A wavelet-based generalization of the multifractal formalism from scalar to vector valued d-dimensional random fields: from theoretical concepts to experimental applications

P. Kestener and A. Arneodo

Laboratoire de Physique
Ecole Normale Supérieure de Lyon
46, allée d’Italie
69364 Lyon cedex 07, FRANCE
A wavelet-based generalization of the multifractal formalism from scalar to vector valued d-dimensional random fields: from theoretical concepts to experimental applications

Laboratoire de Physique Ecole Normale Superieure de Lyon 46,allee d'Italie 69364 Lyon cedex 07, FRANCE
2D WTMM Methodology: PhD work of N. Decoster

2D data: \( I \)

Smoothing scale \( a \)

Gradient \( \nabla \)

\[
\begin{bmatrix}
\frac{\partial}{\partial x'} \\
\frac{\partial}{\partial y'}
\end{bmatrix}
\]

Wavelet Transform

\[ T_\psi(r, a) = \begin{pmatrix}
I * \frac{\partial \phi_a}{\partial x}(r) \\
I * \frac{\partial \phi_a}{\partial y}(r)
\end{pmatrix} \]

\[ T_\psi(r, a) = \nabla (I*\phi_a)(r) = (\mathcal{M}_\psi(r, a), \mathcal{A}_\psi(r, a)) \]
WTMM Methodology: Skeleton

WTMM Chains

WTMM Chains at 3 different scales

Maxima lines: WT skeleton
Local roughness characterization: Hölder exponent

\[ f(x_0 + \lambda u) - f(x_0) \sim \lambda^{h(x_0)} (f(x_0 + u) - f(x_0)) \]

- Monofractal Image
- Multifractal Image

\[ \mathcal{M} \sim a^h, \text{ single } h \]

\[ \mathcal{M} \sim a^h, a^h, a^h, h \in [h_{\text{min}}, h_{\text{max}}] \]
WTMM Method: multifractal formalism

Singularity spectrum:

$$D(h) = d_H \{ r \in \mathbb{R}^d, h(r) = h \}$$

Legendre transform:

$$D(h) = \min_q (qh - \tau(q))$$

Analogy with statistical physics: compute partition functions

$$\mathcal{Z}(q, \alpha) = \sum_{\mathcal{L}(\alpha)} (\mathcal{M}_\psi(r, \alpha))^q \sim \alpha^{\tau(q)}$$
WTMM Method: multifractal formalism

Singularity spectrum:

\[ D(h) = d_H \{ r \in \mathbb{R}^d, h(r) = h \} \]

Legendre transform:

\[ D(h) = \min_q (qh - \tau(q)) \]

Analogy with statistical physics: compute partition functions

\[ \mathcal{Z}(q, \alpha) = \sum_{\mathcal{L}(\alpha)} (\mathcal{M}_\psi(r, \alpha))^q \sim \alpha^{\tau(q)} \]

\[ \mathcal{H}(q, \alpha) = \sum_{\mathcal{L}(\alpha)} \ln |\mathcal{M}_\psi(r, \alpha)| \mathcal{W}_\psi(r, \alpha) \sim \alpha^{h(q)} \]

\[ \mathcal{D}(q, \alpha) = \sum_{\mathcal{L}(\alpha)} \ln |\mathcal{W}_\psi(r, \alpha)| \mathcal{W}_\psi(r, \alpha) \sim \alpha^{D(q)} \]
Application to synthetic monofractal surfaces

Fractional Brownian surfaces: $B^H(r)$

- $H < 0.5$: anti-correlated increments
- $H = 0.5$: uncorrelated increments
- $H > 0.5$: correlated increments

Theoretical Predictions:

- $\tau(q)$ is linear:
  \[ \tau(q) = qH - 2 \]
- Multifractal spectrum is degenerated:
  \[ D(h = H) = 2 \]
Application to synthetic multifractal surfaces

Multifractal (Fractionally Integrated Singular Cascades) surfaces

\[ \tau(q) \text{ is non-linear} \]
\[ \tau(q) = -2 - q(1 - H^*) - \log_2(p_1^q + p_2^q) \]

\[ \text{Singularity spectrum is a non-degenerated convex curve} \]
Goals: using WTMM method to help in diagnosis of breast cancer
What is breast cancer?

- malignant tumor of mammal gland
- incidence: 30000 new case each year in France
- prevention is very difficult (as opposed to lung cancer)
- hereditarity: 5 to 10 % only (BRCA1/2 genes)
- forecast depends on the tumoral volume at diagnosis

⇒ SCREENING using mammography
Radiological anomalies

- Opacities
- Calcifications
- Architectural distortions
Digitalized mammographies : texture analysis

- dense breasts : more difficult to diagnose
- only 2 classes of monofractal properties

Digital Database for Screening Mammography:
http://marathon.csee.usf.edu/Mammography/Database.html
Application of 2D WTMM methodology in mammography

Tissue classification: dense

Dense breast:
monofractal, $H = 0.65$
persistent correlations
Application of 2D WTMM methodology in mammography

Tissue classification: dense vs fatty

- **Dense breast:**
  - Monofractal, $H = 0.65$
  - Persistent correlations

- **Fatty breast:**
  - Monofractal, $H = 0.30$
  - Anti-persistent correlations
Application to digitalized mammographies

Colored Maps:
segmentation of dense $h > 0.52$ areas and fatty $h < 0.38$ areas
Application to digitalized mammographies

Colored Maps:
segmentation of dense $h > 0.52$ areas and fatty $h < 0.38$ areas
Microcalcifications detection

Segmentation of WT skeleton lines:
microcalcifications vs background texture

- Background lines
- Microcalcifications
  almost-punctual objects behave like 'Dirac' shapes ($h = -1$)
Cluster of microcalcifications

Study of microcalcification spatial distribution

Partition functions:

\[ D_F = 1.3 \]

observation:
fractal ramification of
cluster of microcalcifications
\((1 < D_F < 2)\) seems to be correlated to the pathology’s malignancy
the 2D WTMM method provides a framework for an automated measure of the breast radio-density and for studying the fractal geometry of clusters of microcalcifications.

Further study is necessary to validate quantitatively how far measuring the fractal dimension $D_F$ could improve computer-aided diagnosis systems benign/malignant.
3D scalar WTMM method

\[ T_\psi(r, a) = \begin{pmatrix} I \star \frac{\partial \phi_a}{\partial x}(r) \\ I \star \frac{\partial \phi_a}{\partial y}(r) \\ I \star \frac{\partial \phi_a}{\partial z}(r) \end{pmatrix} = \nabla (I \star \phi_a)(r) \]
3D scalar WTMM method: skeleton

WTMM surfaces

WTMM surfaces at 3 different scales

Maxima lines: WT Skeleton (projection along z)

WTMM points

\[ \log(a) \]
Test-application to synthetic 3D monofractal fields

fractional Brownian fields: $B_H(r)$

- $H < 0.5$: anti-correlated increments
- $H = 0.5$: uncorrelated increments
- $H > 0.5$: correlated increments

Theoretical predictions:
- $\tau(q)$ is linear:
  \[ \tau(q) = qH - 3 \]
- Multifractal spectrum is degenerated:
  \[ D(h = H) = 3 \]
Test-application to synthetic 3D multifractal fields

3D multifractal fields (Fractionally Integrated Singular Cascades)

Theoretical predictions:

\[ \tau(q) = -2 - q(1 - H^*) - \log_2(p_1^q + p_2^q) \]

with \( p_1 + p_2 = 1 \)

singularity spectrum is a non-degenerated convex curve
3D dissipation field: isotropic turbulence DNS

pseudo-spectral code, \((512)^3\) grid, \(R_\lambda = 216\) (M. Meneguzzi)
3D WTMM methodology vs Box-Counting algorithms

“Box-Counting” algorithm, binomial fit with $p_1 = 0.3$ and $p_2 = 0.7 \rightarrow p_1 + p_2 = 1$ : diagnoses a conservative multiplicative structure

dissipation

binomial model:
\[
\tau(q) = -2 - q - \log_2(p_1^q + p_2^q)
\]
**3D WTMM methodology vs Box-Counting algorithms**

- “Box-Counting” algorithm, binomial fit with \( p_1 = 0.3 \) and \( p_2 = 0.7 \) \( \rightarrow \) \( p_1 + p_2 = 1 \): diagnoses a **conservative** multiplicative structure

- “3D WTMM” method reveals a **non-conservative** multiplicative structure:
  
  binomial fit with \( p_1 = 0.36 \) and \( p_2 = 0.80 \) \( \Rightarrow \) \( p_1 + p_2 \neq 1 \)

**dissipation**

binomial model:

\[
\tau(q) = -2 - q - \log_2 (p_1^q + p_2^q)
\]

remark:

\[
p_1 = \frac{p_1}{p_1 + p_2}
\]

\( \Rightarrow \) inconsistent box-counting!
Self-similar multifractal vector-valued measure (2D case)


scalar measure \( \{ r : \lim_{l \to 0} \frac{\log \mu(B(r,l))}{\log l} = \alpha \} \), \( \alpha = h + 2 \)

\[ \mathcal{Z}(q, l) = \sum_i \mu_i^q(l) \sim l^{\tau_\mu(q)} \]

vector-valued measure \( \left\{ r : \lim_{l \to 0} \frac{\log \int_{B(r,l)} \| \Phi_l \mu(s) \| \ d\mathcal{L}_d(s)}{\log l} = \alpha \right\} \), \( \alpha = h + 2 \)

\[ \mathcal{Z}(q, l) = \sum_i \| \Phi_l \mu_i \|_i^q \sim l^{\tau_\mu(q)} \]
Self-similar multifractal vector-valued measure (2D case)

\[ \tau_\mu(q) = -\frac{\log(p_1^q + p_2^q + p_3^q + p_4^q)}{\log 2} \]

\[ D_\mu(h) = f_\mu(\alpha - 2) = \inf_q (qh - \tau_\mu(q)) \]
1. Tensorial wavelet transform of field $V = (V_1, V_2)$:

$$
\mathbb{T}_\psi [V](b, a) = (\mathbb{T}_{\psi_1}[V_j](b, a)) = \begin{pmatrix} T_{\psi_1}[V_1] & T_{\psi_1}[V_2] \\ T_{\psi_2}[V_1] & T_{\psi_2}[V_2] \end{pmatrix}
$$

$$
T_{\psi_i}[V_j](b, a) = a^{-3} \int d^3r \, \psi_i (a^{-1} (r-b)) V_j(r), j = 1, 2
$$

2. Direction of greatest variation of vector field:

$$
|\mathbb{T}_\psi [V]| = \sup_{C \neq 0} \frac{||\mathbb{T}_\psi [V] \cdot C||}{||C||}
$$

3. Singular value decomposition of WT tensor:

$$
\mathbb{T}_\psi [V] = (G) \cdot \begin{pmatrix} \sigma_{\text{max}} & 0 \\ 0 & \sigma_{\text{min}} \end{pmatrix} \cdot (D)^T
$$

4. Tensorial wavelet transform:

$$
\mathbb{T}_{\psi, \text{max}} [V](b, a) = \sigma_{\text{max}} G_{\sigma_{\text{max}}}
$$
Tensorial 2D WTMM methodology

Data

Tensorial wavelet transform

\[ T_{\psi, \text{max}}[V](b, a) = \sigma_{\text{max}} G_{\sigma_{\text{max}}} \]

Modulus Maxima \( \sigma_{\text{max}} \) chains of tensorial wavelet transform at scale \( a \):

\[ \left\{ \frac{(b, a)}{\partial \sigma_{\text{max}} / \partial G_{\text{max}}} = 0 \text{ et } \frac{\partial^2 \sigma_{\text{max}}}{\partial G^2_{\text{max}}} < 0 \right\} \]
Tensorial 2D WTMM methodology: Skeleton

WTMM Chains at 3 different scales

Maxima lines: WT Skeleton
Monofractal 2D vector fields

fractional Brownian fields: $B_H(r)$

Spectral method simulation

---

Theoretical predictions:
- linear $\tau(q)$: $\tau(q) = qH - 2$
- degenerated singularity spectrum:
  $D(h = H) = 2$
2D self-similar multifractal vector-valued measures

Self-similar **multifractal** vector-valued measures
(Falconer and O’Neil’s model)

- vectorial box-counting
- vectorial 2D WTMM method

Theoretical predictions:

\[ \tau(q) = -\log_2(p_1^q + p_2^q + p_3^q + p_4^q) \]

\[ p_1 = p_4 = 0.5, \quad p_2 = 2 \text{ and } p_3 = 1 \]

- vectorial box-counting is less accurate
Tensorial 3D WTMM method: turbulent velocity field ($R_{\lambda} = 140$)
Tensorial 3D WTMM method: singularity spectrum of velocity

- parabolic fit: \( \tau(q) = -C_0 - C_1 q - C_2 \frac{q^2}{2} \)
- intermittency coefficient \( C_2 = 0.049 \pm 0.004 \)

1D increments method:
- longitudinal: \( C_2(\delta v_L) \sim 0.025 \)
- transverse: \( C_2(\delta v_T) \sim 0.040 \)
Tensorial 3D WTMM method: turbulent vorticity field \((R_A = 140)\)
Tensorial 3D WTMM method: singularity spectrum of vorticity
Tensorial 3D WTMM method: singularity spectrum of vorticity

- **vorticity**
  - $D_v (h + 1)$ spectrum translated velocity

$\Rightarrow$ same 3D intermittency coefficient!
Conclusion

assessment:

- WTMM multifractal analysis: moving towards vector fields

outlooks:

- better understanding of the information embedded in the WT tensor.
- identification of coherent structures in turbulence using WT tensor’s smallest singular value: vorticity filaments or sheets.
- others applications : astrophysics (interstellar medium, interstellar turbulence), MHD, geophysics, ...

Thanks:

- E. Lévéque, Laboratoire de Physique, ENS Lyon (Turbulent flows DNS).
References

Generalizing the wavelet-based multifractal formalism to vector-valued random fields: application to turbulent velocity and vorticity 3D numerical data.

A wavelet-based method for multifractal analysis of 3D random fields: application to turbulence simulation data.

Three-dimensional wavelet-based multifractal method: the need for revisiting the multifractal description of turbulence dissipation data.

A wavelet-based method for multifractal image analysis: from theoretical concepts to experimental applications.

Wavelet-based multifractal formalism to assist in diagnosis in digitized mamograms.

A wavelet-based method for multifractal image analysis. I. Methodology and test applications on isotropic rough surfaces.

A wavelet-based method for multifractal image analysis. II. Application to synthetic multifractal rough surfaces.

A wavelet-based method for multifractal image analysis. III. Application to high resolution satellite images of cloud structure.

Intermittency, log-normal statistics and multifractal cascade process in high-resolution satellite images of cloud structure.