Wavelets and Affine Distributions
A Time-Frequency Perspective

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OUTLINE

- The notion of time-frequency analysis
- Linear and quadratic time-frequency analysis
- Short-time Fourier transform and wavelet transform; spectrogram and scalogram
- Constant-bandwidth analysis vs. constant-Q analysis
- The affine class
- Affine time-frequency smoothing
- Hyperbolic time-frequency localization
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The notion of time-frequency (TF) analysis
Auditory perception as TF analysis

The TF plane

- Visualize **time-frequency location/concentration** of signal \( x(t) \):
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Linear TF analysis

• **TF analysis**: Measure contribution of TF point \((t_0, f_0)\) to signal \(x(t)\)

• **General approach**: Inner product of \(x(t)\) with “test signal” or “sounding signal” \(\psi_{t_0,f_0}(t)\) located about \((t_0, f_0)\):

\[
\text{LTFR}_{t_0,f_0} := \langle x, \psi_{t_0,f_0} \rangle = \int_{-\infty}^{\infty} x(t) \psi_{t_0,f_0}^*(t) \, dt
\]

LTFR = Linear TF Representation

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<th>WAMA-04</th>
<th>Cargèse, France</th>
<th>7</th>
</tr>
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<tbody>
<tr>
<td>WAMA-04</td>
<td>Cargèse, France</td>
<td>8</td>
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Linear TF synthesis

- TF synthesis (inversion of LTFR): Recover (“synthesize”) signal \( x(t) \) from \( \text{LTFR}_x(t_0, f_0) \)
- General approach:
  \[
  x(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{LTFR}_x(t_0, f_0) \psi_{t_0,f_0}(t) \, dt_0 df_0
  \]
  \( x(t) \) is represented as superposition of TF localized signal components, weighted by "TF coefficient function" \( \text{LTFR}_x(t_0, f_0) \)
- Problem: How to construct test (analysis) functions \( \psi_{t_0,f_0}(t) \) and synthesis functions \( \tilde{\psi}_{t_0,f_0}(t) \)?

Quadratic TF analysis

- TF analysis: Measure “energy contribution” of TF point \( (t_0, f_0) \) to signal \( x(t) \)
- Simple approach:
  \[
  \text{QTFR}_x(t_0, f_0) := \left| \text{LTFR}_x(t_0, f_0) \right|^2 = \left| \langle x, \psi_{t_0,f_0} \rangle \right|^2 \\
  = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1) x^*(t_2) \psi_{t_0,f_0}^*(t_1) \psi_{t_0,f_0}(t_2) \, dt_1 dt_2
  \]
  QTFR = Quadratic TF Representation
- Want QTFR to distribute signal energy \( E_x \) over TF plane:
  \[
  \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{QTFR}_x(t, f) \, dt \, df = E_x \quad \text{“TF energy distribution”}
  \]
- Problem: How to construct test (analysis) functions \( \psi_{t_0,f_0}(t) \)?
Construction of analysis/synthesis functions

- **Problem:** Construct family of analysis functions \( \{ \psi_{t_0,f_0}(t) \} \) such that \( \psi_{t_0,f_0}(t) \) is localized about TF point \((t_0, f_0)\).

- **Systematic approach:** \( \psi_{t_0,f_0}(t) \) derived from “prototype function” \( \psi(t) \) via unitary “TF displacement operator” \( U_{t_0,f_0} \):
  \[
  \psi_{t_0,f_0}(t) := (U_{t_0,f_0} \psi)(t)
  \]

- Same for synthesis functions \( \{ \tilde{\psi}_{t_0,f_0}(t) \} \):
  \[
  \tilde{\psi}_{t_0,f_0}(t) := (U_{t_0,f_0} \tilde{\psi})(t)
  \]

- **Two classical definitions** of \( U_{t_0,f_0} \):
  - TF shift
  - TF scaling (compression/dilatation) + time shift

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Two classical definitions of operator \( U \)

- **TF shift:**
  \[
  \psi_{t_0,f_0}(t) = (U_{t_0,f_0} \psi)(t) = \psi(t-t_0) e^{j2\pi f_0 t}
  \]

- **TF scaling + time shift:**
  \[
  \psi_{t_0,f_0}(t) = (U_{t_0,f_0} \psi)(t) = \sqrt{\frac{f_0}{f_0}} \psi\left(\frac{f_0}{f_0}(t-t_0)\right) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-t_0}{a}\right)|_{a=f_0/f_c}
  \]

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WAMA-04 Cargèse, France

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**Short-Time Fourier Transform (STFT)**

• Recall TF shift:
  \[ \psi_{t_0,f_0}(t) = \left[U_{t_0,f_0}\psi\right](t) = \psi(t-t_0)e^{j2\pi f_0 t} \]

  \[ \Rightarrow \text{LTFR} = \text{STFT}: \]
  \[ \text{STFT}_x(t_0, f_0) = \left\langle x, U_{t_0,f_0}\psi \right\rangle = \int_{-\infty}^{\infty} x(t) \psi^*(t-t_0)e^{-j2\pi f_0 t} dt \]

  **STFT = FT of local (windowed) segment of** \(x(t)\):
STFT signal synthesis

• Recall STFT analysis:

\[
\text{STFT}_x(t_0, f_0) = \langle x, U_{t_0,f_0}\psi \rangle = \int_{-\infty}^{\infty} x(t) \psi^\star(t-t_0)e^{-j2\pi f_0 t} dt
\]

• STFT signal synthesis:

\[
x(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{STFT}_x(t_0, f_0) (U_{t_0,f_0}\tilde{\psi})(t) dt_0 df_0
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{STFT}_x(t_0, f_0) \tilde{\psi}(t-t_0)e^{j2\pi f_0 t} dt_0 df_0
\]

\[x(t)\text{ is weighted superposition of TF shifted versions of } \tilde{\psi}(t)\]

Wavelet Transform (WT)

• Recall TF scaling + time shift:

\[
\psi_{t_0,f_0}(t) = (U_{t_0,f_0}\psi)(t) = \sqrt{f_0 \over f_\psi} \psi\left( f_0 \over f_\psi \right)(t-t_0)
\]

• \( \Rightarrow \) LTFR = WT:

\[
\text{WT}_x(t_0, f_0) = \langle x, U_{t_0,f_0}\psi \rangle = \int_{-\infty}^{\infty} x(t) \sqrt{f_0 \over f_\psi} \psi^\star\left( f_0 \over f_\psi \right)(t-t_0) dt
\]
WT signal synthesis

- Recall WT analysis:
  \[ \text{WT}_x(t_0, f_0) = \langle x, U_{t_0, f_0} \psi \rangle = \int_{-\infty}^{\infty} x(t) \left( \frac{f_0}{f_\psi} \right) \left( \frac{f_0}{f_\psi} \right)(t-t_0) dt \]

- WT signal synthesis:
  \[
  x(t) = \int_{0}^{\infty} \int_{-\infty}^{\infty} \text{WT}_x(t_0, f_0) \left( U_{t_0, f_0} \tilde{\psi} \right)(t) dt_0 df_0 \\
  = \int_{0}^{\infty} \int_{-\infty}^{\infty} \text{WT}_x(t_0, f_0) \left( \frac{f_0}{f_\psi} \right) \left( \frac{f_0}{f_\psi} \right)(t-t_0) dt_0 df_0
  \]

  \(x(t)\) is weighted superposition of TF scaled and time shifted versions of \(\tilde{\psi}(t)\)

Spectrogram and scalogram

- Recall LTFR → QTFR:
  \[ \text{QTFR}_x(t_0, f_0) = \left| \text{LTFR}_x(t_0, f_0) \right|^2 = \left| \langle x, U_{t_0, f_0} \psi \rangle \right|^2 \]

- STFT → spectrogram:
  \[ \text{SPEC}_x(t_0, f_0) := \left| \text{STFT}_x(t_0, f_0) \right|^2 = \left| \int_{-\infty}^{\infty} x(t) \psi^*(t-t_0) e^{-j2\pi f_0 t} dt \right|^2 \]

- WT → scalogram:
  \[ \text{SCAL}_x(t_0, f_0) := \left| \text{WT}_x(t_0, f_0) \right|^2 = \left| \int_{-\infty}^{\infty} x(t) \left( \frac{f_0}{f_\psi} \right) \left( \frac{f_0}{f_\psi} \right)(t-t_0) dt \right|^2 \]
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STFT and constant-BW filterbank: analysis

• STFT analysis as convolution:
  \[
  \text{STFT}_x(t_0, f_0) = \int_{-\infty}^{\infty} x(t) \psi^*(t-t_0) e^{-j2\pi f_0 t} \, dt \\
  = \left[ x(t) \ast \psi^*(-t) e^{j2\pi f_0 t} \right] \cdot e^{-j2\pi f_0 t} \bigg|_{t=t_0}
  \]

• \( \Rightarrow \) Filterbank interpretation/implementation:
**STFT and constant-BW filterbank: synthesis**

- STFT synthesis as convolution:

\[
x(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{STFT}_x(t_0, f_0) \tilde{\psi}(t-t_0) e^{j2\pi f_0 t} \, dt_0 df_0
\]

\[
= \int_{-\infty}^{\infty} \left[ \text{STFT}_x(t_0, f_0) e^{j2\pi f_0 t_0} \ast \tilde{\psi}(t_0) e^{j2\pi f_0 t_0} \right]_{t_0=t} \, df_0
\]

- Filterbank interpretation/implementation:

**Spectrogram analysis as constant-BW filterbank**

- Spectrogram analysis as convolution:

\[
\text{SPEC}_x(t_0, f_0) = \left| \text{STFT}_x(t_0, f_0) \right|^2 = \left| [x(t) \ast \psi^*(-t) e^{j2\pi f_0 t}]_{t=t_0} \right|^2
\]

- Filterbank interpretation/implementation:
**STFT / spectrogram: example**

\[
x(t) \text{ is sum of } \{ f_0, f_1, f_2 \}
\]

**WT and constant-Q filterbank: analysis**

- WT analysis as convolution:

\[
\text{WT}_x(t_0, f_0) = \int_{-\infty}^{\infty} x(t) \left[ \frac{f_0}{f_\psi} \psi^* \left( \frac{f_0}{f_\psi} (t-t_0) \right) \right] dt
\]

\[
= \left[ x(t) * \left[ \frac{f_0}{f_\psi} \psi^* \left( - \frac{f_0}{f_\psi} t \right) \right] \right]_{t=t_0}
\]

- \( \Rightarrow \) Filterbank interpretation/implementation:
WT and constant-Q filterbank: synthesis

- WT synthesis as convolution:
  \[ x(t) = \int_{0}^{\infty} \int_{-\infty}^{\infty} \text{WT}_x(t_0, f_0) \left( \frac{f_0}{f_\psi} \right) \left( \frac{f_0}{f_\psi} (t-t_0) \right) d\omega_0 df_0 \]
  \[ = \int_{0}^{\infty} \left[ \text{WT}_x(t_0, f_0) \ast \left( \frac{f_0}{f_\psi} \right) \left( \frac{f_0}{f_\psi} (t-t_0) \right) \right]_{t_0=t} df_0 \]

- ⇒ Filterbank interpretation/implementation:

Scalogram analysis as constant-Q filterbank

- Scalogram analysis as convolution:
  \[ \text{SCAL}_x(t_0, f_0) = |\text{WT}_x(t_0, f_0)|^2 = \left[ x(t) \ast \left( \frac{f_0}{f_\psi} \psi^* \left( - \frac{f_0}{f_\psi} t \right) \right) \right]_{t=t_0}^2 \]

- ⇒ Filterbank interpretation/implementation:
**WT / scalogram: example**

$x(t)$ is sum of

\[ \left\{ \begin{array}{l}
(f_1) \\
(f_2) \\
(f_3)
\end{array} \right. \]

**STFT / spectrogram vs. WT / scalogram**

**STFT / spectrogram**

**WT / scalogram**
Good-bye and hello

• Good-bye to:
  – STFT
  – spectrogram
  – constant-BW analysis

• Hello to:
  – affine class of QTFRs
  – Wigner distribution and Bertrand distribution
  – hyperbolic TF localization

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**Axiomatic (covariance-based) definition of WT**

- Generic LTFR expression:
  \[
  \text{LTFR}_x(t, f) = \int_{-\infty}^{\infty} x(t') K(t'; t, f) \, dt'
  \]

- **Covariance** of LTFR to TF scalings + time shifts:
  \[
  y(t) = \frac{1}{\sqrt{a}} x\left(\frac{t-\tau}{a}\right) \quad \leftrightarrow \quad Y(f) = \sqrt{a} X(a f) e^{-j2\pi f}
  \]
  \[
  \Rightarrow \text{LTFR}_y(t, f) = \text{LTFR}_x\left(\frac{t-\tau}{a}, af\right)
  \]

- Can show that **covariant LTFRs are given by WT**
  \[
  \text{WT}_x(t, f) = \int_{-\infty}^{\infty} x(t') \sqrt{f} \phi^* \left(\int_{-\infty}^{\infty} x(t') \phi(t'-t) \, dt'\right) \, dt' = \int_{-\infty}^{\infty} x(t') \phi(f(t'-t)) \, dt'
  \]

**Axiomatic (covariance-based) definition of the affine class of QTFRs**

- Generic QTFR expression:
  \[
  \text{QTFR}_x(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1) x^*(t_2) K(t_1, t_2; t, f) \, dt_1 \, dt_2
  \]

- **Covariance** of QTFR to TF scalings + time shifts:
  \[
  y(t) = \frac{1}{\sqrt{a}} x\left(\frac{t-\tau}{a}\right) \quad \leftrightarrow \quad Y(f) = \sqrt{a} X(a f) e^{-j2\pi f}
  \]
  \[
  \Rightarrow \text{QTFR}_y(t, f) = \text{QTFR}_x\left(\frac{t-\tau}{a}, af\right)
  \]

- Can show that **covariant QTFRs are given by**
  \[
  \text{AC}_x(t, f) = f \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1) x^*(t_2) \phi(f(t_1-t), f(t_2-t)) \, dt_1 \, dt_2
  \]
  \[
  \text{AC} = \text{Affine Class}
  \]
The affine class of QTFRs

- Affine class of QTFRs:
  \[ A\mathcal{C}_x(t, f) = f \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1)x^*(t_2) \phi(f(t_1-t), f(t_2-t)) \, dt_1 \, dt_2 \]

- 2-D "kernel" \( \phi(\alpha_1, \alpha_2) \) specifies QTFR of the AC

- Scalogram is a member of the AC; its kernel is separable:
  \[ \phi(\alpha_1, \alpha_2) = \frac{1}{f_\psi} \psi^*\left(\frac{\alpha_1}{f_\psi}\right) \psi\left(\frac{\alpha_2}{f_\psi}\right) \]

- Expression of AC QTFRs in terms of signal's FT:
  \[ A\mathcal{C}_x(t, f) = \frac{1}{f} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f_1)X^*(f_2) \Phi\left(\frac{f_1}{f}, \frac{f_2}{f}\right) e^{i2\pi(f_1-f_2)t} \, df_1 \, df_2 \]

Affine class and affine group

- TF scaling + time shift:
  \( (U_{a, \tau} x)(t) = \frac{1}{\sqrt{a}} x\left(\frac{t-\tau}{a}\right) = \sqrt{\beta} x(\beta t + \gamma) =: (\bar{U}_{\beta, \gamma} x)(t) \)

- Affine time transformation \( t \rightarrow \beta t + \gamma \) ("clock change")

- Composition of clock changes is another clock change:
  \[ \bar{U}_{\beta_2, \gamma_2} \bar{U}_{\beta_1, \gamma_1} = \bar{U}_{\beta_1 \beta_2, \gamma_1 + \beta_1 \gamma_2} \]

- \( \Rightarrow \bar{U}_{\beta, \gamma} \) is unitary representation of the affine group:
  - Set: \( (\beta, \gamma) \in \mathbb{R}^+ \times \mathbb{R} \)
  - Group operation: \( (\beta_1, \gamma_1) \circ (\beta_2, \gamma_2) = (\beta_1 \beta_2, \gamma_1 + \beta_1 \gamma_2) \)
  - Neutral element: \( (\beta_0, \gamma_0) = (1, 0) \)
The Wigner-Ville Distribution (WVD)

- Prominent member of the AC: the WVD
  \[ \text{WVD}_x(t, f) = \int_{-\infty}^{\infty} x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi ft} d\tau \]

- Properties of the WVD:
  - Covariant to TF scaling and time shift (of course)
  - Covariant to frequency shift ⇒ not constant-Q
  - Real for any (real or complex) signal \( x(t) \)
  - Marginal properties: e.g., \( \int_{-\infty}^{\infty} \text{WVD}_x(t, f) dt = |X(f)|^2 \)
  - Localization properties: e.g., \( \text{WVD}_x(t, f) = \delta(f - f_0) \) for \( x(t) = e^{j2\pi f_0 t} \)
  - Many more…
Interference terms in the WVD

\[ \text{WVD}_{x_1 + x_2}(t, f) = \text{WVD}_{x_1}(t, f) + \text{WVD}_{x_2}(t, f) + 2 \text{Re}\{\text{WVD}_{x_1 x_2}(t, f)\} \]

- Interference/cross term


Constant-BW smoothing of the WVD

- Smaller/less interference terms
- Poorer TF resolution

AC expression in terms of WVD

- Any QTFR of the AC can be expressed in terms of the WVD:
  \[
  AC_x(t, f) = \int_{-\infty}^{\infty} \int_{0}^{\infty} WVD_x(t', f') \sigma(f(t'-t), \frac{f'}{f}) \, dt' \, df'.
  \]
  where \( \sigma(\alpha, \beta) \) is related to \( \phi(\alpha_1, \alpha_2) \) and \( \phi(\beta_1, \beta_2) \) by FTs

- If \( \sigma(\alpha, \beta) \) is a smooth function, then \( AC_x(t, f) \) is a smoothed version of \( WVD_x(t, f) \)

- Smoothing causes...
  - smaller/less interference terms
  - poorer TF resolution

Affine (constant-Q) smoothing, different from constant-BW smoothing shown on previous slide!

Affine (constant-Q) smoothing of the WVD

- Recall: 
  \[
  AC_x(t, f) = \int_{-\infty}^{\infty} \int_{0}^{\infty} WVD_x(t', f') \sigma(f(t'-t), \frac{f'}{f}) \, dt' \, df'.
  \]

- Smoothing function \( \sigma(f(t'-t), \frac{f'}{f}) \) at various TF positions:

  ![Diagram showing good time resolution with poor frequency resolution and poor time resolution with good frequency resolution]
Affine smoothing: example

![Diagram](image)

- Smaller/less interference terms
- Poorer TF resolution


Scalogram as smoothed WVD

- Recall scalogram:
\[ \text{SCAL}_{x}(t, f) := \left| \text{WT}_{x}(t, f) \right|^2 = \left| \int_{-\infty}^{\infty} x(t') \sqrt{f_f \psi^* (f_{f_f^*}) dt'} \right|^2 \]

- Expression of scalogram as smoothed WVD:
\[ \text{SCAL}_{x}(t, f) = \int_{0}^{\infty} \int_{-\infty}^{\infty} \text{WVD}_{x}(t', f') \text{WVD}_{\psi} \left( f_f \psi^* \left( f_{f_f^*} \right) \right) dt' df' \]

Smoothing function
is WVD of wavelet:
\[ \sigma(\alpha, \beta) = \text{WVD}_{\psi} \left( \frac{\alpha \beta}{f_f^*} \right) \]
Affine WVD smoothing and constant-Q analysis

- Scalogram as smoothed WVD:

\[
SCAL_{\omega}(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} WVD_x(t', f') WVD_\psi \left( \frac{f}{f_\psi} \left( t' - t \right) \right) dt' df'
\]

\[
= \left| \int_{-\infty}^{\infty} x(t') \frac{f}{f_\psi} \psi^* \left( \frac{f}{f_\psi} \left( t' - t \right) \right) dt' \right|^2
\]

Constant-BW vs. affine (constant-Q) smoothing

Smaller/less interference terms

Poorer TF resolution

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• *Hyperbolic time-frequency localization*

Doppler-tolerant signals

• TF scaling / Doppler effect:
  
  \( (C_{\alpha} x)(t) = \frac{1}{\sqrt{\alpha}} x\left(\frac{t}{\alpha}\right) \leftrightarrow \sqrt{\alpha} X(\alpha f) \)

• "Doppler-tolerant" signal = eigenfunction of \( C_{\alpha} \):
  
  \( (C_{\alpha} x)(t) = \lambda_{\alpha} x(t) \)

• Solution: "hyperbolic impulse"
  
  \[ X(f) = H_{c}(f) = \frac{1}{\sqrt{f}} e^{-j2\pi c \ln(f/f_0)}, \quad f > 0, \quad c \in \mathbb{R} \]

• Group delay:
  
  \[ \tau(f) = -\frac{1}{2\pi} \frac{d}{df} \arg\{H_{c}(f)\} = \frac{c}{f} \]

Hyperbola in the TF plane
Example: Bat sonar signals

- HUNTING
- APPROACH
- PURSUIT
- CAPTURE

Source: P. Flandrin

Hyperbolic TF localization

- Want AC QTFR to satisfy hyperbolic TF localization property:

\[ X(f) = \frac{1}{\sqrt{f}} e^{-j2\pi \ln(f/f_0)} \Rightarrow \text{AC}_x(t, f) = \frac{1}{f} \delta \left( t - \frac{c}{f} \right) \]

- Not satisfied by WVD!
The Bertrand $P_0$ distribution

- The hyperbolic TF localization property is satisfied by the (unitary) Bertrand $P_0$ distribution

$$BER_x(t, f) = f \int_{-\infty}^{\infty} X(f \lambda(u)) X^*(f \lambda(-u)) e^{j2\pi ftu} \mu(u) \, du, \quad f > 0$$

with

$$\lambda(u) = \frac{e^{u/2} \, u/2}{\sinh(u/2)}, \quad \mu(u) = \frac{u/2}{\sinh(u/2)}$$

- The Bertrand $P_0$ distribution is a central member of the AC. It satisfies several important properties (besides the hyperbolic TF localization property).

Bertrand $P_0$ distribution as generator of the AC

- Any QTFR of the AC can be expressed in terms of the Bertrand $P_0$ distribution:

$$AC_x(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} BER_x(t', f') \tilde{\sigma} \left( f(f'-t), \frac{f'}{f} \right) \, dt' \, df'$$

where $\tilde{\sigma}(\alpha, \beta)$ is related to $\sigma(\alpha, \beta)$

- Special case: scalogram

$$SCAL_x(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} BER_x(t', f') \left( \frac{f}{f_\psi(t'-t)}, \frac{f'}{f/f_\psi} \right) \, dt' \, df'$$

Smoothing function is BER of wavelet:

$$\tilde{\sigma}(\alpha, \beta) = BER_{f_\psi} \left( \frac{\alpha}{f_\psi}, \frac{\beta}{f_\psi} \right)$$
Mellin transform and hyperbolic marginals

- Recall *hyperbolic impulse* \( H_c(f) = \frac{1}{\sqrt{f}} e^{-j2\pi c \ln(f/f_r)} \)

- Mellin transform:
  \[
  M_x(c) = \langle X, H_c \rangle = \int_{0}^{\infty} X(f) e^{j2\pi c \ln(f/f_r)} \frac{df}{\sqrt{f}}
  \]

- Hyperbolic marginal property:
  \[
  \int_{0}^{\infty} AC_x\left(\frac{c}{f}, f\right) \frac{df}{f} = |M_x(c)|^2
  \]
  Integrate \( AC_x(t, f) \) over TF hyperbola \( t=cf \)

- *Not* satisfied by WVD... but *satisfied* by Bertrand \( P_0 \) distribution!

Application: TF analysis of gravitational wave

Conclusion

- Linear and quadratic TF analysis
- Short-time Fourier transform and spectrogram
- Wavelet transform and scalogram
- Filterbank interpretation: constant-BW analysis versus constant-Q analysis
- Scaling/shift covariance and affine class of QTFRs
- Wigner-Ville distribution and affine smoothing
- Doppler tolerance and hyperbolic impulses
- Hyperbolic TF localization and Bertrand $P_0$ distribution
- Mellin transform and hyperbolic marginal property

WARNING

YOU ARE LEAVING THE TIME-FREQUENCY PLANE