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Numerical techniques for approximating the solution of matrix ODE on the general linear group

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Outline

- The matrix ODE we deal with
- Theoretical results
- Examples
- Numerical tools:
  - Substituting approach
  - Solution via Riccati equation
  - SVD approach
- Rectangular Case
- Numerical examples

The differential system

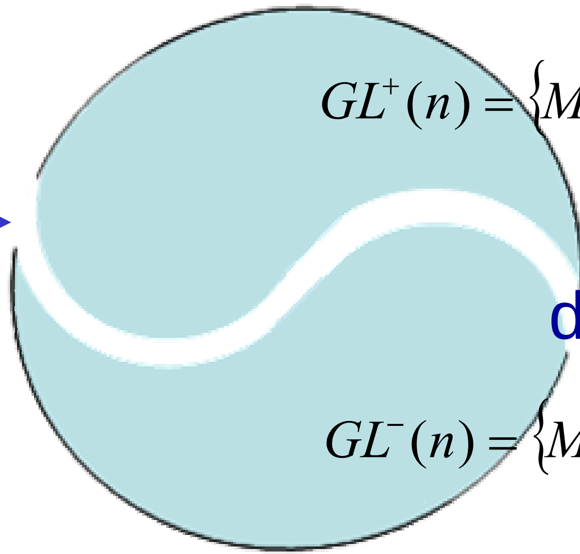
- Consider the matrix differential equation

$$\dot{Y}(t) = Y(t)^{-T} F(Y(t), Y(t)^{-T})$$
$$Y(0) = Y_0 \in GL(n)$$

- F is a continuous matrix function, globally Lipschitz on a subdomain of $GL(n)$
- the solution $Y(t)$ exists and is unique in a neighborhood **]-τ τ[** of the origin 0

The structure of GL(n)

- Two maximal connected and disjoint open subsets comprising $GL(n)$



$$GL^{+}(n)=\left\{M\in\Re^{n\times n}\mid\det(M)>0\right\}$$

$$GL^{-}(n)=\left\{M\in\Re^{n\times n}\mid\det(M)<0\right\}$$

Theoretical results

- The existence of the solution $Y(t)$ for all $\boldsymbol{t}$ is not guaranteed ***a priori*** and the presence of a finite escape time behavior is not precluded.

- The value of the escape point depends on the function F
  - If the escape point $\boldsymbol{\tau}$ is finite then $Y(t)$ approaches a singular matrix as $\boldsymbol{t} \to \boldsymbol{\tau}$
  - if $\boldsymbol{\tau} < \infty$ then $Y(t)$ exists for all $\boldsymbol{t} > \boldsymbol{0}$

Theoretical results

❖ Example: ***F*** **constant** function with ***trace(F) = 0***

$$\dot{Y} = Y^{-T}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad Y(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

solution

$$Y(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{1+t} & -\sqrt{1+t} \\ \sqrt{1-t} & \sqrt{1-t} \end{bmatrix}$$

Existence interval
(-1,1)
Escape point 1

Theoretical results

- Relationship between the singular values of the solution $Y(t)$, the initial condition $Y(0)$ and the symmetric matrix function:

$$E(t) = \int_0^t [F^T(Y(s), Y^{-T}(s)) + F(Y(s), Y^{-T}(s))]\, ds$$

$$\sigma_{\min}(t) \geq \sigma_{\min}^0 + \lambda_{\min}(E(t))$$

Smallest Singular Value of $Y(t)$

Smallest Singular Value of $Y(0)$

Smallest Eigenvalue of $E(t)$

Systems with structure

- If the matrix function $\boldsymbol{F}$ maps all matrices into the Lie algebra of skew-symmetric matrices

  ⇒ $\boldsymbol{Y(t)}$ belongs to the orthogonal manifold (whenever $Y(0)$ is orthogonal)

- If $\boldsymbol{diag}(\boldsymbol{F}) = 0$ for all nonsingular matrices

  ⇒ $\boldsymbol{diag}(Y(t)^{\mathrm{T}}Y(t)) = \boldsymbol{diag}(Y(0)^{\mathrm{T}}Y(0))$

Examples

- **Control Theory**
  - Optimal system assignment via Output Feedback Control
  - Balanced Matrix Factorizations
  - Balanced realizations (Isodynamical flows)
- **Multivariate Data Analysis**
  - Weighted Oblique Procrustes problem
- **Inverse Eigenvalue Problem**
  - Pole placement or eigenvalue assignment problem via output feedback
  - Prescribed Entries Inverse Eigenvalue Problem

Examples in Control Theory

❖ Output Feedback Control of linear system

- Consider the linear dynamical system defined by the triple $(A,B,C)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{p\times n}$

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

- The process of "*feeding back*" the output or the state variables in a dynamical system configuration through the input channels
- Output Feedback: $u(t)$ is replaced by $u(t)=Ky(t)+v(t)$

$K\in\mathbb{R}^{m\times p}$ feedback gain matrix

Examples in Control Theory

- **Output Feedback Control of linear system**
  - The feedback system is

$$\dot{x}(t) = (A + BKC)x(t) + Bv(t)$$
$$y(t) = Cx(t)$$

- **Optimal system assignment**
  - Given a target system described by the triple $(F,G,H) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{p\times n}$ find an *optimal feedback transformation* of (A,B,C) which results the best approximation of (F,G,H).

Examples in Control Theory

- The set $GL(n) \times P^{m\times p}$ of feedback transformation is a Lie group under the operation

$$(T_1, K_1) \circ (T_2, K_2) = (T_1 T_2, K_1 + K_2)$$

- We can consider action on the output feedback group and orbits, particularly:

$$\Phi(A,B,C) = \{(T(A+BKC)T^{-1}, TB, CT^{-1} \mid (T,K) \in GL(n) \times \mathrm{P}^{m\times p}\}$$

- The distance function

$$\Phi = \| T(A+BKC)T^{-1} - F \|^2 + \| TB - G \|^2 + \| CT^{-1} - H \|^2$$

Examples in Control Theory

- The gradient flow of this distance function with respect to a specific Riemannian metric on $\Phi(A,B,C)$ can be written as:

$$\dot{T} = T^{-T} f(T, T^{-T}, K)$$

$$\dot{K} = -B^T T^T (T(A + BKC)T^{-1} - F)T^{-T} C^T$$

Examples in Control Theory

- **Balanced matrix factorizations**
  - General matrix factorization problem:

    Given a matrix $H \in P^{k\times l}$ find two $X \in P^{k\times n}$ and $Y \in P^{n\times l}$ such that $H=XY$

    - balanced factorization $X^{T}X=YY^{T}$
    - diagonal balanced factorization $X^{T}X=YY^{T}=D$

- Balanced and diagonal balanced factorization can be characterized as critical points of cost functions defined on the orbit

$$\mathrm{O}(X,Y) = \{(XT^{-1}, TY) \in \mathrm{P}^{k\times n} \times \mathrm{P}^{n\times l} \mid T \in GL(n)\}$$

Examples in Control Theory

- The cost functions are respectively:

$$\Phi : \mathrm{O}(X,Y) \to \mathrm{P} \qquad \Phi(XT^{-1},TY) = \| XT^{-1} \|^2 + \| TY \|^2$$

$$\Phi_N : \mathrm{O}(X,Y) \to \mathrm{P} \qquad \Phi_N(XT^{-1},TY) = tr(NT^{-T}X^TXT^{-1} + NTYY^TT^T)$$

- Applying a gradient flow techniques differential systems on $GL(n)$ can be constructed:

balanced

$$\dot{T} = T^{-T}(X^TX(T^TT)^{-1} - T^TTYY^T) \qquad T(0) = T_0$$

$$\dot{T} = T^{-T}(X^TXT^{-1}NT^{-T} - T^TNTYY^T) \qquad T(0) = T_0$$

diagonal balanced

Examples in Control Theory

❖ Balanced realizations in linear system theory

- Consider the linear dynamical system defined by the triple $(A,B,C)\in \mathrm{P}^{n\times n}\times \mathrm{P}^{n\times m}\times \mathrm{P}^{p\times n}$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

- Gramians: $W_C = \int_0^\infty e^{At} BB^T e^{A^T t} dt \quad W_O = \int_0^\infty e^{A^T t} C^T C e^{At} dt$
- (A,B,C) is a **balanced realization** if $W_C=W_O$
- (A,B,C) is a **diagonal balanced realization** if $W_C=W_O=D$

Examples in Control Theory

- Any $T \in GL(n)$ changes a realization by

$$(A,B,C) \rightarrow (TAT^{-1},\ TB, CT^{-1})$$

- and the Gramians via

$$W_C \rightarrow T\, W_C\, T^{-1} \qquad\qquad W_0 \rightarrow T^{-T} W_0 T^{-1}$$

- Balanced and diagonal balanced realizations have been proved to be critical points of costs functions defined on the orbit

$$\mathrm{O}(A,B,C) = \{(TAT^{-1}, TB, CT^{-1}) \in \mathrm{P}^{n\times n} \times \mathrm{P}^{n\times m} \times \mathrm{P}^{k\times n} \mid T \in GL(n)\}$$

Examples in Control Theory

- The cost functions are respectively:

$$\Phi : \mathrm{O}(A,B,C) \to \mathrm{P} \qquad \Phi(T) = tr(TW_C T^{-1} + T^{-T} W_O T^{-1})$$

$$\Phi_N : \mathrm{O}(A,B,C) \to \mathrm{P} \qquad \Phi_N(T) = tr(NTW_C T^{-1} + NT^{-T} W_O T^{-1})$$

- All balancing transformation $T \in GL(n)$ for a given asymptotically stable system (A,B,C) can be obtained solving the gradient flow

balanced

$$\dot{T} = T^{-T}(W_O (T^T T)^{-1} - T^T T W_C) \qquad T(0) = T_0$$

$$\dot{T} = T^{-T}(W_O T^{-1} N T^{-T} - T^T N T W_C) \qquad T(0) = T_0$$

diagonal balanced

Examples in Multivariate Data Analysis

- **Weighted oblique Procrustes problem (WObPP)**
  - **Manifold of the oblique rotation matrices**

$$OB(n) = \{X \in \mathbb{R}^{n \times n} \mid \det(X) \neq 0, diag(X^T X) = I\}$$

- Given A, B, C fixed matrices with conformal dimensions
  - Minimize $\| AXC - B \|$ subject to $X \in OB(n)$
    - Problem in factor analysis known as a "rotation to *factor-structure* matrix"
  - Minimize $\| AX^{-T}C - B \|$ subject to $X \in OB(n)$
    - Problem of finding an approximation to a "*factor-pattern*" matrix

Examples in Multivariate Data Analysis

- The solution of the WObPP problem can be obtained solving a **descent matrix ODE**:

$$\boxed{\frac{dX}{dt} = -\pi_{OB(n)}(\nabla) = -X^{-T} off\,(X^{T}\nabla)}$$

- being ∇ the gradient of the function to be minimize with respect to the chosen metric

(N. Trendafilov FGCS 2003)

Examples in Inverse Eigenvalue Problem and control theory

- **Pole placement or eigenvalue assignment via output feedback:**
  - Given a linear system described by the triple $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ and a self-conjugate set of complex points $\{\lambda_1 \lambda_2 \ldots \lambda_{n,}\}$
  - find a feedback gain matrix $\boldsymbol{K}$ such that $\boldsymbol{A+BKC}$ has eigenvalues λ_i
- Denoted by Λ a fixed matrix with eigenvalues λ_i the pole placement task is equivalent to find a matrix $T \in GL(n)$ and $K \in \mathrm{P}^{m\times p}$ minimizing the distance μι $\| \Lambda - T(A+BKC)T^{-1} \|$

Examples in Inverse Eigenvalue Problem and control theory

- Using a gradient flow techniques the solution can be obtained solving

$$\dot{T} = T^{-T}[(A+BKC)^T, T^T(\Lambda - (A+BKC))T^{-T}]$$

$$\dot{K} = -B^T T^T (T(A+BKC)T^{-1} - F)T^{-T} C^T$$

Examples in Inverse Eigenvalue Problem

- **Matrix completion with prescribed eigenvalues**
- PEIEP (prescribed entries inverse eigenvalue problem) :

Given

- $\Lambda=\{(i_v,j_v) \mid v=1,\ldots,m\}$ m pairs of integers $1 \le i_v < j_v \le n$
- $\boldsymbol{a}=\{a_1,\ldots,a_m\} \subset P$
- $\{\lambda_1,\ldots,\lambda_n\} \subset X$ closed under conjugation

Find a matrix $X \in P^{n\times n}$ such that $\sigma(X)=\{\lambda_1,\ldots,\lambda_n\}$

and $x_{i_v j_v} = a_v \quad v=1,\ldots,m$

Examples in Inverse Eigenvalue Problem

- Let Λ a matrix with eigenvalues λ_i and denoting

$$\mathrm{M}(\Lambda)=\{V\Lambda V^{-1} \mid V\in GL(n)\}$$

the orbit of matrices isospectral to Λ under the action group of $GL(n)$ and

$$\Sigma(\Lambda,\boldsymbol{a})=\{X=[x_{ij}]\in \mathrm{P}^{n\times n} \mid x_{i_\nu j_\nu} = a_\nu \quad \nu = 1,\dots,m\}$$

- Solving the PEIEP is to find intersection of the two geometric entities $\mathrm{M}(\Lambda)$ and $\Sigma(\Lambda,\boldsymbol{a})$

Examples in Inverse Eigenvalue Problem

- Minimize for each given $X \in \mathrm{M}(\Lambda)$ the distance between X and $\Sigma(\Lambda,\boldsymbol{a})$

$$\min_{V \in \mathrm{M}(\Lambda)} \frac{1}{2} < V\Lambda V^{-1} - P(V\Lambda V^{-1}), V\Lambda V^{-1} - P(V\Lambda V^{-1}) >$$

Projection on $\Sigma(\Lambda,\boldsymbol{a})$

- Using a descent flow approach we get

$$\frac{dV}{dt} = \kappa(V\Lambda V^{-1})V^{-T} \quad \text{with} \quad \kappa(X) = [X^T, X - P(X)]$$

(M.T. Chu et al. FGCS 2003)

Numerical Approximation: substituting approach

- Consider our system:

$$\boxed{\begin{aligned}&\dot{Y}(t) = Y(t)^{-T} F(Y(t), Y(t)^{-T})\\&Y(0) = Y_0 \in GL(n)\end{aligned}}$$

- Setting $\boldsymbol{Z=Y^{-T}}$ from $\boldsymbol{Y^T Z=I}$ we get

$$\boxed{\dot{Y}^T Z + Y^T \dot{Z} = 0 \Leftrightarrow \dot{Z} = -Y^{-T} \dot{Y}^T Z}$$

$$\begin{cases}\dot{Y} = ZF(Y,Z) = H(Y,Z), & Y(0) = Y_0\\[2ex] \dot{Z} = -ZF^T(Y,Z)Z^T Z = -ZH^T(Y,Z)Z, & Z(0) = Y_0^{-T}\end{cases}$$

Substituting Approach

- **Advantages**:
  - No direct use of the inverse of $Y(t)$ (computational advantages)
- **Drawbacks:**
  - Solution of a new matrix ODE with **double dimension** with respect to the original system;
  - High stiffness (when $\boldsymbol{Y(t)}$ tends to a singular matrix or the Lipschitz constant of $\boldsymbol{H}$ is large);
  - The presence of an additional structure of the solution matrix $Y(t)$ is not considered ➡ need of *ad hoc* numerical scheme

Solution via Riccati equation

- When the matrix function F does not depend explicitely on Y^{-T}, i.e.:

$$\begin{aligned} \dot{Y}(t) &= Y(t)^{-T} F(Y(t)) \\ Y(0) &= Y_0 \in GL(n) \end{aligned}$$

- It could **be convenient** work with the implicit equation

$$\begin{aligned} Y(t)\dot{Y}(t) &= F(Y(t)) \\ Y(0) &= Y_0 \in GL(n) \end{aligned}$$

Solution via Riccati equation

- Applying the second order Gauss Legendre method, we get:

$$Y_{n+1}^T Y_{n+1} + Y_n^T Y_{n+1} - Y_{n+1}^T Y_n - Y_n^T Y_n - 2hF\left(\frac{Y_n + Y_{n+1}}{2}\right) = 0$$

- The previous equation can be iteratively solved starting from an initial approximation $Y_{n+1}^{(0)}$ (avoiding the nonlinearity of ***F***)

$$\boxed{Y_{n+1}^T Y_{n+1} + Y_n^T Y_{n+1} - Y_{n+1}^T Y_n - Y_n^T Y_n - 2hF\left(\frac{Y_n + Y_{n+1}^{(0)}}{2}\right) = 0}$$

Solution via Riccati Equation

- The latter equation is the prototype of an Algebraic Riccati equation, in fact setting

$$A = Y_n \quad \text{and} \quad C = Y_n^T Y_n + 2hF\left(\frac{Y_n + Y_{n+1}^{(0)}}{2}\right)$$

- we get

$$\boxed{R(X) = X^T X + A^T X - X^T A + C = 0}$$

Solution via Algebraic Riccati equation

- Numerical methods to solve **Algebraic Riccati equation** are based on fixed point or Newton iteration:
  - **Picard iteration**: $\boxed{A^T X_{k+1} - X_{k+1}^T A = -C - X_k^T X_k}$
  - **Newton method**:
    - $R : \mathrm{P}^{n\times n} \longrightarrow \mathrm{P}^{n\times n}$
    - its Frechét derivatitive is: $R'_X(H) = H^T(X - A) + (X + A)^T H$
    - the Newton iteration starts from $\boldsymbol{X_0}$ and solves $\boldsymbol{R}(\boldsymbol{X})=0$ via $\boldsymbol{X_{k+1}} = \boldsymbol{X_k} + \boldsymbol{D_k}$ being $\boldsymbol{D_k}$ the solution of Sylvester equation

$$\boxed{R'_X(D_k) = -R(X_k) \Leftrightarrow (X_k + A)^T D_k + D_k^T (X_k - A) = -R(X_k)}$$

Solution via Riccati equation

- Solving Riccati equation implies the numerical treatment of the Sylvester equation

$$\mathrm{A}X + X^{\mathrm{T}}\mathrm{B} = \mathrm{X}$$

with A, B, X given $n \times n$ matrices

Existence: there exists a solution X of the Sylvester equation iff

$$\begin{bmatrix} \mathrm{X} & \mathrm{A} \\ \mathrm{B} & O \end{bmatrix} \text{ and } \begin{bmatrix} O & \mathrm{A} \\ \mathrm{B} & O \end{bmatrix}$$

are equivalent

Solution via Riccati equation

- To obtain conditions for uniqueness of solution and for constructing it, we reformulate the Sylvester equation as a ***$n^2 \times n^2$*** linear system:

$$(I \otimes A)\mathrm{vec}(X) + (B^T \otimes I)\mathrm{vec}(X^T) = \mathrm{vec}(X)$$

$$\mathrm{vec}(X^T) = P(n,n)\mathrm{vec}(X)$$

$$P(n,n) = \sum_{i=1}^{n}\sum_{j=1}^{n} E_{ij} \otimes E_{ij}^T$$

$$\underbrace{\left[(I \otimes A) + (B^T \otimes I)P(n,n)\right]}_{M} \mathrm{vec}(X) = \mathrm{vec}(X)$$

Solution via Riccati equation

$$M = \begin{bmatrix} A + e_1 b_1^T & e_2 b_1^T & \cdots & e_n b_1^T \\ e_1 b_2^T & A + e_2 b_2^T & \cdots & e_n b_2^T \\ \vdots & \vdots & \ddots & \vdots \\ e_1 b_n^T & e_2 b_n^T & \cdots & A + e_n b_n^T \end{bmatrix}$$

being $\boldsymbol{b_i}$ the columns of the matrix B

Uniqueness: there exists a unique solution X of the Sylvester equation $A\boldsymbol{X}+\boldsymbol{X}^T B = X$ if the matrix $\boldsymbol{M}$ is non-singular (rank $(\boldsymbol{M}) = \boldsymbol{n}^2$)

Solution via Riccati equation

- Considering the linear equation derived from:
  - **Picard iteration:** A=A^T and B =A $\Rightarrow$ M is singular
  - **Newton iteration:** A=X_k+A^T and B =$X_k - A$ $\Rightarrow$ M is non-singular $\Rightarrow$ **unique solution** !
- Newton method converges in a reasonable number of iterations
- Numerical solution of Sylvester equation :
  - Direct methods (QR, Gaussian Elimination);
  - Iterative algorithms;
  - Generalize Conjugate Residual method.

Singular Value Decomposition

- To avoid the inverse matrix computations and to control the singularities of the matrix solution *Y(t)* we can adopt a continuous Singular Value Decomposition approach
- The continuous SVD of $Y(t)$ is a continuous factorization

$$Y(t) = U(t)\,\Sigma(t) V^{\mathrm{T}}(\mathrm{t})$$

  - $U(t), V(t)$ orthogonal matrices ($U^T U = I_n$ and $V^T V = I_n$)
  - $\Sigma(t)$ diagonal matrix with diagonal elements the singular values $\sigma_i(t)$ of $Y(t)$
- The motion of $Y(t)$ is now described by the variables $U(t)$, $\Sigma(t)$, $V(t)$ giving more information on the flow

Singular Value Decomposition

- Suppose that the solution $Y(t)$ possesses dinstinct and nonzero singular values $\sigma_i(t)$, for $i=1,\ldots, n$ and t in $[0, \tau)$ then there exists a continuous SVD of $Y(t)$ and the factors $U(t)$, $\Sigma(t)$, $V(t)$ of such a decomposition satisfy the following ODEs:

$$\dot{\Sigma} = \Sigma^{-1} V^T F(Y, Y^{-T}) V - H\Sigma + \Sigma K, \quad \Sigma(0) = \Sigma_0$$

$$\dot{U} = UH, \quad U(0) = U_0$$

$$\dot{V} = VK, \quad V(0) = V_0$$

Singular Value Decomposition

- The differential equations for the singular values are

$$\boxed{\dot{\sigma}_i = \frac{1}{\sigma_i}\left(V^T F(Y, Y^{-T})V\right)_{ii}, \qquad i = 1, \cdots, n}$$

- The elements of the skew-symmetric matrices ***H, K*** are

$$H_{ij} = \frac{1}{\sigma_i \sigma_j (\sigma_j^2 - \sigma_i^2)}\left[\sigma_j^2 \left(V^T FV\right)_{ij} + \sigma_i^2 \left(V^T FV\right)_{ji}\right]$$

$$K_{ij} = \frac{1}{(\sigma_j^2 - \sigma_i^2)}\left[\left(V^T FV\right)_{ij} + \left(V^T FV\right)_{ji}\right]$$

Singular Value Decomposition

- Numerical solution of:
  - a diagonal system in σ_i (information on the conditioning of the matrix solution $Y(t)$)
  - two linear systems in H_{ij} K_{ij}
  - two orthogonal systems in U and V
    - **our aim** is to preserve the non-singular behavior of the numerical solution → explicit integration of the systems in U and V (orthogonality preserved up to the order of the method)
- **Drawback** ⇒ **distinct singular values**
  - **Block Continuous SVD**

Rectangular case

- Some of the previous results can be extended to differential problems on the manifold

$$\boldsymbol{GL}(m,n)=\{\boldsymbol{Y}\in\boldsymbol{P}^{\boldsymbol{m\times n}}\mid \mathrm{rank}(Y)=n\},\quad n\le m$$

- Differential systems on $GL(m,n)$ have the following form:

$$\boxed{\dot{Y}=G(Y),\quad Y(0)=Y_0\in GL(n,p)}$$

- with $\boldsymbol{G}$ belonging to the tangent space of $GL(m,n)$:

$$\boxed{G(Y)=Y\left(Y^TY\right)^{-1}\underbrace{F_1(Y)}_{n\times n}+\left[I_n-Y\left(Y^TY\right)^{-1}Y^T\right]\underbrace{F_2(Y)}_{m\times n}}$$

Rectangular Case: numerical treatment

- **Continuous SVD (economy)**

$$\underbrace{Y(t)}_{m\times n} = U_1(t)\Sigma_1(t)V^T(t)$$

$m\times n$ matrix
$U_1^T U_1 = I_n$

$diag(\sigma_1, \ldots, \sigma_n)$

$n\times n$ matrix
$V^T V = V V^T = I_n$

- Differentiating we obtain the differential systems satisfied by the three factors:

Rectangular Case: numerical treatment

$$\dot{\sigma}_i = \frac{1}{\sigma_i}\left(V^T F_1(Y)V\right)_{ij} \qquad i = 1,\cdots,n$$

$$\dot{V} = VK, \qquad V(0) = V_0$$

$$\dot{U}_1 = U_1 H + (I_n - U_1 U_1^T T) F_2(Y)\Sigma_1^{-1}, \qquad U(0) = U_0$$

Differential System on the Stiefel manifold

$$H_{ij} = \frac{1}{\sigma_i \sigma_j (\sigma_j^2 - \sigma_i^2)}\left[\sigma_j^2 \left(V^T F_1(Y)V\right)_{ij} + \sigma_i^2 \left(V^T F_1(Y)V\right)_{ji}\right]$$

$$K_{ij} = \frac{1}{(\sigma_j^2 - \sigma_i^2)}\left[\left(V^T F_1(Y)V\right)_{ij} + \left(V^T F_1(Y)V\right)_{ji}\right]$$

Rectangular Case: numerical treatment

- **Substituting approach:**

$$\dot{Y} = Y\left(Y^T Y\right)^{-1} F_1(Y) + \left[I_n - Y\left(Y^T Y\right)^{-1} Y^T\right] F_2(Y)$$

- Setting $\boldsymbol{Z = (Y^T Y)^{-1}}$ we obtain

$$\dot{Y} = YZF_1(Y) + \left[I - YZY^T\right] F_2(Y)$$
$$\dot{Z} = -Z\left[F_1(Y) + F_1^T(Y)\right] Z$$

Numerical Illustrations

- First example:

$$\dot{Y} = Y^{-T}\begin{bmatrix} 0 & -\frac{\delta}{2} \\ -\frac{\delta}{2} & 0 \end{bmatrix} \qquad Y(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

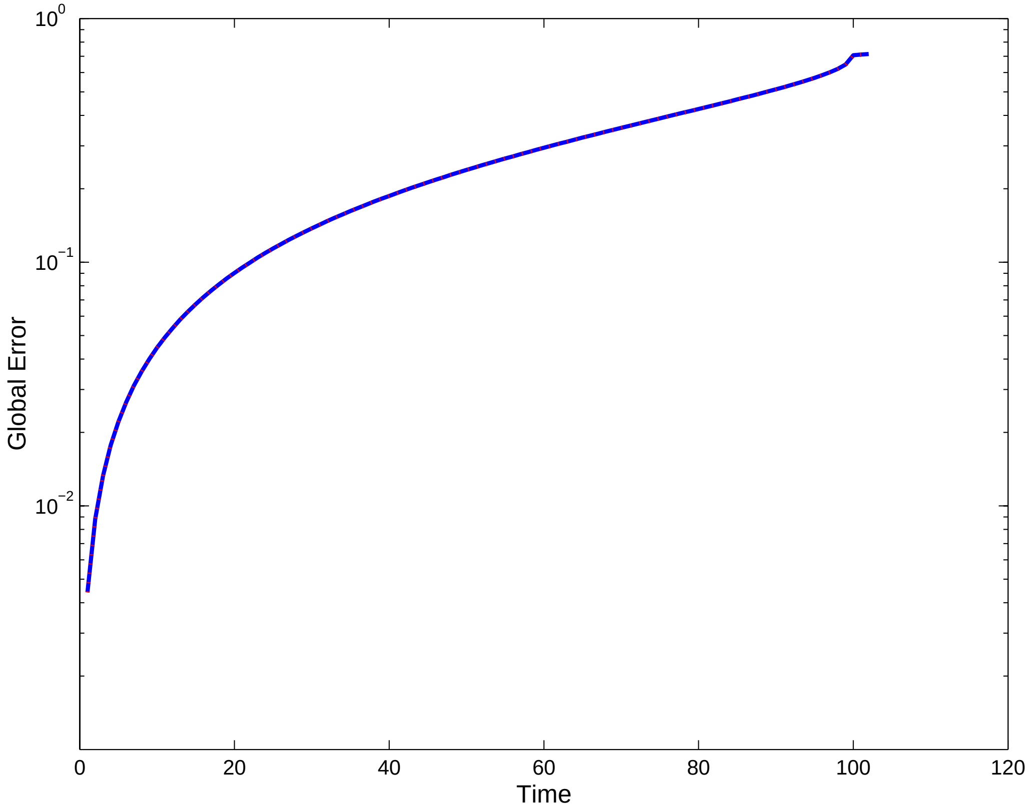
- With solution existing in **(-1/$\boldsymbol{\delta}$, 1/$\boldsymbol{\delta}$)**

$$Y(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{1+\delta t} & -\sqrt{1+\delta t} \\ \sqrt{1-\delta t} & \sqrt{1-\delta t} \end{bmatrix}$$

- We solve the problem with $\delta = 1/2$

Numerical Illustrations

❖ Behaviour of the global error on [0 2)



Numerical Illustrations

- Second example

$$\dot{Y} = Y^{-T}\begin{bmatrix} -\sin(t)\cos(t) & \cos(t) \\ -t\sin(t) & t \end{bmatrix} \qquad Y(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

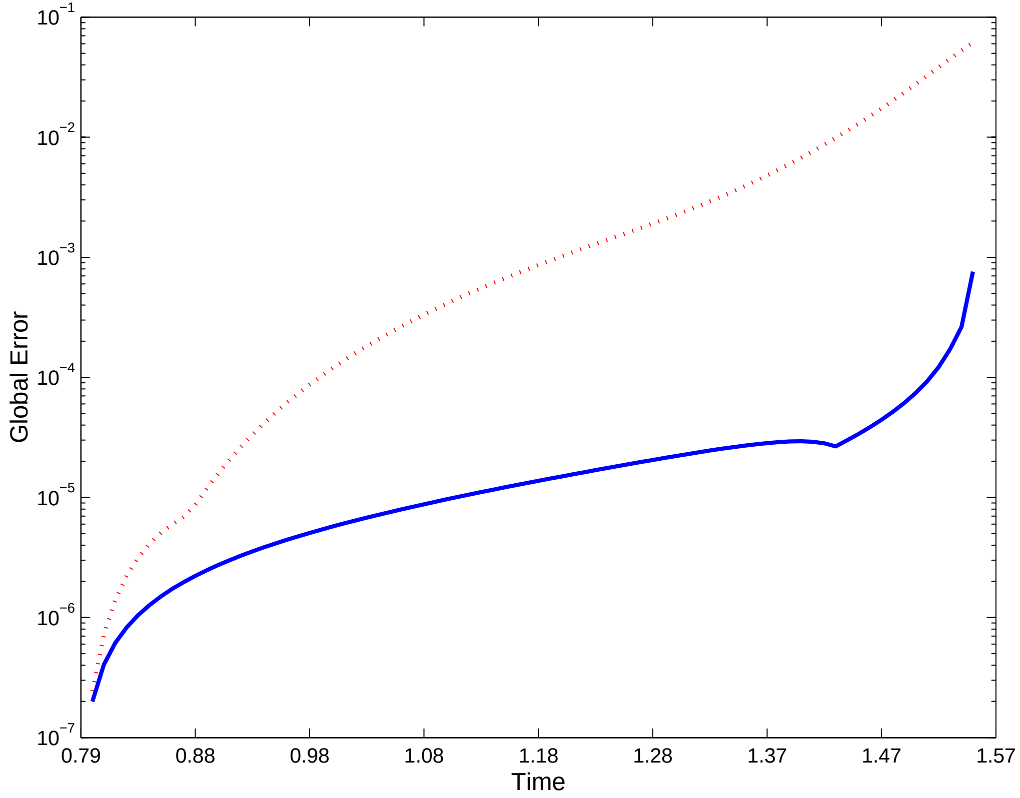
- with solution

$$Y(t) = \begin{bmatrix} \cos(t) & t \\ 0 & 1 \end{bmatrix}$$

- periodically singular (for each $\tau_k = k\,\pi/2$)

Numerical Illustrations

- Semilog plot of the global error on (π/4,π/2)



Conclusions

- We have considered a particular ODEs on $GL(n)$ often occurring in applications
- Several problems modeled by such ODEs
- Different numerical approaches avoiding the direct use of matrix inversion and detection of singular behavior
- Future works:
  - Improving the validation of the proposed approaches by tackling numerical tests on real examples