition of $\omega_{i,j}$. Similarly, when $i = N$ or $j = N$, we replace x_{i+1} by x_i or y_{j+1} by y_i , accordingly. Then $\{\phi_{i,j}\}$ satisfy (3.1)–(3.4), in particular, they are a partition of unity on Ω . For local approximation spaces $V_{i,j}$, $0 \leq i, j \leq N$, we take

$$V_{i,j} = \text{span} \{(x - x_i)^l (y - y_j)^m, l = 0, 1, m = 0, 1\}.$$

Thus $V_{i,j}$ is the space of all bilinear functions defined on $\omega_{i,j}$. We now define the GFEM space as

$$S^{GFEM} = \{\psi : \psi(x, y) = \sum_{i,j=0}^N \phi_{i,j}(x, y) l_{i,j}(x, y)\}, \quad (4.12)$$

where

$$l_{i,j}(x, y) = a_{ij} + b_{ij}(x - x_i) + c_{ij}(y - y_j) + d_{ij}(x - x_i)(y - y_j), \quad a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}.$$

We note that $l_{i,j}$ is defined only on $\omega_{i,j}$, but since $\phi_{i,j}|_{\partial\omega_{i,j}} = 0$, $\phi_{i,j}l_{i,j}$ has a natural continuous extension to Ω . Thus S^{GFEM} is equivalently given by

$$S^{GFEM} = \text{span} \{\phi_{i,j}, (x - x_i)\phi_{i,j}, (y - y_j)\phi_{i,j}, (x - x_i)(y - y_j)\phi_{i,j}\}_{i,j=0}^N. \quad (4.13)$$

We now show that $S^{FEM} = S^{GFEM}$. Since the functions $\phi_{i,j}l_{i,j}$ are continuous in Ω , it is clear from (4.12) that the functions in S^{GFEM} are continuous in Ω . Also since $\phi_{i,j}, l_{i,j}$ are bilinear on each rectangle of the triangulation, the functions in S^{GFEM} are C^0 piecewise biquadratic functions, and hence $S^{GFEM} \subset S^{FEM}$. It remains to show that $S^{FEM} \subset S^{GFEM}$.

We will do this by proving that every element of a basis of S^{FEM} is contained in S^{GFEM} . For $k = 2$, a well-known basis of S^{FEM} consists of the following functions, which can be grouped into four categories:

- (a) The hat functions $\phi_{i,j}(x, y)$ corresponding to the nodes $A_{i,j}$, $0 \leq i, j \leq N$.
- (b) The functions $S_{i,j}^{(1)}(x, y)$ corresponding to the line segments $(A_{i,j}, A_{i+1,j})$, $0 \leq i \leq N-1, 0 \leq j \leq N$, defined by

$$S_{i,j}^{(1)}(x, y) = B_i(x)\phi_j(y). \quad (4.14)$$

Here, $B_i(x)$ is the one dimensional quadratic bubble defined in (4.6). We note that, for $1 \leq j$, $\text{supp } S_{i,j}^{(1)} = [x_i, x_{i+1}] \times [y_{j-1}, y_{j+1}]$. For $j = 0$, the support is $[x_i, x_{i+1}] \times [y_0, y_1]$.

- (c) The functions $S_{i,j}^{(2)}(x, y)$ corresponding to the line segments $(A_{i,j}, A_{i,j+1})$, $0 \leq i \leq N, 0 \leq j \leq N-1$, defined by

$$S_{i,j}^{(2)}(x, y) = \phi_i(x)B_j(y). \quad (4.15)$$

We note that, for $1 \leq i$, $\text{supp } S_{i,j}^{(2)} = [x_{i-1}, x_{i+1}] \times [y_i, y_{i+1}]$. For $i = 0$, the support is $[x_0, x_1] \times [y_i, y_{i+1}]$.

- (d) The functions $B_{i,j}(x, y)$, corresponding to the rectangles $\Omega_{i,j}$, $0 \leq i, j \leq N-1$, defined by

$$B_{i,j}(x, y) = B_i(x)B_j(y). \quad (4.16)$$

We note that $\text{supp } B_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$.

It is immediate from (4.13) that $\phi_{i,j} \in S^{GFEM}$ for $0 \leq i, j \leq N$. Using (4.14), (4.9), and (4.11), we have

$$\begin{aligned} S_{i,j}^{(1)}(x, y) &= B_i(x)\phi_j(y) \\ &= \frac{1}{h} \sum_{l=0}^i \phi_l(x)(x - x_l)\phi_j(y) \\ &= \frac{1}{h} \sum_{l=0}^i (x - x_l)\phi_{l,j}(x, y), \end{aligned}$$

and therefore from (4.13), we have

$$S_{i,j}^{(1)} \in S^{GFEM}, \quad \text{for } 0 \leq i \leq N-1 \text{ and } 0 \leq j \leq N.$$

Similarly, using (4.15), (4.9), (4.11), and (4.13), we have

$$S_{i,j}^{(2)}(x, y) = \frac{1}{h} \sum_{l=0}^j (y - y_l) \phi_{i,l}(x, y) \in S^{GFEM},$$

for $0 \leq i \leq N$ and $0 \leq j \leq N - 1$. Finally, from (4.16), (4.9), (4.11), and (4.13), we have

$$\begin{aligned} B_{i,j}(x, y) &= B_i(x)B_j(y) \\ &= \left[\frac{1}{h} \sum_{l=0}^i \phi_l(x)(x - x_l) \right] \left[\frac{1}{h} \sum_{m=0}^j \phi_m(y)(y - y_m) \right] \\ &= \sum_{l=0}^i \sum_{m=0}^j \phi_l(x)\phi_m(y) \left[\frac{1}{h^2} (x - x_l)(y - y_m) \right] \\ &= \frac{1}{h^2} \sum_{l=0}^i \sum_{m=0}^j (x - x_l)(y - y_m) \phi_{l,m}(x, y) \in S^{GFEM}, \end{aligned}$$

for $0 \leq i \leq N - 1$ and $0 \leq j \leq N - 1$. Thus we have shown that all the basis elements for S^{FEM} belong to S^{GFEM} . Therefore, $S^{FEM} = S^{GFEM}$.

Remark 4.2. We note that the local approximation $V_{i,j}$ in Example 2, for the indices i, j where $\bar{\omega}_{i,j} \cap \Gamma_1 \neq \emptyset$, can be chosen such that all $l_{i,j}(x, y) \in V_{i,j}$ satisfy the Dirichlet boundary condition on $\bar{\omega}_{i,j} \cap \Gamma_1$, i.e., $l_{i,j}(x, y) = 0$ for $(x, y) \in \bar{\omega}_{i,j} \cap \Gamma_1$, and $V_{i,j}$ do not contain constant functions for these indices i and j . Moreover, $S^{GFEM} = S^{FEM}$ for any k in (4.10), and thus, in this example (also in Example 1), the GFEM spaces are same as the FEM spaces corresponding to the h - as well as p - version of FEM. We further note that for any polygonal domain Ω and for any triangulation of Ω , the classical FEM space of C^0 piecewise linear polynomials, can be viewed as a GFEM space with standard hat functions serving as the partition of unity functions, and where the local approximation spaces contain only constant functions.

Through Examples 1 and 2, we have shown that certain classical FEMs can be cast in the framework of a GFEM. But we do not claim that, for any domain Ω , every FEM relative to every triangulation of Ω can be cast in this framework. Our main reason for presenting these examples is to illustrate that the idea of GFEM is a generalization of the idea of the FEM.

The framework of a GFEM offers more freedom in choosing shape functions with relatively simpler supports, when compared to classical FEMs. A FEM uses a triangulation of the domain Ω , or a mesh, to construct piecewise polynomial approximating functions. The supports of the shape functions (used in FEMs) are union of “triangles” relative to the triangulation or the mesh. But for domains Ω in 3-d, with complicated geometry (e.g., domains with voids and cracks), it is quite difficult to generate a good mesh on Ω . One of the important

aspects of the GFEM is that it permits the use of partition of unity functions (in contrast to those used in Examples 1 and 2), whose supports may not depend on any mesh (*e.g.*, Shepard functions discussed in Section 2), or may depend on a simple mesh that does not conform to the geometry of Ω (see [37]). In this sense, the GFEM is also a meshless method (see [4]) and this feature allows us to avoid the use of a sophisticated mesh generator. We mention, in particular, that for the partition of unity functions for a GFEM, we may use one the particle shape functions, *e.g.*, RKP shape functions (see [26]), used in meshless methods. Another important aspect of GFEM is that local approximation spaces can have functions other than polynomials (in contrast to the $V_{i,j}$ used in Example 2), which locally approximate the unknown solution of (2.1) well. Thus the shape functions in a GFEM need not be piecewise polynomials (in contrast to classical FEM), and the approximating functions can be tailored to approximate the unknown solution well.

The shape functions of S^{GFEM} may be linearly dependent giving rise to a singular linear system (3.33). This can be easily seen in Example 1 ($k = 2$), where there are $2(N + 1)$ shape functions in S^{GFEM} , given by $\eta_{ij} = \phi_i(x)(x - x_i)^j$, $j = 0, 1$, $0 \leq i \leq N$. But the dimension of S^{FEM} in (4.1) with $k = 2$ is $2N + 1$, and since $S^{FEM} = S^{GFEM}$, we have

$$\dim S^{GFEM} = \dim S^{FEM} = 2N + 1 < 2(N + 1).$$

Thus the number of shape functions in S^{GFEM} is greater than its dimensions; the shape functions $\{\eta_{ij}; j = 0, 1\}_{i=0}^N$ must be linearly dependent. Similar conclusion is also true for the shape functions of S^{GFEM} , given by (4.10), in Example 2 (also see [38]). There are other situations in which the shape functions of S^{GFEM} are linearly independent, *e.g.*, with another choice of partition of unity functions as shown in [28, 34]. But the shape functions could be “almost linearly dependent” giving rise to a severely ill-conditioned linear system. We will discuss the solution of singular or ill- conditioned linear system, obtained from GFEM, in Section 6.

Finally we comment on Theorems 3.2 and 3.3 in Section 3, in the context of the FEM, when the FEM space can also be viewed as a GFEM space. These theorems are fundamental approximation results for GFEM. In the examples presented in this section, we have seen that $S^{GFEM} = S^{FEM}$, but application of these theorems on S^{GFEM} does not yield the well known error estimates for the FEM.

In Example 1, the FEM approximating space S^{FEM} ((4.1) with $k = 2$) is the space of C^0 piecewise quadratic polynomials. It is well known that

$$\|u - u_{FEM}\|_{\mathcal{E}(\Omega)} \leq Ch^2 \|u\|_{H^3(\Omega)}, \quad (4.17)$$

where u_{FEM} is the FEM solution relative to S^{FEM} . Here u is the smooth (in $H^3(\Omega)$) solution of an elliptic linear Dirichlet BVP posed on $\Omega = I = (0, 1)$ with $u(0) = u(1) = 0$. Since in this example, $S^{FEM} = S^{GFEM}$, we can use Theorem 3.2 or 3.3 to obtain an error estimate. Towards this end, we choose $\xi_j^u \in V_j$,

$0 \leq j \leq N$, such that

$$\|u - \xi_j^u\|_{\mathcal{E}(\omega_j)} \leq Ch\|u\|_{H^2(\omega_j)} \equiv \epsilon_2(j). \quad (4.18)$$

Recall that $V_j = \text{span}\{1, (x - x_j)\}$, $1 \leq j \leq N - 1$, $V_0 = \text{span}\{x\}$, and $V_N = \text{span}\{(x - 1)\}$. Let $\xi^u \equiv \sum_{j=0}^N \phi_j(x)\xi_j^u(x)$ as in (3.11). It is easy to check that (3.17) and (3.18) hold in this example, and thus from Theorem 3.3 and the above inequality, we get

$$\begin{aligned} \|u - \tilde{\xi}^u\|_{\mathcal{E}(\Omega)}^2 &\leq C \sum_{j=0}^N (\epsilon_2(j))^2 \\ &\leq Ch^2 \sum_{j=0}^N \|u\|_{H^2(\omega_j)}^2 \\ &\leq Ch^2 \|u\|_{H^2(\Omega)}^2. \end{aligned}$$

Thus, using (2.10) with $\xi = \tilde{\xi}^u$, we have

$$\begin{aligned} \|u - u_{GFEM}\|_{\mathcal{E}(\Omega)} &\leq \|u - \tilde{\xi}^u\|_{\mathcal{E}(\Omega)} \\ &\leq Ch\|u\|_{H^2(\Omega)}, \end{aligned} \quad (4.19)$$

where, u_{GFEM} is the solution of (2.7) with $S = S^{GFEM}$. We note that since $S^{FEM} = S^{GFEM}$, $u_{FEM} = u_{GFEM}$. But (4.19), which is based on Theorem 3.3, gives only $O(h)$, whereas the classical estimate (4.17) gives $O(h^2)$. Thus Theorem 3.3 does not give the correct order of convergence in this situation. The reason for this loss of a power of h in (4.19) can be explained as follows: The only assumptions on partition of unity functions $\{\phi_j\}$ are (3.1)–(3.4). It was not assumed that $\{\phi_j\}$ “reproduce” linear polynomials, *i.e.*, that $\sum_{j=0}^N x_j \phi_j(x) = x$, for $x \in I$. But the partition of unity functions $\{\phi_j\}$ used in Example 1 were hat functions, which do “reproduce” the linear polynomials, *i.e.*, $\sum_{j=0}^N x_j \phi_j(x) = x$, for $x \in I$. An approximation result for the GFEM, with partition of unity functions that are assumed to reproduce linear or higher degree polynomials, will be reported in a forthcoming paper. This result will yield an $O(h^2)$ error estimate for Example 1.

5 Selection of Local Approximation Spaces

As we have seen in Sections 2 and 3, the local approximation spaces play a central role in the GFEM. We discuss the selection of effective local approximation spaces in this section.

5.1 Selection of the spaces V_j using the available information on the solution u

As mentioned in Section 3, the selection of local approximation spaces V_j is governed by the available information on the exact solution u of Problem (2.1).

In this subsection we discuss some types of available information, and show how it can be used in the process of selecting V_j .

(a) The available information on u is in terms of the Sobolev spaces:

In this case we assume that the only available information on u is that it lies in $H^m(\omega_j)$ and

$$\|u\|_{H^m(\omega_j)} = \left(\int_{\omega_j} \sum_{|k| \leq m} (D^k u)^2 dx \right)^{1/2} \leq K_j^{(m)}, \quad m = 0, 1, \dots, j = 1, \dots, N, \quad (5.1)$$

where $k = (k_1, k_2)$, $k_i \geq 0$, and $|k| = k_1 + k_2$. We wish to select the spaces V_j so that

$$\sup_{u \in \mathcal{E}(\omega_j)} \inf_{\xi_j \in V_j} \|u - \xi_j\|_{\mathcal{E}(\omega_j)} \text{ is small.}$$

$$\|u\|_{H^m(\omega_j)} \leq K_j^{(m)}$$

In [3] we showed that if we know only (5.1), then the space of polynomials of degree $\leq p$ on ω_j is a good choice for V_j ; denote this space by $V_j = W_j^{(p)}$. Then, for $m \geq 1$,

$$\epsilon_2(j) \leq C K_j^{(m)} \frac{h_j^{\min(p, m-1)}}{p^{m-1}}, \quad (5.2)$$

where $h_j = \text{diam } \omega_j$ and C is independent of u, h, p , and m .

Remark 5.1. The estimate (5.2) is the best possible under the assumption that the only available information is (5.1).

Remark 5.2. From (5.2) and Theorem 3.2, specifically (3.13), we obtain an error estimate for u_{GFEM} . Comparing this estimate with the classical FEM estimate, we see that we lose one power of h . This is because we have assumed only that $\{\phi_j\}$ is a partition of unity, *i.e.*, that it reproduces constants, but possibly not linear functions (see Section 4).

(b) The available information on u is in terms the BVP:

So far we have assumed only that u is the solution of the BVP (2.1), *i.e.*, we know nothing other than that it satisfies (5.1). Often we know more. For example, if u is the solution of (2.1) with $a = 1$ and $f = 0$; *i.e.*, that

$$\begin{cases} \Delta u = 0, & \text{for } (x, y) \in \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_2, \end{cases} \quad (5.3)$$

then u is a harmonic function. Therefore, in this situation, we use harmonic polynomials, instead of all the polynomials in $W_j^{(p)}$. Let

$$\mathcal{H}W_j^{(p)} = \left\{ v \in W_j^{(p)} : v \text{ is harmonic on } \omega_j \right\},$$

the left superscript \mathcal{H} denoting harmonic. Suppose ω_j is star-shaped with respect to a ball and $\partial\omega_j$ is piecewise analytic with internal angles $\alpha_j = \beta_j\pi$, with

$0 \leq \beta_j < 2 - \lambda, \lambda > 0$. Then, with shape functions in $\mathcal{H}W_j^{(p)}$, it is known (see [29]) that

$$\epsilon_2(j) \leq CK_j^{(m)} h^{m-1} \left(\frac{\log p}{p} \right)^{(2-\lambda)(m-1)}, \quad p \geq m-1, \quad m \geq 1, \quad (5.4)$$

where $K_j^{(m)}$ is as in (5.1) and C is independent of u , but does depend on the shape of ω_j . We note that rate of convergence in p depends on the angle of corners of the boundary.

Remark 5.3. The dimension of $\mathcal{H}W_j^{(p)}$ is $2p+1$, whereas the dimension of $W_j^{(p)}$ is $\frac{(p+1)(p+2)}{2}$. Hence for a given asymptotic rate of convergence, the space of harmonic polynomials has a smaller number of degrees of freedom than the space of standard polynomials.

Remark 5.4. If the right-hand side is not zero, then we have to add additional shape functions. For example, if $f=1$, we add the shape function $\xi = x^2 + y^2$.

Remark 5.5. Because there is a known relation between the norm $\|u\|_{H^m(\omega_j)}$ of a harmonic function and its trace on $\partial\omega_j$, we can express (5.4) in terms of an appropriate norm of u on $\partial\omega_j$.

Remark 5.6. We have here addressed the selection of shape functions for the special form of the equation in (2.1), namely $\Delta u = 0$. V.I. Vekua ([42]) and I.N Bergman ([9]) have developed a theory of generalized harmonic polynomials for differential equations with analytic coefficients, *i.e.*, functions that are related to the differential equation as are harmonic polynomials related to Laplace's equation. For a discussion of generalized harmonic polynomials in connection with the equation

$$\Delta u + k^2 u = 0,$$

see [28].

Remark 5.7. Analogous results can be obtained for systems of PDEs, *e.g.* the elasticity equations, and higher order equations, *e.g.* the biharmonic equation.

5.2 Selection of the spaces V_j when ω_j has a complicated structure

In Section 5.1 we tacitly assumed that ω_j is simply connected. Assume now that ω_j has a "circular" hole centered at some point $(\bar{x}, \bar{y}) \in \Omega$ and consider the problem (5.3). Suppose

$$\Omega \supset \omega_j = \omega_j^{(1)} \setminus \omega_j^{(2)},$$

where

$$\omega_j^{(1)} = \{(x, y) : |x - \bar{x}| < h, |y - \bar{y}| < h\}$$

and

$$\omega_j^{(2)} = \{(x, y) : (x - \bar{x})^2 + (y - \bar{y})^2 < \delta^2, \text{ where } \delta < h\},$$

and assume that $\partial\omega_j^{(2)} \subset \Gamma_2$ and $g = 0$ on $\partial\omega_j^{(2)}$, *i.e.*, in (5.3) we have $\frac{\partial u}{\partial n} = 0$ on $\partial\omega_j^{(2)}$. We consider the functions

$$\begin{aligned}\xi_{j,l}^{(1)}(r, \theta) &= (r^l + r^{-l}\delta^{2l}) \sin l\theta, \quad l = 1, 2, \dots \\ \xi_{j,l}^{(2)}(r, \theta) &= (r^l + r^{-l}\delta^{2l}) \cos l\theta, \quad l = 0, 1, \dots,\end{aligned}\tag{5.5}$$

where (r, θ) are polar coordinates with respect to (\bar{x}, \bar{y}) . Clearly, $\xi_{j,l}^{(i)}, i = 1, 2$, are harmonic polynomials satisfying $\frac{\partial \xi_{j,l}^{(i)}}{\partial n} = 0$ on $\partial\omega_j^{(2)}$. Since u is harmonic in ω_j , it can be expanded in an infinite (Laurent) series in terms of the functions in (5.5):

$$u(r, \theta) = 2a_0 + \sum_{l=1}^{\infty} a_l \xi_{j,l}^{(2)}(r, \theta) + \sum_{l=1}^{\infty} b_l \xi_{j,l}^{(1)}(r, \theta).\tag{5.6}$$

Thus the functions in (5.5) can be used as shape functions and linear combinations of the first few functions in (5.5) provide accurate approximations to u . Because $\frac{\partial u}{\partial n} = 0$ is a natural boundary condition, which need not be explicitly imposed, we can use the functions

$$r^l \sin l\theta, r^{-l} \sin l\theta, r^l \cos l\theta, r^{-l} \cos l\theta.\tag{5.7}$$

The family (5.7) also provides accurate approximations to u on ω_j .

Remark 5.8. We have constructed the shape functions on the whole plane with one hole. If the domain is more complex, *e.g.*, has multiple holes as in a perforated domain, then the construction of the shape functions is more complicated. In these situations we can use (a) numerical construction; (b) analytical construction based on conformal mappings. With procedure (b) we utilize the facts that

1. Conformal mappings preserve the harmonicity of the functions; and
2. Conformal mappings preserve the H^1 -seminorm.

Now we can use mapped harmonic polynomials as the shape functions. For a discussion of conformal mappings, we refer to [22]. These special functions are the solutions of a boundary value problem on the domains ω_j or on a bigger domain $\tilde{\omega}_j \supset \omega_j$. We call these problems Handbook Problems because they are reminiscent of the handbook problems used in engineering. These problems (which are local) can be solved numerically by *e.g.*, GFEM. It is also possible to use certain analytic formulas similar to (5.6), determining numerically the parameters in the analytical form of these functions.

So far we have assumed that ω_j is a domain, *i.e.*, a connected set. In applications the GFEM is used for crack propagation problems. Then ω_j is “cut” by a line into two domains $\omega_j^{(1)}$ and $\omega_j^{(2)}$: $\omega_j = \omega_j^{(1)} \cup \omega_j^{(2)}$ and $\omega_j^{(1)} \cap \omega_j^{(2)} = \emptyset$. The exact solution u is smooth or possibly harmonic separately on $\omega_j^{(1)}$ and

$\omega_j^{(2)}$, but not on ω_j itself; u and its normal derivative are discontinuous across $\gamma = \partial\omega_j^{(1)} \cap \partial\omega_j^{(2)}$.

Here we have to create the space

$$V_j = V_j^{(p)} = \begin{cases} W_{j,1}^{(p)}, & \text{on } \omega_j^{(1)} \\ W_{j,2}^{(p)}, & \text{on } \omega_j^{(2)} \end{cases} .$$

so that there is a $\xi_j = (\xi_j^{(1)}, \xi_j^{(2)}) \in V_j^{(p)}$ so that

$$\|u - \xi_j^{(1)}\|_{\mathcal{E}(\omega_j^{(1)})}^2 + \|u - \xi_j^{(2)}\|_{\mathcal{E}(\omega_j^{(2)})}^2 \text{ is small.}$$

The basic Theorem 3.3 still holds. Denoting by $\chi_j^{(i)}$ the characteristic function for $\omega_j^{(i)}$, the constant function mentioned in proof of Theorem 3.3 must be replaced by $(\chi_j^{(1)}, \chi_j^{(2)})$. Then $W_{j,i}^{(p)} = W_j^p \chi_j^{(i)}$, $i = 1, 2$ (respectively, $\mathcal{H}W_{j,i}^{(p)} = \mathcal{H}W_j^p \chi_j^{(i)}$, $i = 1, 2$). We emphasize that in $V_j^{(p)}$ we have to use shape functions in $\omega_j^{(1)}$ and $\omega_j^{(2)}$ separately. Then we get analogous results as before.

5.3 Selection of the spaces V_j when the solution u has singularities

In the applications, the solution of (2.1) can be singular because of one or more of the following reasons:

1. the boundary $\partial\Omega$ has corners;
2. the boundary condition changes, *e.g.*, from Dirichlet to Neumann;
3. the coefficient $a(x, y)$ is rough, *e.g.* it is piecewise constant;
4. the right-hand side is not smooth;
5. the solution has a boundary layer.

We address Items 1 and 2 only. The character of the singular behavior of the solution of (2.1) is well-known. We will assume that the boundary $\partial\Omega$ has a corner at A , located at the origin, and that the boundary of $\partial\Omega$ near A consists of two straight lines; this assumption is only for the sake of simplicity. If f and g in (2.1) are sufficiently smooth, then in a neighborhood of A ,

$$u(r, \theta) = \sum_{k=0}^s a_k r^{\lambda_k} \log^{\mu_k} r \psi_j(\theta) + \zeta(r, \theta), \quad (5.8)$$

where $\lambda_{k+1} \geq \lambda_k$, $\mu_{k+1} \geq \mu_k$, $\psi_j(\theta)$ is a smooth function of θ , and $\zeta(r, \theta)$ is smoother than any of the terms in the sum. Here (r, θ) are polar coordinates with origin at A . We note that (5.8) is also true when $\Gamma_1 \cap \Gamma_2 = A$, which is relevant for Item 2.

Now we select the shape functions in V_j to be the functions $r^{\lambda_k} \log^{\mu_k} r \psi_k(\theta)$, $k = 0, 1, \dots, s$, together with polynomials. Then the error of the approximation of u by functions in V_j is only the error in the approximation of $\zeta(r, \theta)$ by polynomials.

Remark 5.9. There is a large literature on expansions of the form (5.8), *e.g.*, [12, 20, 21, 31].

Remark 5.10. An expansion similar to (5.8) is also valid for elasticity problems.

Remark 5.11. If we have $g = 0$ in (5.3), then $\mu_k = 0$ in (5.8).

Construction of these singular functions may not be simple, especially in the elasticity problem. Hence a numerical treatment is unavoidable. Either we solve the associated Handbook Problem (local) problem numerically (with the GFEM) or use analytic formulae with numerically determined parameters; see, *e.g.*, [33]. We always have $\zeta \in \mathcal{E}(\Omega)$ and hence it is not necessary to use the special functions in (5.8) as shape functions, *i.e.*, we can take $s = 0$ in (5.8). However, the accuracy when using only polynomial shape functions is very low.

The use of special shape functions in ω_j for which $A \in \bar{\omega}_j$ is very important. Also, we have to use some of the special shape functions in patches ω_j when $A \notin \bar{\omega}_j$, but ω_j is close to A . The number of special shape function needed depends on the accuracy requirement. Determining the optimal number of terms as well as in which elements special shape functions are needed is not simple. Usually, two terms in patches ω_j for which $\bar{\omega}_j \in A$ and one term in all ω_j that are the direct neighbors of these patches is sufficient.

5.4 Selection of the spaces V_j satisfying the Dirichlet boundary condition

If $\bar{\omega}_j \cap \Gamma_1 = \emptyset$, then there are no restrictions on the approximation functions on $\partial\omega_j$. But, if $\bar{\omega}_j \cap \Gamma_1 \neq \emptyset$, then functions in V_j must equal 0 on $\bar{\omega}_j \cap \Gamma_1$. Usually, it is not difficult to create such functions. For example, if the boundary Γ_1 is a straight line, or a circle and we are solving the Laplace's equation, $\Delta u = 0$, then it is easy to construct such functions.

The error estimate for ϵ_2 then depends, as before, on the approximation properties of the space V_j .

Remark 5.13. If the Dirichlet conditions is not homogeneous, functions in V_j must satisfy this condition; then all the results hold.

Remark 5.14. GFEM constructs ω_j so that the condition $|\bar{\omega}_j \cap \Gamma_1| \geq \gamma \text{diam } \omega_j$ is satisfied. This is easily accomplished. Then there are no difficulties with imposing the Dirichlet boundary conditions. This is an issue with meshfree method; see [4] for a discussion of techniques to overcome it.

6 Implementational Issues in the GFEM

Implementation of the GFEM consists of four major parts, namely:

- (a) the selection of local approximating functions;
- (b) the selection of partition of unity (PU) functions;
- (c) the construction of the stiffness matrix;
- (d) the solution of the linear system; and,
- (e) the computation of data of interest.

(a) We have already discussed the selection of local approximating functions, $\{\xi_{ji}\}$, in Section 5, which depends on the available information on the unknown solution u of the problem (2.1) or (2.6).

(b) The primary role of PU functions, $\{\phi_j\}$, in GFEM is to paste together the local approximation functions, $\{\xi_{ji}\}$, to form global approximation functions that are conforming, *i.e.*, global approximation functions that are in \mathcal{E}_{Γ_1} . In theory, any partition of unity, satisfying (3.1)–(3.4), will suffice; we may consider Shepard functions with disks as their supports, as described in Section 2, or finite element hat functions, or any family of particle shape functions used in meshless methods (see [4, 26]).

But the choice of patches $\{\omega_j\}$ and the associated PU functions $\{\phi_j\}$ affects many aspects of the implementation of GFEM, *e.g.*, (c) and (d). We first discuss the effect of patches and the PU functions on the work involved in (c), in constructing the stiffness matrix. From (3.34), a typical element of the stiffness matrix is of the form

$$\int_{\omega_j \cap \omega_l} \nabla \eta_{lk} \cdot \nabla \eta_{ji} \, dx. \quad (6.1)$$

Since these integrals are evaluated by numerical integration, it is important to choose $\{\omega_j\}$ such that the sets $\{\omega_j \cap \omega_l\}, 1 \leq j, l \leq N$ are simple domains, in which numerical integration could be performed efficiently. For example, if the ω_j 's are disks (in \mathbb{R}^2) or balls (in \mathbb{R}^3), a typical $\omega_j \cap \omega_l$ is a “lens shaped” domain, and accurate numerical integration over such domains is known to be difficult. We note however, that an efficient numerical integration scheme for such domains was reported in [13]. In [34, 37], ω_j 's were chosen to be rectangles, and a typical $\omega_j \cap \omega_l$ was also a rectangle. It is much easier to perform numerical integration on rectangular domains. Thus the patches $\{\omega_j\}$ should be chosen so that the sets $\omega_j \cap \omega_l$ are simple enough to perform numerical integration. Moreover, since $\eta_{ji} = \phi_j \xi_{ji}$, the integrand in (6.1) has terms involving $\{\phi_j\}$ and $\{\nabla \phi_j\}$, and thus the numerical evaluation of (6.1) depends also on the smoothness of the PU functions $\{\phi_j\}$ and their derivatives $\{\nabla \phi_j\}$.

The choice of PU functions $\{\phi_j\}$ also affects the linear system (3.33). We have mentioned in Section 4 that the shape functions of S^{GFEM} could be linearly dependent or independent, depending on the the choice of PU functions. This,

in turn, leads to either a singular or a non-singular linear system. We further note that the condition number of the stiffness matrix, when the linear system is non-singular, depends on the choice of the PU functions. Thus the choice of PU functions affects the choice of the linear solver used in (e), since the choice of linear solvers depends on linear systems. Finally, the constants C_1 and C_2 , in (3.3) and (3.4) respectively, are directly related to the choice of $\{\phi_j\}$, and these constants, in turn, affect the constants in the error estimates (3.13), (3.14), and (3.21). We note, however, that it may not be wise to choose the PU functions $\{\phi_j\}$ based only on any one of these effects. The choice of $\{\phi_j\}$ should be balanced with respect to several other aspects of the GFEM, *e.g.*, the selection of local shape functions.

(c) Evaluation of the elements of the stiffness matrix A , in (3.33), involves more than just ensuring that the sets $\{\omega_j \cap \omega_l\}$ are simple domains. The success of GFEM depends on evaluating the elements of A with high accuracy. Since A is symmetric, only the upper triangular part of A is evaluated. In [37, 38], the same numerical integration was used simultaneously to evaluate all the elements in the same row (the diagonal element and the elements to the right of the diagonal in the same row). Also numerical integration, based on adaptive procedure, was used to evaluate these elements. In the problems considered in [37, 38], the diagonal elements of A were always dominant and a low tolerance requirement in the adaptive quadrature for evaluating diagonal elements ensured the accuracy of evaluation of off-diagonal elements. The tolerance, for the relative error in the evaluation of the diagonal elements, was prescribed as 0.01, or less, of the required relative accuracy of the computed solution.

(d) We now comment on solving the linear system (3.33). We have mentioned before that the stiffness matrix A in (3.33) could be positive semi-definite or severely ill-conditioned. When A is positive semi-definite, the system (3.33) has non-unique solutions. We have mentioned before in Section 3 that the lack of unique solvability of (3.33) does not imply that the GFEM has non-unique solutions.

A solution of (3.33) can be obtained with (i) a specialized direct solver based on elimination, or (ii) an iterative solver.

(i) The linear system (3.33) was successfully solved in [37] using the direct method of multi-frontal sparse Gaussian elimination for symmetric, indefinite systems that was developed in [17] and implemented in subroutines MA47 and MA48 in the Hartwell Subroutine Library.

(ii) An iterative scheme was also used in [37] to solve (3.33), which we describe here. We first perturb the matrix A by ϵI , where $\epsilon > 0$ is small. Let

$$A_\epsilon \equiv A + \epsilon I.$$

Clearly, A_ϵ is positive definite. We first compute

$$\begin{aligned} c_0 &= A_\epsilon^{-1} b, \\ r_0 &= b - A c_0, \\ z_0 &= A_\epsilon^{-1} r_0, \\ v_0 &= A z_0. \end{aligned}$$

Then, for $i = 1, 2, \dots$, we compute

$$\begin{aligned} c_i &= c_0 + \sum_{j=0}^{i-1} z_j, \\ r_i &= r_0 - \sum_{j=0}^{i-1} v_j, \\ z_i &= A_\epsilon^{-1} r_i, \\ v_i &= A z_i, \end{aligned}$$

until the ratio

$$\frac{|z_i^T A z_i|}{|c_i^T A c_i|}$$

is sufficiently small, which is attained, say, for $i = I$. Then c_I is considered a solution of (3.33). In practice, we have seen that the above ratio becomes sufficiently small in one or two steps. For a numerical example, we refer to [37].

(e) Successful solution of the linear system (3.33) yields the vector \mathbf{c} , which is used to compute various data of interest; for example, approximation of the exact solution or its gradient at a particular point $\bar{x} \in \Omega$. This data is obtained by computing

$$u_{GFEM}(\bar{x}) = \sum_{j=1}^N \sum_{i=1}^{m(j)} c_{ji} \eta_{ji}(\bar{x}) \quad \text{and} \quad \nabla u_{GFEM}(\bar{x}) = \sum_{j=1}^N \sum_{i=1}^{m(j)} c_{ji} \nabla \eta_{ji}(\bar{x}).$$

We note that computation of $\eta_{ji}(x)$ and $\nabla \eta_{ji}(x)$ involve computation of $\phi_j(x)$, $\xi_{ji}(x)$, $\nabla \phi_j(x)$, and $\nabla \xi_{ji}(x)$. There are other data of interest, *e.g.*, stress intensity factors; we will not discuss their evaluation in this paper.

7 Applications, Experience, and Potential of the GFEM

We have discussed the basic ideas in the mathematical foundation of the GFEM in the simple setting of linear elliptic BVPs.

A wide variety of shape functions can be used in the GFEM. This allows the GFEM to successfully approximate non-smooth solutions of BVPs on domains having corners or multiple cracks, or with mixed type of boundary conditions – Dirichlet and Neumann. It is also easy to construct shape functions that are smooth, *i.e.*, with higher regularity. Thus the GFEM can be used to solve higher order problems, *e.g.*, biharmonic or polyharmonic problems. Also, the GFEM with smooth shape functions can be used in problems with boundary conditions involving distributions, in which situation the solution of the BVP is not in the energy space. Furthermore, the capability of choosing appropriate shape functions makes the GFEM well-suited for solving Helmholtz problem

([28, 23]) and certain non-linear problems ([8]). We have mentioned before that the GFEM either does not employ a mesh or uses a mesh only minimally. This allows the GFEM, without re-meshing or with minimal re-meshing, to be used in problems involving domains with changing boundaries, or with an unknown boundary, as in crack propagation problems or free-boundary problems.

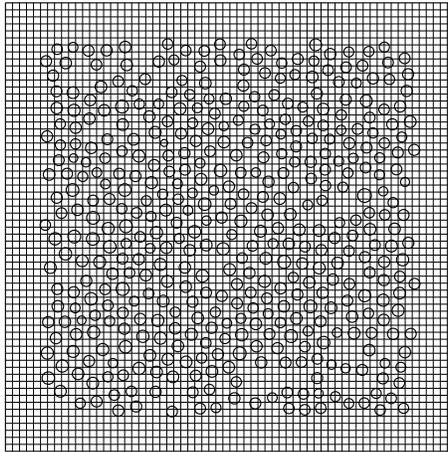


Figure 1: Example of a perforated domain

The GFEM was successfully used on problems with complicated domains in [38, 40, 39] using simple meshes, and thus avoiding complex meshes that conform to the geometry of the domain. An example of one of the domains considered in these papers is given in Figure 1. We note that the voids in this domain could be replaced by fibers. In fact, the positions of the voids in Figure 1 are identical to the positions of fibers in a composite material and were obtained by actual measurement ([2]). Such problems were successfully solved in [39, 40] by the GFEM using simple 8×8 and 16×16 uniform square meshes to cover the perforated domain. Detailed analysis of the accuracy and computational complexity was given in [40]. The problem with perforated domain is a typical example of multi-scale problems. Moreover, the GFEM was used on problems with boundary layers in [16].

The major cost of the GFEM, when applied to problems with complex domains, is the numerical integration. And, as mentioned in Section 6, the success of the GFEM depends on efficient numerical integration based on adaptive procedures. Adaptive numerical integration based on Simpson's rule turned out to be most effective in the problems considered in [38, 40, 39].

The ideas in the GFEM have potential of being used in other frameworks. We have already seen in Section 4 that certain FEM approximation spaces could be viewed as special cases of GFEM spaces. Also, the approximation spaces in certain meshless methods can be viewed as a GFEM space (with constants as local approximating functions and the particle shape functions as PU functions). A GFEM space, S^{GFEM} , has the potential of being used in the

context of mixed formulations of elliptic BVPs. S^{GFEM} can also be used in the framework of collocation methods. Of course, there are many open problems of a mathematical nature in the use of S^{GFEM} in mixed, collocation, or possibly other methods. The problems of implementation of these approaches are also open.

The effectiveness of the performance of the GFEM (or similar methods) on certain benchmark problems has been shown in the literature [1, 25]. But these benchmark problems are so simple that the performance of the classical FEM on these problems is often superior to the GFEM. The future of the GFEM or other similar methods is uncertain unless their superiority is established on appropriate realistic benchmark problems. It is extremely important to classify problems where these methods will outperform the classical FEM.

Finally, we provide a list of problems, where the GFEM and other similar methods have great promise of being efficient and successful:

- Problems with non-smooth solutions, where some information about the solution is known, or could be obtained by a local numerical computation. The non-smoothness of the solution could be due to either the boundary, or the coefficients, or the type of the problem, *e.g.*, the Helmholtz problem.
- Problems where the domain is so complex that creating a mesh by a mesh-generator is either not feasible or not efficient. We note, however, that a lot of progress has been made in creating efficient mesh-generators in the last decade.
- Problems with time dependent boundaries or free boundaries (*i.e.*, problems with unknown boundaries). Typical examples of such problems are crack-propagation problems, seepage problems and parachute problems.
- Certain non-linear problems, *e.g.*, metal forming problems.

8 Appendix: The Poincaré Inequalities

In this appendix we outline the derivation of bounds for the Poincaré constants C_3 and C_4 introduced in Theorem 3.3. These bounds will be in terms of simple geometric data for the patches ω_j . Throughout this section, x and y denote points in \mathbb{R}^2 .

Theorem 8.1 *Suppose ω is convex, d is the diameter of ω , and ω contains a disc of diameter $\tilde{d} \geq \frac{d}{\kappa_1}$. Then*

$$\|v\|_{L^2_a(\omega)} \leq 2\kappa_1 \left(\frac{\beta}{\alpha}\right)^{3/2} d \|v\|_{\mathcal{E}(\omega)}, \text{ for all } v \in \mathcal{E}(\omega) \text{ satisfying } \int_{\omega} av \, dx = 0. \quad (8.1)$$

Proof. We now outline the proof of estimate (8.1). We will use the following result: If ω is convex, then

$$\|v - v_{a,S}\|_{L_a^2(\omega)} \leq \frac{(\pi|\omega|)^{1/2}}{|S|} \left(\frac{\beta}{\alpha}\right)^{3/2} d^2 \|v\|_{\mathcal{E}(\omega)}, \text{ for all } v \in H^1(\omega), \quad (8.2)$$

where S is any measurable set in ω , and

$$v_{a,S} = \frac{1}{|S|_a} \int_S av \, dx, \text{ where } |S|_a = \int_S a \, dy.$$

For $a = 1$, this result is proved in [19]. The proof of (8.2) is a mild extension of the proof of (7.45) in [19]. Now suppose ω contains a disk of diameter $\tilde{d} \geq \frac{d}{\kappa_1}$. Then, taking $S = \omega$ in (8.2) we get

$$\|v\|_{L^2(\omega)} \leq 2\kappa_1 \left(\frac{\beta}{\alpha}\right)^{3/2} d \|v\|_{\mathcal{E}(\omega)}, \text{ for all } v \text{ satisfying } \int_{\omega} av \, dx = 0,$$

which is (8.1). \square

Let ω be an open set in \mathbb{R}^2 and suppose $l \subset \partial\omega$ is an arc. For $x \in \omega$, let

$$s_l(x) = \text{the convex hull of } \{x\} \cup l$$

be the sector subtending l , and let $\gamma_l(x)$ be the angle of $s_l(x)$.

Theorem 8.2 *Suppose ω is convex, $d = \text{diam}(\omega)$, and $\tilde{\omega}$ is a disk of diameter $\tilde{d} \geq \frac{d}{\kappa_2}$, whose closure lies in ω . Suppose*

$$\gamma_l(x) \geq \gamma_0 > 0, \text{ for all } x \in \tilde{\omega}. \quad (8.3)$$

(Such an α_0 exists since the closure of $\tilde{\omega}$ lies in ω .) Then

$$\|v\|_{L_a^2(\omega)} \leq \left\{ \left(\frac{\beta}{\alpha}\right)^{3/2} 2\kappa_1 + \left(\frac{\beta}{\alpha}\right) \frac{\kappa_2\pi}{\gamma_0} \right\} d \|v\|_{\mathcal{E}(\omega)}, \text{ for all } v \in \mathcal{E}(\omega) \text{ with } v|_l = 0. \quad (8.4)$$

Proof. We now outline the proof of estimate (8.4), which is in two steps. We first use estimate (8.2) with $S = \tilde{\omega}$ to get

$$\begin{aligned} \|v\|_{L_a^2(\omega)} &\leq \left(\frac{\beta}{\alpha}\right)^{3/2} \frac{(\pi|\omega|)^{1/2}}{|\tilde{\omega}|} d^2 \|v\|_{\mathcal{E}(\omega)} + \|v_{a,S}\|_{L_a^2(\omega)} \\ &\leq \left(\frac{\beta}{\alpha}\right)^{3/2} \frac{(\pi|\omega|)^{1/2}}{|\tilde{\omega}|} d^2 \|v\|_{\mathcal{E}(\omega)} + \left(\frac{\beta}{\alpha}\right)^{1/2} \frac{|\omega|^{1/2}}{|\tilde{\omega}|^{1/2}} \|v\|_{L_a^2(\tilde{\omega})}. \end{aligned} \quad (8.5)$$

Next we estimate $\|v\|_{L_a^2(\tilde{\omega})}$. For $x \in \tilde{\omega}$ and $y \in l$, we have

$$v(x) - v(y) = - \int_0^{|x-y|} D_r[v(x + r(\cos \theta, \sin \theta))] \, dr,$$

where

$$y = y(\theta) = x + r(\cos \theta, \sin \theta),$$

(r, θ) denoting the polar coordinates of y with respect to x . Now, if $v(y) = 0$ for $y \in l$, we have

$$v(x) = - \int_0^{|x-y|} D_r[v(x + r(\cos \theta, \sin \theta))] dr.$$

Integrating this equality with respect to θ from 0 to $\gamma_l(x)$, we get

$$v(x)\gamma_l(x) = - \int_0^{\gamma_l(x)} \int_0^{|x-y(\theta)|} D_r[v(x + r(\cos \theta, \sin \theta))] dr d\theta.$$

Hence

$$\begin{aligned} |v(x)| &= \frac{1}{\gamma_l(x)} \left| \int_0^{\gamma_l(x)} \int_0^{|x-y(\theta)|} \frac{D_r[v(x + r(\cos \theta, \sin \theta))]}{r} r dr d\theta \right| \\ &= \frac{1}{\gamma_0} \int_{s_l(x)} \frac{|Dv(y)|}{|x-y|} dy. \\ &\leq \frac{1}{\gamma_0} \int_{\omega} \frac{|Dv(y)|}{|x-y|} dy \\ &= \frac{1}{\gamma_0} V_{\frac{1}{2}}(|Dv|)(x), \end{aligned} \tag{8.6}$$

where

$$(V_{\mu}h)(x) = \int_{\omega} |x-y|^{2(\mu-1)} h(y) dy$$

is the Riesz potential of h . Squaring (8.6) and integrating over $\tilde{\omega}$, we get

$$\|v\|_{L^2(\tilde{\omega})} \leq \frac{1}{\gamma_0} \|V_{\frac{1}{2}}(|Dv|)\|_{L^2(\tilde{\omega})} \leq \frac{1}{\gamma_0} \|V_{\frac{1}{2}}(|Dv|)\|_{L^2(\omega)}. \tag{8.7}$$

We have the following estimate for the Riesz potential from Lemma 7.12 in [19]:

$$\|V_{\mu}h\|_{L^2(\omega)} \leq \frac{1}{\mu} \pi^{1/2} |\omega|^{1/2} \|h\|_{L^2(\omega)}. \tag{8.8}$$

Combining (8.7) and (8.8) yields

$$\|v\|_{L_a^2(\tilde{\omega})} \leq \left(\frac{\beta}{\alpha}\right)^{1/2} \frac{\pi d}{\gamma_0} \|v\|_{\mathcal{E}(\omega)}. \tag{8.9}$$

Combining (8.5) and (8.9) we have

$$\|v\|_{L_a^2(\omega)} \leq \left(\frac{\beta}{\alpha}\right)^{3/2} \frac{(\pi|\omega|)^{1/2}}{|\tilde{\omega}|} d^2 \|v\|_{\mathcal{E}(\omega)} + \left(\frac{\beta}{\alpha}\right) \frac{|\omega|^{1/2}}{|\tilde{\omega}|^{1/2}} \frac{\pi d}{\gamma_0} \|v\|_{\mathcal{E}(\omega)}. \tag{8.10}$$

Finally, since $\tilde{\omega}$ is a disk of radius $\tilde{d} \geq \frac{d}{\kappa_2}$, we get

$$\|v\|_{L^2(\omega_j)} \leq \left\{ \left(\frac{\beta}{\alpha} \right)^{3/2} 2\kappa_1 + \left(\frac{\beta}{\alpha} \right) \frac{\kappa_2 \pi}{\gamma_0} \right\} d \|v\|_{\mathcal{E}}, \text{ for all } v \in \mathcal{E}(\omega) \text{ with } v|_l = 0, \quad (8.11)$$

which is (8.4). \square

Remark 8.1. In Theorem 8.2 we assumed that l is an arc. This hypothesis can be considerably weakened; for example, it can be a disconnected set.

References

- [1] S. N. Atluri and S. Shen. *The Meshless Local Petrov Galerkin Method*. Tech. Sci. Press, 2002.
- [2] I. Babuška, B. Anderson, P. J. Smith, and K. Levin. Damage analysis of fiber composites; part 1, Statistical analysis on fiber scale. *Comp. Meth. Appl. Mech. Engrg.*, 172:27–77, 1999.
- [3] I. Babuška, U. Banerjee, and J. Osborn. On principles for the selection of shape functions for the generalized finite element method. *Comput. Methods Appl. Mech. Engrg.*, 191:5595–5629, 2002.
- [4] I. Babuška, U. Banerjee, and J. Osborn. Survey of meshless and generalized finite element methods. *Acta Numerica*, 12:1–125, 2003.
- [5] I. Babuška, G. Caloz, and J. Osborn. Special finite element methods for a class of second order elliptic problems with rough coefficients. *SIAM J. Numer. Anal.*, 31:945–981, 1994.
- [6] I. Babuška and J. M. Melenk. The partition of unity finite element method. *Int. J. Numer. Meth. Engrg.*, 40:727–758, 1997.
- [7] I. Babuška and J. Osborn. Can a finite element method perform arbitrarily badly? *Math. Comp.*, 69:443–462, 2000.
- [8] T. Belytschko, H. Chen, J. Xu, and G. Zi. Dynamic crack propagation based on loss of hyperbolicity and new discontinuous enrichment. *Int. J. Numer. Meth. Engrg.*, 58:1873–1905, 2003.
- [9] S. Bergman. *Integral operators in the theory of linear partial differential equations*. Springer, 1961.
- [10] M. D. Buhmann. Radial basis functions. *Acta Numerica*, 9:1–38, 2000.
- [11] M. D. Buhmann. *Radial Basis Functions: Theory and Implementations*. Cambridge University Press, 2003.
- [12] M. Dauge. *Elliptic Boundary Value Problems on Corner Domains*. Springer, Berlin, 1988. Lecture Notes in Mathematics, 1341.

- [13] S. De and K. J. Bathe. The method of finite spheres. *Computational Mechanics*, 25:329–345, 2000.
- [14] L. F. Demkowicz, A. Karafiat, and T. Liszka. On some convergence results for FDM with irregular mesh. *Comput. Methods Appl. Mech. Engrg.*, 42:343–355, 1984.
- [15] C. A. Duarte and J. T. Oden. An h-p adaptive method using clouds. *Comput. Methods Appl. Mech. Engrg.*, 139:237–262, 1996.
- [16] C. A. M. Duarte and I. Babuška. Mesh independent p-orthotropic enrichment using generalized finite element method. *Int. J. Numer. Meth. Engrg.*, 55:1477–1492, 2002.
- [17] I. S. Duff and J. K. Reid. The multifrontal solution of indefinite sparse symmetric linear systems. *ACM Trans. Math. Softw.*, pages 302–325, 1983.
- [18] C. Franke and R. Schaback. Convergence order estimates of meshless collocation methods using radial basis functions. *Adv. Comp. Math.*, 8:381–399, 1998.
- [19] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer, 1983.
- [20] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston, 1985.
- [21] P. Grisvard. *Singularities in Boundary Value Problems*. Masson, Paris, 1992.
- [22] H. Kober. *Dictionary of Conformal Representation*. Dover, 1957.
- [23] O. Laghrouche and P. Bettès. Solving short wave problems using special finite elements; towards an adaptive approach. In J. Whiteman, editor, *Mathematics of Finite Elements and Applications X*, pages 181–195. Elsevier, 2000.
- [24] S. Li and W. K. Liu. Meshfree and particle methods and their application. *Applied Mechanics Review*, 55:1–34, 2002.
- [25] G. R. Liu. *Meshless Methods*. CRC Press, Boca Raton, 2002.
- [26] W. K. Liu, Y. Chen, S. Jun, J. S. Chen, t. Belytschko, C. Pan, R. A. Uras, and C. T. Chang. Overview and applications of reproducing kernel particle methods. *Archives of Computational Methods in Engineering: State of the art reviews*, 3:3–80, 1996.
- [27] J. M. Melenk. *On Generalized Finite Element Methods*. University of Maryland, 1995. Ph.D. thesis.

- [28] J. M. Melenk and I. Babuška. The partition of unity finite element method: Theory and application. *Comput. Methods Appl. Mech. Engrg.*, 139:289–314, 1996.
- [29] J. M. Melenk and I. Babuška. Approximation with harmonic and generalized harmonic polynomials in the partition of unity method. *Computer Assisted Mechanics and Engineering Sciences*, 4:607–632, 1997.
- [30] S. G. Mikhailin. *Numerical Performance of Variational Methods*. Nordhoff, 1971.
- [31] S. A. Nazarov and B. A. Plamenevsky. *Elliptic Problems in Domains with Piecewise Smooth Boundaries*. Walter de Gruyter, Berlin, New York, 1994.
- [32] E. Onate, F. Perazzo, and J. Miguel. A finite point method for elasticity problems. *Computers & Structures*, 79:2143–2149, 2001.
- [33] P. J. Papadakis and I. Babuška. A numerical procedure for the determination of certain quantities related to the stress intensity factors in two dimensional elasticity. *Comput. Methods Appl. Mech. Engrg.*, 122:69–92, 1995.
- [34] M. A. Schweitzer. *A Parallel Multilevel Partition of Unity Method for Elliptic Partial Differential Equations*. Springer, 2000.
- [35] D. Shepard. A two-dimensional interpolation function for irregularly spaced points. *Proc. ACM Nat. Conf.*, pages 517–524, 1968.
- [36] F. L. Stazi, E. Budyn, J. Chessa, and T. Belytschko. An extended finite element with higher-order elements for curved cracks. *Computational Mechanics*, 31:38–48, 2003.
- [37] T. Stroubolis, I. Babuška, and K. Copps. The design and analysis of the generalized finite element method. *Comput. Methods Appl. Mech. Engrg.*, 181:43–69, 2001.
- [38] T. Stroubolis, K. Copps, and I. Babuška. The generalized finite element method. *Comput. Methods Appl. Mech. Engrg.*, 190:4081–4193, 2001.
- [39] T. Stroubolis, L. Zhang, and I. Babuška. p-version of generalized fem using mesh-based handbooks. *Int. J. Numer. Meth. Engrg.* to appear.
- [40] T. Stroubolis, L. Zhang, and I. Babuška. Generalized finite element method using mesh-based handbooks: application to problems in domains with many voids. *Comput. Methods Appl. Mech. Engrg.*, 192:3109–3161, 2003.
- [41] N. Sukumar, N. Moes, B. Moran, and T. Belytschko. Extended finite element method for three dimensional crack modelling. *Int. J. Numer. Meth. Engrg.*, 48(11):1549–1570, 2000.

- [42] I. N. Vekua. *New Methods for Solving Elliptic Equations*. North-Holland, Amsterdam, 1967.
- [43] J. R. Xiao and M. A. McCarthy. A local heaviside weighted meshless method for two-dimensional linear elasticity. *Computational Mechanics*, 31:301–315, 2003.