Control Theoretic Aspects of Matrix Factorizations

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Joint work with G. Dirr and M. Kleinsteuber.

DAAD-Project: PPP Hong Kong, D/0122045;

partially supported by Marie-Curie CTS Fellowship

Report Documentation Page

Report Documentation Page	Form Approved OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
03 JAN 2005	**N/A**	-

4. TITLE AND SUBTITLE **Control Theoretic Aspects of Matrix Factorizations**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **University of W¨urzburg, Mathematical Institute, Germany**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release, distribution unlimited

13. SUPPLEMENTARY NOTES
See also ADM001749, Lie Group Methods And Control Theory Workshop Held on 28 June 2004 - 1 July 2004., The original document contains color images.

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			
unclassified	**unclassified**	**unclassified**	**UU**	**51**	

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Motivation

- Quantum Computing
- Quantum Control, Control of Spin Systems
- Control of Numerical Algorithms
- Constructive Controllability, Motion Planning in Robotics

Time-optimal Factorization Problem

- G compact connected Lie group with Lie Algebra $\mathfrak{g}$
- $\omega := \{\Omega_1^+, \ldots, \Omega_r^+, \Omega_1^-, \ldots, \Omega_s^-\}$ finite set of LA generators of $\mathfrak{g}$
- Ω_i^+: "slow, cost expensive" directions
  Ω_i^-: "fast, cheap" directions
- Given $X \in G$, define

$$T_{\min}(X) = \inf \left\{ \sum_i |t_i^+| \;\middle|\; X = \prod_{\text{finite}} e^{t_i^\pm \Omega_i^\pm} \right\}$$

Problem:

- Is $T_{\min} < \infty$ always? Compute $T_{\min}$!
- When does there exist a *finite, time-optimal* factorization?

Example

Optimal Condition Numbers

- $G = GL(n)$ general linear group of invertible matrices
- $\omega := \{\Omega_1^+, ..., \Omega_r^+, \Omega_1^-, ..., \Omega_s^-\}$ finite set of LA generators of $\mathfrak{gl}(\mathfrak{n})$
- Ω_i^+: "hyperbolic Jacobi rotations"
  Ω_i^-: "standard Jacobi directions"
- Given $X \in G$, define (κ denotes the condition number)

$$T_{\min}(X) = \inf \left\{ \sum_i \kappa(\mathrm{e}^{t_i^+ \Omega_i^+})| \mid X = \prod_{\text{finite}} \mathrm{e}^{t_i^\pm \Omega_i^\pm} \right\}$$

Problem:

- This factorization task with minimal total condition number!
- Does there exists factorization with better condition numbers than for X?

Lie Groups & Lie Algebras

Intermezzo: Lie Groups and Lie Algebras

Example. General linear group of invertible $n \times n$ matrices

$$GL(n,\mathbb{R}) := \{X \in \mathbb{R}^{n\times n} |\ \det X \neq 0\}.$$

Definition. A matrix *Lie group* is any subgroup $G \subset GL(n,\mathbb{R})$ that is also a (locally closed) submanifold of $\mathbb{R}^{n\times n}$.

Intermezzo: Lie Groups and Lie Algebras

Examples, cont'd:

(a) The *real orthogonal group*

$$O(n) := \{X \in \mathbb{R}^{n\times n} |\ XX^\top = I_n\}$$

(b) The *special unitary group*

$$SU(n) := \{X \in \mathbb{C}^{n\times n} |\ XX^* = I_n, \det X = 1\}$$

(c) The *Euclidean group*

$$E(n) := \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \middle| \ R \in O(n), p \in \mathbb{R}^n \right\}.$$

The first two examples are compact groups, while the third is not.

Intermezzo: Lie Groups and Lie Algebras

Definition. A vector space V with a bilinear operation $[\ ,\ ] : V \times V \to V$ satisfying

(i) $[x, y] = -[y, x]$

(ii) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (Jacobi Identity)

is called a *Lie Algebra*.

Intermezzo: Lie Groups and Lie Algebras

- Lie algebras are the tangent spaces of Lie groups.

- **Theorem.** Let $G \subset GL(n, \mathbb{R})$ be a matrix Lie group. Then the tangent space $\mathfrak{g} := T_I G$ at the identity matrix is a Lie algebra with commutator as the Lie bracket:

$$[X, Y] = XY - YX.$$

Intermezzo: Lie Groups and Lie Algebras

Examples

(a) The Lie algebra of $O(n)$ is

$$\mathfrak{o}(n) := \{\Omega \in \mathbb{R}^{n\times n} |\ \Omega^\top = -\Omega\}.$$

(b) The Lie algebra of $SU(n)$ is

$$\mathfrak{su}(n) := \{\Omega \in \mathbb{C}^{n\times n} |\ \Omega^* = -\Omega, \mathrm{tr}\Omega = 0\}$$

(c) The Lie algebra of $E(n)$ is

$$\mathfrak{e}(n) := \left\{ \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix} \middle|\ \Omega^\top = -\Omega, v \in \mathbb{R}^n \right\}.$$

Control on Lie Groups

Control on Lie Groups

- G Lie Group with Lie Algebra $\mathfrak{g}$.
- Bilinear control system on G

$$(\Sigma) \qquad \dot{X}(t) = \left(A_d + \sum_{j=1}^{m} u_j(t) A_j\right) X(t), \;\; X(0) = I,$$

where $A_d, A_1, ..., A_m \in \mathfrak{g}$.

- Reachable Set at time $T > 0$

$$\mathcal{R}(T) = \{X_F \in G | \; \exists u_1, ..., u_m \text{ and } s \leq T : X(s) = X_F\}$$

- Reachable Set

$$\mathcal{R} = \cup_T \mathcal{R}(T)$$

Control on Lie Groups

Definition

- **Accessibility:** The reachable set $\mathcal{R}(T)$ has an interior point
- **Local Controllability:** The identity $I \in \mathcal{R}(T)$ is an interior point
- **Controllability:** For any $X_F \in G$ there exist controls $u_1(\cdot), ..., u_m(\cdot)$ and $T > 0$ s.t. the solution of (Σ) satisfies $X(0) = I, X(T) = X_F$.

Control on Lie Groups

Problem 1 (Accessibility)

- Definition (*System Lie Algebra*)

  $\mathcal{L} :=$ smallest Lie subalgebra of $\mathfrak{g}$, containing $A_1, ..., A_m, A_d$

  Generators: $([A, B] = AB - BA)$

  $$A_d, A_1, ..., A_m, [A_d, A_i], [A_i, A_j], [A_d, [A_i, A_j]], ...$$

- **Theorem.** (Σ) is accessible if and only if the system Lie algebra is $\mathcal{L} = \mathfrak{g}$.

Control on Lie Groups

- **Theorem** (Lian et al. 1994) Suppose
  (i) For some constant controls $u_1, ..., u_m$
  $$(\Sigma_{const}) \quad \dot{X} = (A_d + \sum_j u_j A_j)X$$
  is weakly positively Poisson stable.
  (ii) The system Lie algebra $\mathcal{L}$ satisfies $\mathcal{L} = \mathfrak{g}$.

  Then the bilinear control system is controllable.

Accessability $+$ **Poisson Stability** $\Rightarrow$ **Controllability**

Control on Lie Groups

Definition (Poisson Stability)

Flow of (Σ_{const}): $\Phi : G \times \mathbb{R} \to G;\ (z,t) \mapsto \Phi(z,t)$

- (Σ_{const}) is **Weakly Positively Poisson Stable** if for all $z \in G$, any neighborhood $B(z)$ of z and all $T > 0$, there exists $t > T$ such that $\Phi(U_z, t) \cap B(z) \neq \emptyset$.

Examples: a swing (no damping), satellite attitude, ball rolling in a bowl.

Control on Lie Groups

- Theorem (Jurdjevic-Sussmann) Assume:

  (i) There exist constant controls such that $A_d + \sum_j u_j A_j$ lies in a **compact** subalgebra $\mathfrak{k}$ of $\mathfrak{g}$.

  (ii) The system Lie algebra $\mathcal{L}$ satisfies $\mathcal{L} = \mathfrak{g}$.

  Then the system (Σ) is controllable.

Control on Lie Groups

- Corollary
  Let G be a **compact** connected Lie group. Then (Σ) is controllable if and only if
  $$\mathcal{L} = \mathfrak{g}.$$

Time-Optimal Control on Lie Groups

Time-Optimal Control on Lie Groups

General Notation:

- Let G be a compact Lie Group with Lie algebra $\mathfrak{g}$; $K \subset G$ a compact connected Lie subgroup with LA $\mathfrak{k}$. Consider the bilinear control system on G

$$(\Sigma) \qquad \dot{X} = \left(A_d + \sum_{j=1}^{m} u_j A_j\right) X, \quad X(0) = I$$

  with $A_d \in \mathfrak{g}, A_1, \ldots, A_m \in \mathfrak{k}$.

- Assumption:
  - Σ is controllable, i.e. $\mathfrak{g}$ =LA generated by $A_d, A_1, \ldots, A_m$
  - $\mathfrak{k}$ = LA generated by $A_1, \ldots, A_m$

Time-Optimal Control on Lie Groups

- Given: Initial state $X_0 = I$, Final state $X_F \in G$
- Problem 1.Find controls $u_1(\cdot), ..., u_m(\cdot)$ s.t. the corresponding solution $X(t)$ of (Σ) satisfies

$$X(0) = X_0, \; X(T) = X_F \quad \text{for some} \quad T > 0$$

- Problem 2.If problem 1 has at least one solution, then find a time-optimal one, i.e. one with *minimal* $T = T_{\text{opt}}(X_F)$.
- Problem 1 is always solvable, provided (Σ) is controllable!

Time-Optimal Control on Lie Groups

Fast versus slow directions

- A_d is called the *drift term*, $A_1, ..., A_m$ the *fast directions*
- Fact 1. If $A_d = 0$ and (Σ) controllable, then can control to X_F in *arbitrarily small time*: $T_{\text{opt}}(X_F) = 0$, always!
- Fact 2. The presence of drift term $A_d \neq 0$ is responsible for $T_{\text{opt}} > 0$.
- Idea: Factor out fast directions!

Time-Optimal Control on Lie Groups

Quotient System and Equivalence Principle

- Consider the quotient space

$$G/K := \{Kg \mid g \in G\}$$

of left co-sets Kg, $\quad K = \exp(\mathfrak{k})$ Lie Group generated by fast controls.

- G/K is a smooth manifold

Time-Optimal Control on Lie Groups

Example: (NMR)

- For the NMR Schrödinger Equation on $G = SU(2^N)$

$$\dot{X} = -\mathrm{i}\left(H_d + \sum_{j=1}^{2N} u_j H_j\right) X, \quad X(0) = I$$

$\mathfrak{k} :=$ LA generated by $\mathrm{i}H_1, ..., \mathrm{i}H_{2N}$
$K := \exp(\mathfrak{k})$ compact, connected Lie subgroup of $SU(2^N)$, generated by $\exp(\mathrm{i}tH_j), t \in \mathbb{R}, j = 1, ..., 2N$.

One verifies $K = SU(2) \otimes ... \otimes SU(2)$

- For $N = 1 : K = SU(2) = G$
- For $N = 2 : K = SU(2) \otimes SU(2) \simeq SO(4) \subset SU(4)$

Time-Optimal Control on Lie Groups

Quotient System and Equivalence Principle

- The *quotient system* of

$$(\Sigma) \qquad \dot{X} = \Big(A_d + \sum_{j=1}^{m} u_j A_j\Big) X, \quad X(0) = I, \quad X(T) = X_F$$

is the control system on G/K

$$(\Sigma/K) \qquad \dot{P} = \mathrm{Ad}_{U(t)}(A_d) P, \quad P(0) = K, \quad P(T) = K X_F$$

$\mathrm{Ad}_g(A_d) = g A_d g^{-1}$, $g \in K$. The control functions for (Σ/K) are arbitrary L^1_{loc} functions $t \mapsto U(t) \in K$.

Time-Optimal Control on Lie Groups

Quotient System and Equivalence Principle

- Theorem (Equivalence Principle).

  (Σ) is controllable on G iff (Σ/K) is controllable on G/K. Moreover, the optimal times on G and G/K coincide.

$$T_{\mathrm{opt}}^{G}(X_F) = T_{\mathrm{opt}}^{G/K}(KX_F)$$

  Proof: PhD thesis by Khaneja

- The optimal time $T_{\mathrm{opt}}^{G/K}$ has an interpretation within Sub-Riemannian Geometry.

Time-Optimal Control on Lie Groups

Sub-Riemannian Geometry

- Let M be a Riemannian manifold, $E \subset TM$ a constant dimensional subbundle that satisfies the *Hörmander Condition*

  For any $p \in M$, the LA of the sections of E evaluated in p is equal to T_pM (controllability cond.)

- For any two points $x, y \in M$, the *Sub-Riemannian distance* is

$$d(x,y) := \inf\Big\{ \int_0^1 ||\dot{\alpha}(t)||dt \mid \alpha(0) = x, \alpha(1) = y, \dot{\alpha}(t) \in E_{\alpha(t)} \Big\}.$$

- Example: $M = G/K, E_p := \mathrm{span}\{kA_dk^{-1} \mid k \in K\}P$, $P \in M$ satisfies the Hörmander Cond. (Equivalence principle)
- NMR: $M = SU(2^N)/SU(2) \otimes ... \otimes SU(2)$ Sub-Riemannian space

Time-Optimal Control on Lie Groups

Sub-Riemannian Geometry

- **Theorem.**

$$T_{\rm opt}^{G/K}(KX_F) = d(K, KX_F)$$

  Sub-Riemannian distance

- **Remark.** The Sub-Riemannian distance $d(x,y)$ is greater than or equal the Riemannian distance on G/K:

$$d(x,y) \geq \text{ geodesic distance between } x, y$$

- There is one case where these distances are equal: *Riemannian symmetric spaces.*

Time-Optimal Control on Lie Groups

Sub-Riemannian Geometry

- **Theorem.** If G/K is a Riemannian Symmetric Space, then

  $$T_{\mathrm{opt}}(X_F) = \text{length of a geodesic in } G/K \text{ that connects } K \text{ with } KX_F$$

- **Main Advantage:** Riemannian distances (i.e. lengths of geodesics) are much easier to compute than Sub-Riemannian distances.

Time-Optimal Control on Lie Groups

- **Theorem.** The homogenous space G/K is a Riemannian symmetric space, provided $(\mathfrak{g}, \mathfrak{k})$ is a Cartan-pair, i.e. $\mathfrak{g}$ is semisimple and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{p} := \mathfrak{k}^{\perp}$$

satisfies

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

Time-Optimal Control on Lie Groups

Riemannian Symmetric Spaces

- $SU(n)/SO(n)$ is a Riemannian Symmetric Space
- $SU(4)/SU(2) \otimes SU(2)$ is a Riemannian Symmetric Space (good! 2-Spin Case)
- $SU(8)/SU(2) \otimes SU(2) \otimes SU(2)$ is *NOT* a Riemannian Symmetric Space (bad!)

Time-Optimal Factorization

Time-optimal Factorization

- Let G be a connected, compact Lie group with Lie algebra $\mathfrak{g}$.
- Let $K \subset G$ be a connected compact subgroup with Lie algebra $\mathfrak{k}$.
- Let $\Delta \in \mathfrak{g}$ be a drift term s.t. $\langle \Delta, \mathfrak{k} \rangle_L = \mathfrak{g}$.
- Consider the discrete control System:

$$(\Sigma_d) \qquad X_{n+1} = K_n \mathrm{e}^{t_n \Delta} L_n X_n, \quad X_0 = I \quad K_n, L_n \in K, t_n \geq 0.$$

For $X \in G$ let $T^d_{\mathrm{opt}}(X) :=$

$$\inf \Big\{ \sum_{n=1}^{\infty} t_n \,\Big|\, \exists\, (K_n, L_n, t_n) : \prod_{n=1}^{\infty} K_n \mathrm{e}^{t_n \Delta} L_n = X \Big\}.$$

Time-optimal Factorization

Problem:

- Is (Σ_d) controllable, i.e. does $T^d_{\mathrm{opt}}(X) < \infty$ hold for all $X \in G$?
- Determine the "minimal" time $T^d_{\mathrm{opt}}(X)$ for $X \in G$.

Time-optimal Factorization

Generalized Version (multiple drifts)

- G compact connected Lie group with LA $\mathfrak{g}$
- $\omega := \{\Omega_1^+, ..., \Omega_r^+, \Omega_1^-, ..., \Omega_s^-\}$ finite set of LA generators of $\mathfrak{k}$
- Ω_i^+: "slow, cost expensive" directions
  Ω_i^-: "fast, cheap" directions
- Given $X \in G$, define

$$T_{\min}(X) = \inf \left\{ \sum_i |t_i^+| \;\Big|\; X = \prod_{\text{finite}} \mathrm{e}^{t_i^\pm \Omega_i^\pm} \right\}$$

Time-optimal Factorization

Problem

- Is $T_{\min} < \infty$ always? Compute $T_{\min}$!
- When does there exist a *finite, time-optimal* factorization?

Time-optimal Factorization

Example 1 (Euler Angles)

- $SO(3)$, $\omega = \{\Omega_1^+, \Omega_1^-\}$,

$$\Omega_1^+ := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \Omega_1^- := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- *Euler Angles*:

$$X = \mathrm{e}^{\theta_1 \Omega_1^-} \mathrm{e}^{\theta_2 \Omega_1^+} \mathrm{e}^{\theta_3 \Omega_1^-}, \quad \theta_i \in [-\pi, \pi]$$

- We will show: Euler Angles are time-optimal and

$$T_{\min} = |\theta_2| \in [0, \pi]$$

Time-optimal Factorization

Example 2 (Euler Angles)

- $SO(3)$, $\omega = \{\Omega_1^+, \Omega_2^+\}$, $\Omega_2^+ := \Omega_1^-$
- Then Euler angles are i.g. *NOT* time-optimal:

$$T_{\min} < \theta_1 + \theta_2 + \theta_3 \ ! \quad \text{(Mittenhuber)}$$

Time-optimal Factorization

Equivalence Principle

- Let G be a connected, compact Lie group with Lie algebra $\mathfrak{g}$.
- Let $\mathfrak{k} := \langle A_1, \ldots, A_m \rangle_L$, $K := \exp \mathfrak{k}$.
- Let $\Delta \in \mathfrak{g}$ be a drift term such that $\langle \Delta, \mathfrak{k} \rangle_L = \mathfrak{g}$.
- Theorem.

  (a) The discrete control system (Σ_d) on G is controllable and thus $T^d_{\mathrm{opt}}(X) < \infty$

  (b) For any $X \in G$ the minimal times $T^d_{\mathrm{opt}}(X) = T_{\mathrm{opt}}(X)$ coincide, where $T_{\mathrm{opt}}(X)$ is the minimal time for the control problem

$$\dot{X} = \Big(\Delta + \sum_{j=1}^{m} u_j A_j\Big) X, \quad X(0) = I, X(T) = X$$

Time-optimal Factorization

- **Problem:** I.g. time optimal factorizations are infinite

  Under what conditions on the drift term Δ are they *finite*?

- **Definition** [Haselgrove, Nielsen, Osborne]: A drift term Δ is called *lazy*, if there exists $\varepsilon > 0$ such that

$$T_{\mathrm{opt}}(\mathrm{e}^{t\Delta}) < t \qquad \text{for all } t \in (0, \varepsilon). \qquad\qquad (**)$$

  If Δ is not lazy, we call it *fast*.

Time-optimal Factorization

- **Theorem.** If Δ is lazy, there are no finite, time optimal factorizations for any element $X \in G - K$.

Time-optimal Factorization

- **Conjecture 1:** There exists a finite, time optimal factorization for all $X \in G$ iff Δ is fast.
- **Conjecture 2:** Δ fast $\iff [\Delta, \Delta^{\perp}] = 0.$
- **Remark:** Conjecture 2 implies Conjecture 1.

Computation of Optimal Time

Theorem (Khaneja). Let $(\mathfrak{g}, \mathfrak{k})$ be a Cartan pair. Let $\Delta^{\perp}$ be the orthogonal projection of Δ onto $\mathfrak{p}$ and let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ that contains $\Delta^{\perp}$. Then:

- Each $X \in G$ has a decomposition of the form

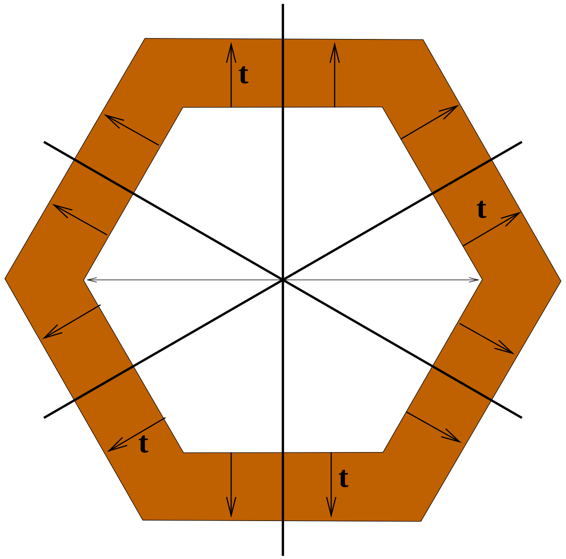
$$X = U\Sigma V \qquad \text{with } U, V \in K \text{ and } \Sigma \in \exp \mathfrak{a}.$$

- The minimal time is given by

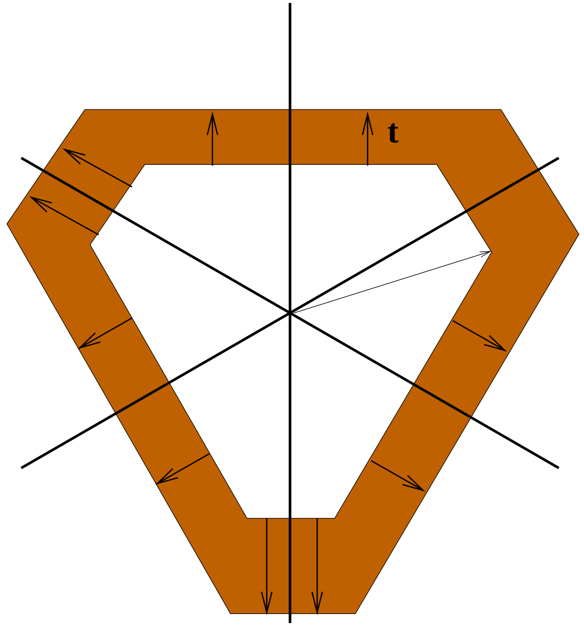
$$T_{\mathrm{opt}}(X) = \min \left\{ t \geq 0 \,\middle|\, \left(t \cdot \mathrm{conv}\, \mathcal{W}(\Delta^{\perp}) \right) \cap \exp^{-1}(\Sigma) \neq \emptyset \right\},$$

  where $X = U\Sigma V$ is an arbitrary factorization of the above type and $\mathcal{W}(\Delta^{\perp})$ denotes the Weyl orbit of $\Delta^{\perp}$.

Computation of Optimal Time



Convex hull of the Weyl Orbit of a "symmetric" drift term Δ



Convex hull of the Weyl Orbit of an arbitrary Δ.

Computation of Optimal Time

Example 1, cont'd:

- $G := \mathrm{SO}(3)$ and $\mathfrak{g} := \mathfrak{so}(3)$,

$$\Omega_1 := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \Omega_2 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $\Delta := \alpha\Omega_1 + \beta\Omega_2$, $\mathfrak{k} := \langle \Omega_2 \rangle$

Euler Angles: $X = \mathrm{e}^{\theta_1\Omega_2}\mathrm{e}^{\theta_2\Omega_1}\mathrm{e}^{\theta_3\Omega_2}$, $\theta_i \in [-\pi, \pi]$

- $T_{\mathrm{opt}}(X) = \alpha^{-1}|\theta_2|$,
- Δ fast $\iff$ $\beta = 0$.

Computation of minimal time

Example: (NMR cont'd)

- NMR-Schrödinger equation on $\mathrm{SU}(4)$

$$\dot{X} = -2\pi \mathrm{i}\Big(H_d + \sum_{i=1}^{4} u_i H_i\Big), \quad X(0) = \mathrm{I},$$

  where $H_d := \sigma_z \otimes \sigma_z$, $H_1 := \mathrm{I}_2 \otimes \sigma_x$, $H_2 := \mathrm{I}_2 \otimes \sigma_y$, $H_3 := \sigma_x \otimes \mathrm{I}_2$, and $H_4 := \sigma_y \otimes \mathrm{I}_2$.

- $K = \mathrm{SU}(2) \otimes \mathrm{SU}(2)$.

- $\Delta = -2\pi \mathrm{i} H_d$ and $\mathfrak{a} := \mathrm{i}\langle \sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z \rangle$.

Computation of minimal time

Example: (NMR cont'd)

Theorem. For all $X = U\Sigma V \in SU(4)$ and $U, V \in K$, and $\Sigma \in \exp \mathfrak{a}$ fixed it holds

- $T(X) = \min \left\{ \sum_{n=1}^{3} |t_n| \middle| e^{t_1 2\pi \mathrm{i}(\sigma_x \otimes \sigma_x)} e^{t_2 2\pi \mathrm{i}(\sigma_y \otimes \sigma_y)} e^{t_3 2\pi \mathrm{i}(\sigma_z \otimes \sigma_z)} = \Sigma \right\}$
- $T(X) \leq \frac{3}{2}$

Computation of minimal time

Optimization Algorithm (NMR cont'd)

Let $X(t,u) = U(u_1, ..., u_6)\Sigma(t_1, t_2, t_3)V(u_7, ..., u_{12})$,

$$U(u_1, \ldots, u_6) = \mathrm{e}^{-\mathrm{i}2\pi u_1 H_1}\mathrm{e}^{-\mathrm{i}2\pi u_2 H_2}\mathrm{e}^{-\mathrm{i}2\pi u_3 H_1}\mathrm{e}^{-\mathrm{i}2\pi u_4 H_3}\mathrm{e}^{-\mathrm{i}2\pi u_5 H_4}\mathrm{e}^{-\mathrm{i}2\pi u_6 H_3}$$

$$V(u_7, \ldots, u_{12}) = \mathrm{e}^{-\mathrm{i}2\pi u_7 H_1}\mathrm{e}^{-\mathrm{i}2\pi u_8 H_2}\mathrm{e}^{-\mathrm{i}2\pi u_9 H_1}\mathrm{e}^{-\mathrm{i}2\pi u_{10} H_3}\mathrm{e}^{-\mathrm{i}2\pi u_{11} H_4}\mathrm{e}^{-\mathrm{i}2\pi u_{12} H_3}$$

$$\Sigma = \mathrm{e}^{t_1 2\pi\mathrm{i}(\sigma_x \otimes \sigma_x)}\mathrm{e}^{t_2 2\pi\mathrm{i}(\sigma_y \otimes \sigma_y)}\mathrm{e}^{t_3 2\pi\mathrm{i}(\sigma_z \otimes \sigma_z)}$$

To compute the minimal time $T(X)$, we combine simulated annealing with gradient methods to solve the nonlinear optimization problem:

$$\begin{aligned} \min \quad & f(t,u) := |t_1| + |t_2| + |t_3|, \\ \text{subject to} \quad & g(t,u) := 4 - \mathrm{Retr}(X_F^* X(t,u)) = 0 \end{aligned}$$

where $t = [t_1, t_2, t_3], u = [u_1, u_2, ..., u_{12}] \in [-1, 1]^{12}$

Computation of Time-optimal Pulse Sequences

Consists of two sub-problems:

- Given $T \geq 0$, solve

$$\begin{aligned} \min_{t,u} \quad & g(t,u), \\ \text{subject to} \quad & f(t,u) \leq T, \\ & t \geq 0. \end{aligned}$$

- Let $V(T)$ be the global optimal value of $g(t,u)$, associated with a given $T \geq 0$.

$$\begin{aligned} \text{Minimize} \quad & T \\ \text{subject to} \quad & V(T) = 0, \\ & T \geq 0. \end{aligned}$$

Computation of Time-optimal Pulse Sequences

Example

$$X_F = \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{aligned} T(X_F) &= 1.499996 \\ t &= [0.499993 \mid 0.500017 \mid 0.499986] \end{aligned}$$