NEW NUMERICAL INTEGRATORS BASED ON SOLVABILITY AND SPLITTING

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**New Numerical Integrators Based On Solvability And Splitting**

Universitat Jaume I, Castellón, Spain

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See also ADM001749, Lie Group Methods And Control Theory Workshop Held on 28 June 2004 - 1 July 2004., The original document contains color images.
...approach suggested by

Arieh Iserles
Outline of the talk

1. Some (well known) Lie group methods for linear problems (Fer and Magnus expansions).

2. Schemes based on triangular matrices (splitting + solvability).

3. Some methods and practical issues in their construction
1 Lie group methods (linear problems)

Let us consider a linear matrix ODE evolving in a Lie group $\mathcal{G}$

$$Y' = A(t)Y, \quad Y(t_0) = Y_0 \in \mathcal{G}$$

(0)

with $A : [t_0, \infty] \times \mathcal{G} \rightarrow \mathfrak{g}$ smooth enough.

$\mathfrak{g}$: Lie algebra associated with $\mathcal{G}$

Examples of $\mathcal{G}$: $\text{SL}(n)$, $\text{O}(n)$, $\text{SU}(n)$, $\text{Sp}(n)$, $\text{SO}(n)$, ...

$Y(t) \in \text{Lie group } \mathcal{G} \text{ if } A(t) \in \text{Lie algebra } \mathfrak{g}$

* There are several schemes preserving this feature (Magnus, Fer, Cayley,...)
1.1 Magnus expansion

For the equation

\[ Y' = A(t)Y, \quad Y(t_0) = I, \]

\[ \text{Magnus (1954) proposed} \]

\[ Y(t) = e^{\Omega(t)}, \quad \Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t) \]  

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\[ Y(t) = e^{\Omega(t)}, \quad \Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t) \quad (1) \]

with \( \log(Y(t)) \) satisfying

\[ \Omega' = d \exp_{\Omega}^{-1} A(t) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \text{ad}_\Omega^k A(t), \quad \Omega(t_0) = 0, \quad (2) \]
1.1 Magnus expansion (II)

Here

\[ \text{ad}_{\Omega}^0 A = A \]
\[ \text{ad}_{\Omega}^k A = [\Omega, \text{ad}_{\Omega}^{k-1} A] \]

\[ [\Omega, A] \equiv \Omega A - A\Omega \]

and \( B_k \) are Bernoulli numbers.
1.1 Magnus expansion (III)

First terms in the expansion \((A_i \equiv A(t_i))\):

\[
\begin{align*}
\Omega_1(t) &= \int_{t_0}^{t} A(t_1) dt_1 \\
\Omega_2(t) &= \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 [A_1, A_2] \\
\Omega_3(t) &= \frac{1}{6} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 ([A_1, [A_2, A_3]] + [A_3, [A_2, A_1]])
\end{align*}
\]

\(e^{\Omega(t)} \in G\) even if the series \(\Omega\) is truncated
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\[e^{\Omega(t)} \in G\text{ even if the series }\Omega\text{ is truncated}\]

* Expansion widely used in Quantum Mechanics, NMR spectroscopy, infrared divergences in QED, control theory,...
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1. Evaluation of $\exp(\Omega)$
   
   (Moler & Van Loan, Celledoni & Iserles,...)

2. Number of commutators involved in the expansion

To reduce this number is particularly useful the concept of **graded free Lie algebra** (Munthe-Kaas, Owren 1999)
As a result,
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* Efficient in applications
1.2 Other schemes

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* proposed (as an exercise!) by R. Bellman, ‘Introduction to Matrix Analysis’, 1960, page 204:

"The solution of \( \frac{dX}{dt} = Q(t)X, \ X(0) = I \), can be put in the form
\[ e^P e^{P_1} \ldots e^{P_n} \ldots, \]
where \( P = \int_0^t Q(s)\,ds \), and \( P_n = \int_0^1 Q_n\,ds \), with

\[ Q_n = e^{-P_{n-1}} Q_{n-1} e^{P_{n-1}} + \int_0^{-1} e^{s P_{n-1}} Q_{n-1} e^{-s P_{n-1}}\,ds \]

The infinite product converges if \( t \) is sufficiently small."

(See also Mathematical Reviews 21 2771, review done by R. Bellman)
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* This class of methods can actually be built from Magnus.

* They require the computation of several matrix exponentials.
1.3 Methods based on the Cayley transform

Let us suppose that $Y' = A(t)Y$ is defined in a $J$-orthogonal Lie group,

$$O_J(n) = \{ A \in \text{GL}_n(\mathbb{R}) : A^T J A = J \},$$

$J$: constant matrix
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$J$: constant matrix

Examples: orthogonal group ($J = I$), symplectic group, Lorentz group ($J = \text{diag}(1, -1, -1, -1)$).

Solution:

$$Y(t) = \left( I - \frac{1}{2}C(t) \right)^{-1} \left( I + \frac{1}{2}C(t) \right)$$
1.3 Methods based on the Cayley transform (II)

with $C(t) \in o_J(n)$ satisfying (Iserles 2001)

$$\frac{dC}{dt} = A - \frac{1}{2}[C,A] - \frac{1}{4}CAC, \quad t \geq t_0, \quad C(t_0) = 0.$$

⇒ efficient methods without matrix exponentials!
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* It is possible to construct methods which are more efficient than those based on the Cayley transform (Blanes, C., Ros 2002).
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* In some cases, if the exponential is approximated by rational functions the method does not preserve the Lie group structure, in particular, when $\mathcal{G} = \text{SL}(n)$

⇒⇒ Another class of methods is required.
2 Solvability + splitting

The procedure

For the linear system

\[ Y' = A(t)Y, \quad Y(0) = I, \]

we denote \( Y_0 \equiv Y, \ A_0 \equiv A \) and suppose that

\[ A_0(t) = A_{0+}(t) + A_{0-}(t), \]

where

\( A_{0+} \in \bigtriangledown_n \) is strictly upper-triangular

\( A_{0-} \in \bigtriangleup_n \) is weakly lower-triangular.
The idea is to write the solution as a product of upper and lower triangular matrices.
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More specifically, we propose the following factorization:

\[ Y_0(t) = L_0(t)U_0(t)Y_1(t) \]

such that

\[ L'_0 = A_0(t)L_0, \quad L_0(0) = I \]
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$$Y_0(t) = L_0(t)U_0(t)Y_1(t)$$

such that

$$L_0' = A_0(t)L_0, \quad L_0(0) = I$$

Observe then that $L_0(t)$ can be obtained by quadratures and $L_0(t) \in \tilde{\Delta}_n$. 
Now we form the matrix

\[ C_0 = L_0^{-1} A_0 + L_0 \]
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which can also be split as

\[ C_0(t) = C_{0+}(t) + C_{0-}(t), \]

where

\[ C_{0+} \in \tilde{\nabla}_n \] is weakly upper-triangular

\[ C_{0-} \in \triangle_n \] is strictly lower-triangular.
Next we choose $U_0$ as the solution of

$$U'_0 = C_0^+(t) U_0, \quad U_0(0) = I$$

so that $U_0(t)$ can also be obtained by quadratures.
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It is easy to show that $Y_1$ satisfies

$$Y'_1 = A_1(t)Y_1, \quad Y_1(0) = I,$$

with

$$A_1 = U_0^{-1}C_0U_0.$$
This gives a single step of the *solvable cycle*, which we repeat with $A_1$. 

\[ A_1 = A_{1+} + A_{1-}, \quad A_{1+} \in \nabla_n, \quad A_{1-} \in \tilde{\Delta}_n \]

\[ Y_1 = L_1 U_1 Y_2 \]

\[ L'_1 = A_{1-} L_1, \quad L_1(0) = I \]

etc.
In this way one has the following algorithm:

\[ Y \equiv Y_0 = L_0 U_0 L_1 U_1 \cdots L_k U_k Y_{k+1} \]

with \((k = 0, 1, 2, \ldots)\)

\[ A_k = A_{k+} + A_{k-}, \quad A_{k+} \in \nabla_n, \quad A_{k-} \in \tilde{\Delta}_n \]

\[ L'_k = A_{k-} L_k, \quad L_k(0) = I \]

\[ C_k \equiv L_k^{-1} A_{k+} L_k = C_{k+} + C_{k-} \]

\[ C_{k+} \in \tilde{\nabla}_n, \quad C_{k-} \in \Delta_n \]

\[ U'_k = C_{k+} U_k, \quad U_k(0) = I \]
and finally

\[ A_{k+1} \equiv U_k^{-1} C_k U_k, \quad Y'_{k+1} = A_{k+1} Y_{k+1} \]
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Usually the factorization is truncated by taking \( Y_{k+1} = I \).
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Usually the factorization is truncated by taking \( Y_{k+1} = I \).

In what follows we will analyse the main features of this procedure as a numerical integrator.
2.1 Order of the method

Suppose that \( A(t) = \varepsilon(a_0 + a_1 t + a_2 t^2 + \cdots) \) for some parameter \( \varepsilon > 0 \).
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Suppose that $A(t) = \varepsilon(a_0 + a_1 t + a_2 t^2 + \cdots)$ for some parameter $\varepsilon > 0$.

Then

\[
A_{j-} = t^{n_j} \varepsilon^{n_j} (\varepsilon \alpha_1 + t (\varepsilon \alpha_2 + \varepsilon^2 \alpha_3) + O(t^2))
\]
\[
A_{j+} = t^{m_j} \varepsilon^{m_j} (\varepsilon \beta_1 + t (\varepsilon \beta_2 + \varepsilon^2 \beta_3) + O(t^2))
\]

for $j = 1, 2, \ldots$, so that

\[
L_j(t) = I + \frac{1}{n_j + 1} (t \varepsilon)^{n_j+1} \alpha_1 + \frac{1}{n_j + 2} t^{n_j+2} \varepsilon^{n_j} (\varepsilon \alpha_2 + \varepsilon^2 \alpha_3) + \cdots
\]
\[
U_j(t) = I + \frac{1}{m_j + 1} (t \varepsilon)^{m_j+1} \beta_1 + \frac{1}{m_j + 2} t^{m_j+2} \varepsilon^{m_j} (\varepsilon \beta_2 + \varepsilon^2 \beta_3) + \cdots
\]
Furthermore,

\[ n_{j+1} = n_j + m_j + 1 \]
\[ m_{j+1} = n_j + 2m_j + 2 \quad j = 1, 2, \ldots \]
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2.1 Order of the method (III)

In consequence,
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(1) This algorithm could be useful for problems of the form

\[ Y' = (B_0 + \epsilon B_1)Y \]

if the system \( Y' = B_0 Y \) can be solved exactly.
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\[ Y' = (B_0 + \epsilon B_1)Y \]

if the system \( Y' = B_0 Y \) can be solved exactly.

(2) The order of approximation is...
### 2.1 Order of the method (IV)

\[ Y_0 \approx L_0 U_0 \quad \text{is order 1} \]
\[ Y_0 \approx L_0 U_0 L_1 \quad \text{2} \]
\[ Y_0 \approx L_0 U_0 L_1 U_1 \quad \text{4} \]
\[ Y_0 \approx L_0 U_0 L_1 U_1 L_2 \quad \text{7} \]
\[ Y_0 \approx L_0 U_0 L_1 U_1 L_2 U_2 \quad \text{12} \]
\[ Y_0 \approx L_0 U_0 L_1 U_1 L_2 U_2 L_3 \quad \text{20} \]
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With only 4 solvable cycles we get order 33!
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...if we can compute \( L_k \) and \( U_k \) up to this order...
2.2 Questions

Several problems involved
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(1) Does the approximate solution evolve in the Lie group if $A$ is in the Lie algebra, i.e., is it a Lie group method?
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(1) Does the approximate solution evolve in the Lie group if \( A \) is in the Lie algebra, i.e., is it a Lie group method?

(2) Solve explicitly the systems \( L'_k = A_k L_k \) and \( U'_k = C_k U_k \)
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(1) Does the approximate solution evolve in the Lie group if $A$ is in the Lie algebra, i.e., is it a Lie group method?

(2) Solve explicitly the systems $L'_k = A_k - L_k$ and $U'_k = C_k + U_k$

(3) Approximate efficiently the (multiple) integrals involved.
(1) Preservation of the Lie-group structure

If $A(t) \in \mathfrak{sl}(n)$, the algorithm provides by construction approximations to $Y(t)$ in $\text{SL}(n)$. 
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Proof. $A_k = A_{k+} + A_{k-}$, with $A_{k+} \in \nabla_n, A_{k-} \in \tilde{\Delta}_n$. In fact $A_{k-}$ belongs to a solvable subalgebra of $\mathfrak{sl}(n)$. Therefore the solution of

$$L'_k = A_{k-} L_k, \quad L_k(0) = I$$

$L_k(t) \in \text{SL}(n)$ (in fact, a solvable subgroup of).
3 Practical issues

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If \( A(t) \in \mathfrak{sl}(n) \), the algorithm provides by construction approximations to \( Y(t) \) in \( \text{SL}(n) \).

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\[
L'_k = A_{k-} L_k, \quad L_k(0) = I
\]

\( L_k(t) \in \text{SL}(n) \) (in fact, a solvable subgroup of).

\( \text{Tr}(A_{k+}) = 0 \), and the trace is invariant under similarity, so that

\[
\text{Tr}(C_k) = \text{Tr}(L_k^{-1} A_{k+} L_k) = \text{Tr}(A_{k+}) = 0 \quad \Rightarrow \quad C_k \in \mathfrak{sl}(n)
\]
Next, $C_k = C_{k+} + C_{k-}$, with $C_{k+} \in \tilde{n}$, $C_{k-} \in n$ and $U_k$, solution of

$$U_k' = C_{k+} U_k, \quad U_k(0) = I$$

belongs to $SL(n)$. Finally

$$A_{k+1} \equiv U_k^{-1} C_{k-} U_k \in sl(n)$$

and the process is repeated.
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Other properties (i.e., orthogonality) are preserved only up to the order of the method.
3 Practical issues (III)

(2a) Explicit solution of $L'_k = A_k L_k$
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Consider $k = 0$ and denote $A_0(t) = (a_{ij})$, $i, j = 1, \ldots, n$, $L_0(t) = (L_{ij})$, $j \leq i$

$$A_{ii}(t) \equiv \int_0^t a_{ii}(t_1)dt_1.$$ 

Then the solution of $L'_0 = A_0(t)L_0$, $L_0(0) = I$ is

$$L_{ii}(t) = e^{A_{ii}(t)}, \quad i = 1, \ldots, n$$

$$L_{ij}(t) = e^{A_{ii}(t)} \int_0^t e^{-A_{ii}(t_1)} \left( \sum_{k=j}^{i-1} a_{ik}(t_1)L_{kj}(t_1) \right) dt_1$$

$i = 2, \ldots, n$, $j = 1, \ldots, i - 1$. 
3 Practical issues (IV)

(2b) Explicit solution of $U'_k = C_k U_k$
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Consider $k = 0$ and denote $C_0(t) = (c_{ij}), i, j = 1, \ldots, n$, $U_0(t) = (U_{ij})$, $j \geq i$

$$C_{ii}(t) \equiv \int_0^t c_{ii}(t_1) dt_1.$$ 

Then the solution of $U'_0 = C_0^+(t)U_0$, $U_0(0) = I$ is

$$U_{ii}(t) = e^{C_{ii}(t)}, \quad i = 1, \ldots, n$$

$$U_{ij}(t) = e^{C_{ii}(t)} \int_0^t e^{-C_{ii}(t_1)} \left( \sum_{k=i+1}^j c_{ik}(t_1) U_{kj}(t_1) \right) dt_1$$

$i = 1, \ldots, n - 1, \ j = i + 1, \ldots, n.$
3 Practical issues (V)

⇒ Explicit expressions for the elements of $L_k$ and $U_k$ in terms of multivariate integrals.
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They can be evaluated in sequence as follows:

<table>
<thead>
<tr>
<th>$L_{ii}$</th>
<th>$i = 1, \ldots, n$</th>
<th>$U_{ii}$</th>
<th>$i = 1, \ldots, n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{i,i-1}$</td>
<td>$i = 2, \ldots, n$</td>
<td>$U_{i,i+1}$</td>
<td>$i = 1, \ldots, n-1$</td>
</tr>
<tr>
<td>$L_{i,i-2}$</td>
<td>$i = 3, \ldots, n$</td>
<td>$U_{i,i+2}$</td>
<td>$i = 1, \ldots, n-2$</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>$L_{n1}$</td>
<td></td>
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In principle, the integrals appearing in $L_k$ and $U_k$ can be approximated by quadrature rules.
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**Question:** Is it possible to approximate all the nested integrals with the evaluations required to compute

$$A_{ii} = \int_{0}^{t} a_{ii}(t_1)dt_1,$$

i.e., à la Magnus?
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\textbf{Question}: Is it possible to approximate \textit{all} the nested integrals with the evaluations required to compute

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\textbf{YES!}
3.1 Example

Illustration: method of order 4 with 2 $A$ evaluations

Step $t = 0 \rightarrow t = h$. 
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Illustration: method of order 4 with 2 \( A \) evaluations

Step \( t = 0 \rightarrow t = h \).

1- Approximate \( A_{ii}(h), i = 1, \ldots, n \) up to order 4

\[
A_{ii}(h) = \int_{0}^{h} a_{ii}(t) dt = \frac{h}{3} (a_{ii}(0) + 4a_{ii}(h/2) + a_{ii}(h)) + O(h^5)
\]

\[
\equiv \tilde{A}_{ii}(h) + O(h^5)
\]

and \( A_{ii}(h/2), i = 1, \ldots, n - 1 \), up to order 3 (necessary to approximate \( L_{ij} \)):

\[
A_{ii}(h/2) = \frac{h}{24} (5a_{ii}(0) + 8a_{ii}(h/2) - a_{ii}(h)) + O(h^4)
\]
3.1 Example (II)

2- \( L_{ii}(h) = \exp(\tilde{A}_{ii}(h)) + O(h^5) \) \((i = 1, \ldots, n)\) and \( L_{ii}(h/2) = \exp(\tilde{A}_{ii}(h/2)) + O(h^4) \) \((i = 1, \ldots, n - 1)\).

3- Obtain an approximation to \( L_{ij}(h), j < i \), of order 4 and \( L_{ij}(h/2) \) of order 3

\[
L_{ij}(h) = e^{A_{ii}(h)} \int_0^h F_{ij}(t) dt
\]

with

\[
F_{ij}(t) \equiv e^{-A_{ii}(t)} \sum_{k=j}^{i-1} a_{ik}(t)L_{kj}(t)
\]
3.1 Example (III)

Then

\[ L_{ij}(h) = e^{\tilde{A}_{ii}(h)} \frac{h}{3} (F_{ij}(0) + 4F_{ij}(h/2) + F_{ij}(h)) + O(h^5) \]

where \( F_{ij}(0) = a_{ij}(0) \) and \( F_{ij}(h/2) \) and \( F_{ij}(h) \) have to be approximated up to order \( h^3 \).

The sequence of computation is \( (i = 2, \ldots, n) \):

(a) \[ F_{i,i-1}(h/2) = e^{-\tilde{A}_{ii}(h/2)} a_{i,i-1}(h/2) L_{i-1,i-1}(h/2) + O(h^4) \]

(b) \[ F_{i,i-1}(h) = e^{-\tilde{A}_{ii}(h/2)} a_{i,i-1}(h) L_{i-1,i-1}(h) + O(h^5) \]

(c) \( L_{i,i-1}(h), \ i = 2, \ldots, n \) up to order 4
3.1 Example (IV)

(d) 
\[ L_{i,i-1}(h/2) = e^{\tilde{A}_{ii}(h/2)} \frac{h}{24} (5a_{i,i-1}(0) + 8F_{i,i-1}(h/2) - F_{i,i-1}(h)) + O(h^4) \]

(e) \[ L_{i,i-2}(h), \ i = 3, \ldots, n, \ \text{up to order 4 and} \ L_{i,i-2}(h/2) \ \text{up to order 3} \]

...and so on.
3.1 Example (IV)

(d) 
\[ L_{i,i-1}(h/2) = e^{\tilde{A}_{ii}(h/2)} \frac{h}{24} (5a_{i,i-1}(0) + 8F_{i,i-1}(h/2) - F_{i,i-1}(h)) + O(h^4) \]

(e) \( L_{i,i-2}(h), i = 3, \ldots, n, \) up to order 4 and \( L_{i,i-2}(h/2) \) up to order 3...and so on.

In this way we have \( L_0(h) \) computed up to order \( O(h^5) \) and also \( L_0(h/2) \) up to order \( O(h^4) \) with 2 evaluations of \( A(t) \).
3.1 Example (V)

3- Next we compute $C_0$:

$$C_0(0) = A_{0+}(0) \quad \text{error } O(h^5)$$

$$C_0(h/2) = L_0^{-1}(h/2)A_{0+}(h/2)L_0(h/2) \quad \text{error } O(h^4)$$

$$C_0(h) = L_0^{-1}(h)A_{0+}(h)L_0(h) \quad \text{error } O(h^5)$$

4- $C_{ii}(h) = \frac{h}{3} (c_{ii}(0) + 4c_{ii}(h/2) + c_{ii}(h)) + O(h^5)$

$$C_{ii}(h/2) = \frac{h}{24} (5c_{ii}(0) + 8c_{ii}(h/2) - c_{ii}(h)) + O(h^4)$$
3.1 Example (VI)

5- $U_{i,i+1}(h)$, $i = 1, \ldots, n-1$, up to order $O(h^5)$;

$U_{i,i+1}(h/2)$, $i = 1, \ldots, n-1$, up to order $O(h^4)$;

$U_{i,i+2}(h)$, $i = 1, \ldots, n-2$, up to order $O(h^5)$;

$U_{i,i+2}(h)$, $i = 1, \ldots, n-2$, up to order $O(h^4)$;

... and so on.

Thus we compute $U_0(h)$ with error $O(h^5)$ and also $U_0(h/2)$ with error $O(h^4)$. 
3.1 Example (VII)

6- $A_1$:

$$A_1(0) = C_{0-}(0) \quad \text{error } O(h^5)$$

$$A_1(h/2) = U_0^{-1}(h/2) C_{0-}(h/2) U_0(h/2) \quad \text{error } O(h^4)$$

$$A_1(h) = U_0^{-1}(h) C_{0-}(h) U_0(h) \quad \text{error } O(h^5)$$

... and the process is repeated again for the second cycle
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A_1(0) = C_{0-}(0) \quad \text{error } O(h^5)
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A_1(h) = U_0^{-1}(h)C_{0-}(h)U_0(h) \quad \text{error } O(h^5)
\]

... and the process is repeated again for the second cycle

$\Rightarrow$ it is possible to construct a method of order 4 with only 2 $A(t)$ evaluations (3 for the first step).
3.2 Other possibilities

One could use other quadrature rules instead, for instance Gauss–Legendre, but...
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Solution: use G–L with matrix evaluations in the previous/next step.
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**Solution:** use G–L with matrix evaluations in the previous/next step.

⇒ method of order 4 with 2 evaluations (and 1 in the next step)
3.3 Some methods

Order 4

\[ Y \approx L_0 U_0 L_1 U_1 \]

- Quadratures NC / GL, 2 matrix evaluations per step

Order 6

\[ Y \approx L_0 U_0 L_1 U_1 L_2 \]

- order 6 with a 5 points NC quadrature (4 evaluations per step)
- order 7 with a 7 points NC (6 evaluations)

Order 12

\[ Y \approx L_0 U_0 L_1 U_1 L_2 U_2 \]

- with a 11 points NC (or GL involving several steps)
3.4 Variable step size

Local extrapolation technique is trivial to implement in this setting.
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For instance,

\[ Y_1 \equiv L_0 U_0 L_1 \]

\[ \hat{Y}_1 \equiv L_0 U_0 L_1 U_1 = Y_1 U_1 \]

Then

\[ \hat{Y}_1 - Y_1 = Y_1 U_1 - Y_1 = Y_1 (U_1 - I) \]

and \( \| \hat{Y}_1 - Y_1 \| \) can be used as a measure of the error.
3.5 Future work

* Analyse the convergence of the procedure
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* Analyse the convergence of the procedure
* Consider numerical examples in $SL(n)$ with (very) large $n$
* Highly oscillatory problems (with special quadratures)
* Analyse in practice the preservation of other structures (Blanes & Moan)
* Try to generalize to nonlinear problems
The End

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