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6. AUTHOR(S) Dewey H. Hodges	
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13. ABSTRACT (Maximum 200 words)

A current trend in the development of missiles is in the direction of more flexibility, higher maneuverability, and higher speeds, all of which require a higher level of fidelity for calculations of stability, loads, control, and guidance. To address these issues, the present interdisciplinary basic research was conducted involving structural analysis, dynamics, dynamic stability, aeroelastic stability, and trajectory analysis of missiles, rockets, and projectiles. A computer code for the dynamic stability, structural dynamics and aeroelastic response of the missile has been written using a geometrically-exact, mixed finite element method. The aerodynamic modeling of the loading for the missile body and fins is based on slender body theory and thin-airfoil theory, respectively. Results agree with published results for dynamic stability in addition to limit cycle oscillations for disturbed flight near and above the critical thrust. Parametric studies for specific flexible missile configurations are presented, including effects of flexibility on stability, limit-cycle amplitudes, and missile loads. Although results indicate little potential for affecting aeroelastic stability by use of composite couplings, they do exhibit significant interaction between aeroelastic effects and the thrust, a follower force.

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A New Approach to Aeroelastic Response, Stability and Loads of Missiles and Projectiles

Dewey H. Hodges, Principal Investigator
Professor, School of Aerospace Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332-0150

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Dr. Gary L. Anderson, Technical Monitor

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Summary

Statement of the Problem

A current trend in the development of missiles is in the direction of more flexibility, higher maneuverability, and higher speeds, all of which require a higher level of fidelity for calculations of stability, loads, control, and guidance. Unfortunately, until now has been no integrated tool for preliminary design that considers the related problems that need to be addressed to provide this increase in fidelity. To this end, interdisciplinary basic research has been conducted that involves structural analysis, dynamics, dynamic stability, aeroelastic stability, and trajectory analysis of missiles, rockets, and projectiles.

Most Important Results

A computer code for the dynamic stability, structural dynamics and aeroelastic response of the missile has been written using a geometrically-exact, mixed finite element method. The aerodynamic modeling of the loading for the missile body and fins is based on slender-body theory and thin-airfoil theory, respectively. Results agree with published results for dynamic stability and show the development of limit cycle oscillations for disturbed flight near and above the critical thrust. Parametric studies of the aeroelastic behavior of specific flexible missile configurations are presented, including effects of flexibility on stability, limit-cycle amplitudes, and missile loads. Results indicate little potential for affecting aeroelastic stability by means of composite couplings. However, the results do yield a significant interaction between the thrust, which is a follower force, and the aeroelastic stability. This observation led to additional research on the influence of engine thrust on wing flutter.

1 Introduction

1.1 Background

Missile development design has seen growing emphasis of higher speeds, more demanding maneuvers, and higher flexibility to meet various mission requirements. For example, several U.S. Army programs, such as extended range projectiles, the compact kinetic energy missile (CKEM) and the ARROW system require increased maneuverability and flexibility, thus leading to an increase of the relative importance of structural loads and deformation in the multidisciplinary problem.

Frequently missiles, rockets, and projectiles must deliver the last ounce of performance in order to meet their design objectives. For example, projectile designers strive to maximize the payload that the system delivers to a specified range; alternatively, they may strive for the maximum range for a given payload. Since very small relative changes in total mass may mean large relative changes in payload mass, even very small margins of gain are important. The extended range projectile program calls for the increase of payload mass by the use of composites in structural design. Defensive missiles, projectiles that are rocket powered in portions of their flight profiles (such as the CKEM), and smart missiles may be designed to deliver precise hits with maximum final kinetic energy or velocity in order to effectively knock out an incoming missile, tank, or other enemy weapon. Greater sophistication in such areas as the evasive maneuvering capability of enemy weapon systems, for example, may require new generations of weapon system to deliver higher speeds and sustain higher loads and skin temperatures.

It is important to recognize the potential nonlinearities which can arise in both missiles and projectiles. One source of nonlinearities in missiles is a large axial force, so that even to get the standard linear equations one must linearize about a nontrivial state. Additional nonlinearities in both missiles and projectiles can arise due to free-play in threaded and snap joints. Also, fins on missiles and projectiles have nonlinearities due to large deflections and free-play in the hinges. Further nonlinear effects come about from matter shifting inside the casing. The need to account for imperfections and free-play effects provides motivation to base the approach on exact nonlinear kinematics.

Presently, missile conceptual designers specify the stiffness of the missile to structural designers. This specification without feedback and iteration does not facilitate multidisciplinary design optimization. Furthermore, even though the designs are driven by stiffness and not by strength, no attempt has been made until now to take advantage of the elastic couplings afforded by use of composites. Higher loads are likely to occur due to increased demands placed on modern equipment, and there is a higher probability of the occurrence of static and dynamic aeroelastic instabilities. Nevertheless, present methodologies are incapable of coping with these problems. Designers must wait until the prototype stage to see whether or not there are going to be aeroelastic problems in the various flight regimes of the system. This approach is quite wasteful and inefficient.

The above observations also suggest coupling between the flight mechanics, guidance and control of a missile and its structural dynamics and aeroelasticity. A strictly optimal trajectory may induce higher internal loads and deformation, and aeroelastic phenomena can affect the originally planned missile trajectory. Present methodologies do not allow the exploration of this coupling. The stability problem due to thrust is strictly a dynamic stability issue, but aeroelastic phenomena may influence it. Unlike conventional flight vehicles, however, the static and dynamic aeroelastic instabilities may be coupled with flight dynamics modes. It is well known that a static criterion of stability is not sufficient in systems loaded by follower force such as thrust. Thus, a statically stable missile may be dynamically unstable. Structural deformation may affect the stability, and use of elastic tailoring may allow the designer to avoid aeroelastic instabilities in the design space early in the design effort.

Recent designs have emphasized the use of composite materials to keep the weight down. As the duration of flight and flight velocity are increased, the rise in casing temperature due to aerodynamic heating may become important. A significant rise in temperature may bring about a degradation in the stiffness properties of the composite materials, particularly in the matrix. The increased flexibility may enhance aeroelastic effects, creating non-negligible flexibility effects that interfere with the control system's ability to ensure that the missile or projectile reach precisely the desired destination.

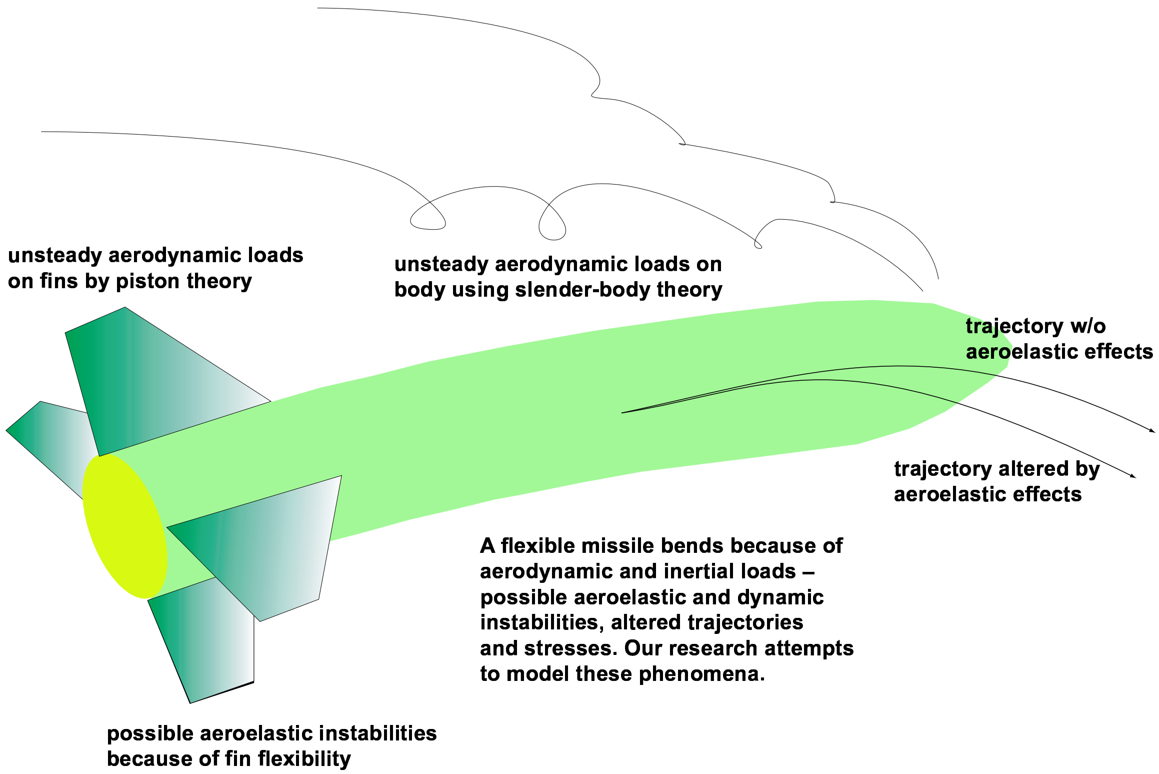

Figure 1: Schematic of missile problem

With these trends in view, it seems imperative that modern analytical tools be created, so that a far better understanding of the influence of missile design parameters and operating conditions on their aeroelastic stability and loads can be obtained. Several issues introduced here will be discussed and their study conducted within one framework, in which the structural part is based on a mixed variational formulation which is geometrically exact and based on finite elements and aerodynamic theories which vary according to the flight regime and missile geometry. The motivations for specific aspects of the missile aeroelasticity study are depicted in Fig. 1.

1.2 Literature Survey

In-flight missiles and projectiles experience various static, dynamic and aeroelastic stability issues with or without thrusts. But those problems have not had sufficient attention and there were at most a couple of published attempts to include the effects of aeroelasticity on the trajectory. Also, the flight mechanics of elastic missiles has seen little attention until recent years. Missiles and projectiles are typical examples of structures that can be represented by beam models. Indeed, several stability problems related to missiles have been solved mostly by linear beam analyses (Euler-Bernoulli or Timoshenko beam analysis analytically or numerically). However, since those approaches are basically linear, they could not assess the nonlinear features arising from the structural dynamics or aeroelasticity. Integrated nonlinear tools that are capable of analysis of multidisciplinary problems such as those mentioned above do not yet exist. In order to place the present work in the right perspective, literature survey is undertaken and divided into several sections.

1.2.1 Stability Problem due to Thrust

Beal (1965) investigated the stability of a uniform free-free beam under controlled follower force. For the case of a constant thrust without directional control system ($K_\theta = 0$; tangential end thrust), he obtained the coalescence branches and the first critical thrust of flutter associated with beam bending. The Galerkin technique gave two zero eigenvalues for all values of thrust at $K_\theta = 0$. Beal concluded from physical reasoning that one eigenvalue was associated with a rigid-body translation mode, and the other with a rigid-body rotation accompanied by translation. So the system is unstable a priori, no matter whether the vibratory modes are stable or not. But here the critical thrust is defined concerning bending vibratory modes. Beal showed that for the constant thrust with directional control, the critical thrust magnitude corresponds to a reduction of the lowest frequency to zero. And finally for the case of pulsating thrusts, he concluded that the longitudinal stiffness plays an important role by showing instabilities due to the variations of the fundamental longitudinal beam frequency.

Peters and Wu (1978) studied the lateral stability of a free flying column subjected to an axial thrust with directional control. They concluded that under the condition of no direction control of the follower force, a pair of zero eigenvalues exist for all forces with an eigenfunction of rank 1 corresponding to rigid-body translation and an eigenfunction of rank 2 corresponding to the rigid-body rotation. But the methods they employed are restricted to models where the rotation sensor is located at the end of the beam. The maximum stable thrust in those models are dependent upon sensor location. Wu (1976) investigated the relation between the critical load and eigencurves by using a finite element method. He concluded that the magnitude and location of a concentrated mass can improve the stability characteristics of a missile. Park and Mote (1985) studied a free-free Euler-Bernoulli beam, transporting a concentrated mass with rotary and transverse inertia under end thrust. The effects of axial location of mass and beam rotation sensor were investigated. For the case of no directional control, flutter or divergence type instability occurs, depending on the magnitude and location of the concentrated mass. The location and the ratio of the concentrated mass to the total beam mass were calculated for force to be maximized. With directional control the instability first occurs with increasing force (called the primary instability). It can be either of the flutter or divergence type depending upon the rotation sensor location. Kirillov and Seyranian (1998) performed study on optimal distribution of mass and stiffness for a beam moving in space under a tangential end force. Their results showed that stability characteristics of the moving beam can be radically improved by using rational distributions of mass and stiffness. But the analysis did not consider aerodynamics.

Park (1987) studied dynamic stability of a free-free Timoshenko beam under a controlled constant follower force. Unlike the above analyses, the effects of rotary inertia and shear deformation on the stability of the beam with a controlled follower force were investigated. In the case of no directional control, he concluded that the instability at the critical force is of the flutter type, and the critical force increases as shear flexibility increases. With directional control, the primary instability type is either flutter or divergence, depending upon the rotation sensor location and the magnitude of the sensor gain. From a practical point of view, the effect of rotary inertia was negligible.

Because of difficulties in realizing follower forces such as thrust in the laboratory, there has been little progress on finding flutter limit experimentally. Sugiyama *et al.* (1995) experimentally verified the effect of damping on the flutter of cantilevered column under rocket thrust and experiment was conducted by the direct installation of a solid rocket motor to the tip end of the columns.

Kim and Choo (1998) investigated a Timoshenko beam subjected to a pulsating follower force, previously addressed only by Beal. The effects of axial location and translation inertia of the concentrated mass are studied, and the relationship between critical forces and widths of instability regions in the vicinity of $2\omega_1$ (twice the first natural frequency of bending vibration) are also examined. They concluded that the variation of the instability region near $2\omega_1$ is closely related to the type of critical force.

It is well known that spinning has a stabilization effect against the directional change of the spinning axis. In a rigid body the stabilization characteristics vary as the spinning speed is increased. However, in case of flexible beam model, the stability region may vary due to the effects of elastic modes. Yoon and Kim (2002) analyzed the dynamic stability of a spinning beam subjected to a pulsating thrust. They concluded that the critical load of a free-free beam under constant thrust was not affected by spinning motion, but as the spinning speed was increased, the instability regions were reduced.

Leipholz and Piche (1984) studied the effect of weight and follower forces on the stability of elastic rods using a two-term Galerkin approximation. Their study included pinned-pinned, clamped-free, and free-free rods. They argued that the representation of the missile mass by assuming a point-mass model cannot lead to critical loads for divergence and flutter, and that such a problem can be avoided by making the more general assumption that the mass per unit length is strictly positive along the entire length of the rod. They showed that instability could be avoided by careful choice of load direction.

1.2.2 Static and Dynamic Aeroelastic Instability

Linear flight mechanics of spinning projectiles dates back to early 20th century and was extended after World War II; see, for example, Foweler *et al.* (1920), McShane *et al.* (1953), and Nicolaides (1953). Platus (1982) reformulated these results in missile-fixed coordinates for reentry vehicles. Later, nonlinear flight mechanics was extensively addressed by Nicolaides (1959), Murphy (1963), Clare (1971), Pepitone and Jacobson (1978), Murphy (1981), and Murphy (1989). Nonlinear flight mechanics of flying missiles still holds an important place in identifying various in-flight problems.

Most material on missile aeroelasticity in the literature is concerned with missile fins than with missile bodies, because missile fins are more flexible and movable and thereby more likely to be in a condition of flutter before the missile body. For more discussion, Vahdati and Imregun (1997), Cayson and Berry (1990), Murray *et al.* (1975), Bae and Lee (2004) deal with the aeroelastic issues on missile fins.

For increasingly flexible missiles there is an increasingly important coupling between so-called flight dynamics phenomena and aeroelasticity. For example, Matejka (1970) conducted both analytical studies and wind tunnel tests of a two stage Terrier-Tomahawk 9 rocket vehicle. Aeroelastic bending (or more specifically, the adverse movement of the system center of pressure due to vehicle flexibility) explains an observed severe reduction in static stability, rendering the rigid-body static stability criteria insufficient. Both the results of the analytical procedure and the wind tunnel tests verified that it was possible for the flexible flight vehicle to be in a condition of roll resonance during powered flight, while highly stable flight is predicted based on rigid-body considerations alone.

Moreover, Elyada (1989) studied the aeroelastic divergence of a rocket vehicle in closed form, where roll resonance and trajectory errors can be predicted. Assuming that the accelerations associated with deformation are negligible compared to the ones connected with rigid-body motion, general divergence analyses are considerably simplified. He showed that the short-period mode angular frequency for the flexible missile is always less than that of the associated rigid vehicle. Thus, in a vehicle designed to roll at a frequency smaller than its rigid short-period mode angular frequency, failure to consider this may result in an unexpected roll resonance. In aerodynamically misaligned vehicles, moderating nonlinear effects (or structural failures) occur at substantially lower dynamic pressures.

There are two kinds of misalignment in missile. One is aerodynamic and the other is thrust. Nakano (1968) conducted study on the bending load due to thrust misalignment. Body divergence, regarded as a phenomenon where the aeroelastic equilibrium without stabilizing moment is lost, was analyzed in terms of dynamic pressure and load factor. He assumed steady-flow aerodynamics and a straight beam for the missile body, showing the relationship between loads and misalignment values. He concluded that in unguided missiles, the ratio of flight dynamic pressure over divergence dynamic pressure should be kept far below than unity because of prediction uncertainty of aeroelastic parameters or performances and load due to wind.

Crimi (1984) derived from Lagrange's equations the linear equations of motion for a spinning, aeroelastic missile; however, structural damping was not included in the formulation. He showed that divergence and dynamic stability are functions of velocity, spin rate and bending stiffness, and that aeroelastic effects cause degradation of vehicle static longitudinal stability as bending stiffness is decreased. Platus (1992) derived a nonlinear equation of motion for slender, spinning missiles using a Lagrangean approach that yields a nonlinear terms that produce nonlinear coupling between the elastic deflections and the rigid-body motions. But no attempt is made to assess the relative importance of the nonlinear terms. He showed that missile flexibility on static stability reduces the critical frequency for pitch-roll coupling, and viscous structural damping has a destabilizing effect on stability at roll rates above the critical frequency for roll-pitch coupling. One should be able to predict the spin and deflection history at any time in flight for a given projectile under given flight conditions. Stearns *et al.* (1988) provides such results but details of the analysis and model are not available for verification or review. Legner *et al.* (1994) studied the primary effects of segmentation (which is used for enhancing the penetrating characteristics of the projectile) and flexure on hypervelocity projectiles, but the details of that analysis are also unavailable. They showed effect of the fundamental bending frequency on the angle of attack and the displacement of the projectile tip and concluded that the most significant tip displacement corresponds to region in time when the angle of attack is maximized, and that increase of bending frequency leads to an increase of angle of attack. Livshits *et al.* (1996) studied dynamic aeroelastic analysis of free-flight rockets, incorporating effects of follower forces together with imperfection factors (dynamic imbalance, thrust misalignment and nonlinear fittings) excluding only gyroscopic effects, which are typical for spin-stabilized types of rockets only. All the loads acting on the rocket were considered as follower forces, including the centrifugal forces coupled with the rocket bending. This is not correct, as such forces are not dissipative as follower forces are. They showed the resonance type of instability; *i.e.*, when the spin rate crosses the rocket's fundamental frequency in bending, the rocket continues to accelerate in roll, developing growing angles of attack after the burnout. They also demonstrated the importance of the imperfections, especially the dynamic imbalance and thrust misalignment.

Even though structural dynamics of flying missiles is essential in getting structural design requirements leading to high performance, it has not been dealt with much in the literature in comparison with its importance. The range of missile stiffness should be known at the preliminary design phase for optimum design in terms of maneuverability and stability. Maloney *et al.* (1970) made an extensive study of mechanical joints in common use and investigated their effects on the flight modes and stiffness. They concluded that tactical missile joints play a major role in dissipating vibratory energy and the energy dissipation comes from both sliding friction and gas pumping.

Some evidence shows that long-finned missiles, such as some anti-tank kinetic energy projectiles, have been forced to spin at rates close to their lowest elastic frequency and have therefore been subject to large inelastic deformations. Special solutions showing spin lock-in at the lowest elastic frequency were developed by Mikhail (1996) and Murphy and Mermagen (2000). Mikhail showed examples of spin lock-in when fin damage produces a roll inducing moment sufficient to cause a steady state spin greater than the lowest elastic frequency and the initial spin rate was zero. However, Murphy and Mermagen (2000) insisted that results obtained by the former should be dismissed due to incorrect expressions for the angular momentum. Murphy and Mermagen (2000) approximated the elastic missile by three rigid bodies connected by two massless elastic beams and showed that it is impossible to cause spin lock-in by roll inducing moment and zero initial spin alone. It should be noted that the use of the three-body model is a major simplification of the actual physical problem. Later they replaced the three-body model with a continuous elastic model using differential equations in Murphy and Mermagen (2000) and obtained numerical results for the natural frequencies, flexing waveforms and equilibrium spins for a specific missile.

Reis and Sundberg (1967) investigated the causes of large coning angle that a Nike-Tomahawk sounding rocket experienced during flight. They assumed that Magnus forces, aeroelastic bending, and/or lee-side boundary separation were probable reasons. Based on flight data they showed that aeroelastic bending was one of the causes. Cochran and Christensen (1979) studied the post-launch effect of transverse bending of a spinning free-flight rocket during the guidance phase. They used two different methods which are simple two-body model and sophisticated assumed-modes model.

1.2.3 Trajectory Optimization

The optimal trajectory is usually found by minimizing a performance index that contains constraints on state and control variables as well as a minimum time structure, and is based on a point-mass model. The simplistic models that are often used are unable to capture coupling between optimal trajectories and the stability and loads that can be provided by powerful simulation programs using a full 3-D finite element method. Nasuti and Innocenti (1996) included maneuverability and agility considerations in the optimization process, with a kinematic model and constraints obtained from dynamic limits. A maneuver envelope was proposed that would allow the incorporation of design parameters into agility optimization. The speed from propulsive considerations, the load factor from structural limitations, and the turn rate from stall characteristics were bounded for the maneuverability envelope; another constraint was an estimated upper bound on turn rate in the post-stall condition function of the maximum propulsive control.

Muzumdar and Hull (1996) developed an optimal midcourse guidance law for a high-thrust, bank-to-turn, short-range attack missile. The analytical guidance law was obtained by making approximations in the optimal control problem works for midcourse guidance but needs terminal guidance to hit the target. The error compensation (EC) guidance law enables the missile to hit the target without terminal guidance. The EC guidance law is obtained by replacing the approximation terms by bounded controls, where the bounds are handled indirectly by adding penalty terms to the performance index. The EC weights are determined by using the EC control in the trajectory optimization problem and minimizing the flight time with respect to the weights.

Wang *et al.* (1993) developed an optimality-based feedback trajectory shaping guidance law. The guidance law is assumed to be in some feedback form. The optimal solution involves solving a nonlinear two-point, boundary-value problem, which is formidable, expensive, and fragile (*i.e.* not robust). A common practice is to parameterize the control and solve a suboptimal control problem through parameter optimization. The approach combines the design of guidance parameters and control gains into the optimization process. It was shown that this control law would achieve better performance and be robust with respect to the initial condition perturbations although the open loop control has the shortcoming that control is less responsive to the perturbations.

Hallman (1990) studied how the optimal solution is affected by changes to design parameters that are held fixed during the optimization, after determining an optimum trajectory design. This area of study is called postoptimality or sensitivity analysis. As opposed to the conventional brute force approach where repeated optimization problems are solved, sensitivity analysis allows an efficient, accurate, and systematic methodology for studying perturbations about an optimal design.

Han and Balakrishnan (1999) investigated the use of an “adaptive critic” controller to steer an agile missile to completely reverse its flight path angle in minimum time starting from given initial and final Mach numbers and with a constraint on the minimum flight Mach number. This was undertaken for optimal solutions that encompass perturbations to the assumed initial conditions or a family of initial conditions. The neighboring optimal control allows pointwise solutions of an optimal two-point, boundary-value problem to be used with a linearized approximation over a range of initial conditions but can fail outside the regime in which linearization is valid. Dynamic programming can handle a family of initial conditions for linear as well as nonlinear problems. Both solution methods are computationally intensive, and the solution is not available in feedback form. For implementation this becomes a drawback. Outside of dynamic programming, there is no unified mathematical formalism under which a controller can be designed for nonlinear systems. They proposed a formulation that (1) solves a nonlinear control problem directly without any approximation to the system model, (2) yields a control law in a feedback from as a function of the current states, and (3) maintains the same structure regardless of the type or problem. Such a formulation is afforded by the field of neural networks, specifically, the adaptive critic architecture. They showed that this method provides optimal control to the missile from an envelope of initial Mach numbers in minimum time. An added advantage in using these neurocontrollers is that they provide minimum time solutions even when one changes the initial flight path angle from zero to any nonzero (positive) value. Dynamic programming has been the main tool for such solutions.

Imado *et al.* (1990) studied optimal midcourse guidance laws for medium-range, air-to-air missiles that employ different guidance modes depending on the required missile velocity and navigation time. This was done for two separate problems: (1) against a faraway or low-altitude target where missile velocity is a prime factor, so that the midcourse guidance law that maximizes the residual velocity is preferable; (2) against a near target where the time margin is most important so that the midcourse guidance law that minimizes the interception time is preferable. After the required missile residual velocity is analyzed against a conventional and an advanced target, four types of midcourse guidance laws depending on objectives are presented, each with its merits and demerits.

1.3 Present Approach

The aim of the current research is to investigate the effects of follower forces on aeroelastic stability of missiles. Missile aerodynamics is quite complex, and analytical models cannot exactly simulate the complex flow under various flight conditions. However, an aerodynamic model that is representative of some typical flight conditions sufficient to see how aerodynamics interacts with thrust. The structural model for the missile is based on the mixed variational formulation. It should be noted that most missile flutter problems shown in the literature have only to do with missile fins. In the present research, our efforts to understand missile body flutter have gone through some difficulty due to extremely limited literature; a rigorous validation thus appears to be impossible. Furthermore, several authors have revealed there are two zero eigenvalues in planar deformation problems for a free-free beam with a follower force. These zero eigenvalues rigid-body modes; thus one can say, a priori, that free-free beams are neutrally stable in these two rigid-body modes. This does not involve bending of the beam structure. Here the stability analysis is only concerned with bending modes.

The ideas embedded above along with all the works in the literature survey could be explored within one framework which consists of structural formulation and aerodynamic model. The structural formulation is based on finite-element based nonlinear one-dimensional analysis. This finite element analysis is very powerful in that it is geometrically exact and allows the use of very simple shape functions. The most challenging part comes from the aerodynamics which is very dependent on the missile geometry and flight conditions. The main idea concerning the present approach to aerodynamic modeling is to build *representative* aerodynamic models that are suited to serve the current research purposes with the sacrifice of some accuracy. Thus, analytical, closed-form aerodynamic expressions or at least less computational methods such as the modified Newtonian method or piston theory are preferred. Another thing to be said about missile aerodynamics expressions is that differentiated variables should be expressed in other kinematic variables in keeping with the lowest order of differentiations in the structural formulation. The follower force caused by missile thrust has its own instabilities without consideration of an aerodynamic model, and the same observation applies also to aeroelastic instabilities without consideration of the effects of the follower force. Therefore, the interactions of follower forces and aerodynamic forces make one think of the possible stability boundaries suggested by Leipholz (1980) and Huseyin (1978).

2 Effect of Thrust on Missile Stability

Missile flight can be divided into two phases, powered flight and free flight without thrust. During powered flight, loads and dynamic stability are the main issues. Projectiles are under severe stresses. Total mass varies from propellant consumption and aerodynamic center changes as well due to possible bending deformation from considerable lift on both nose and tail and the variation of aerodynamic coefficients. There is a certain velocity where the shifted location of the aerodynamic center coincides with the location of the center of mass. But this situation can be avoided by keeping the burnout velocity below that velocity. Such a variation trend that is depicted in Livshits and Yaniv (1999) is introduced for clarification in Fig. 2 for movement of the missile center of mass and aerodynamic center for both rigid- and flexible-body models as a function of velocity.

During powered flight, the missile reaches its maximum speed, the so-called burnout velocity. After reaching the burnout velocity, the missile decelerates. Therefore, accelerating flight with thrust, steady flight when the thrust magnitude is equal to drag, and ballistic flight without thrust are of interest to current research efforts.

2.1 Structural Formulation

The structural part of the formulation comes from the mixed variational formulation based on the exact intrinsic equations for dynamics of moving beams presented by Hodges (1990). Modifications of the original variational principle necessary for the present study are the inclusion of the gravitational potential energy and appropriate energy variation for dealing with rigid-body dynamics, the analysis of which is needed for the missile time-marching scheme. The frames presented

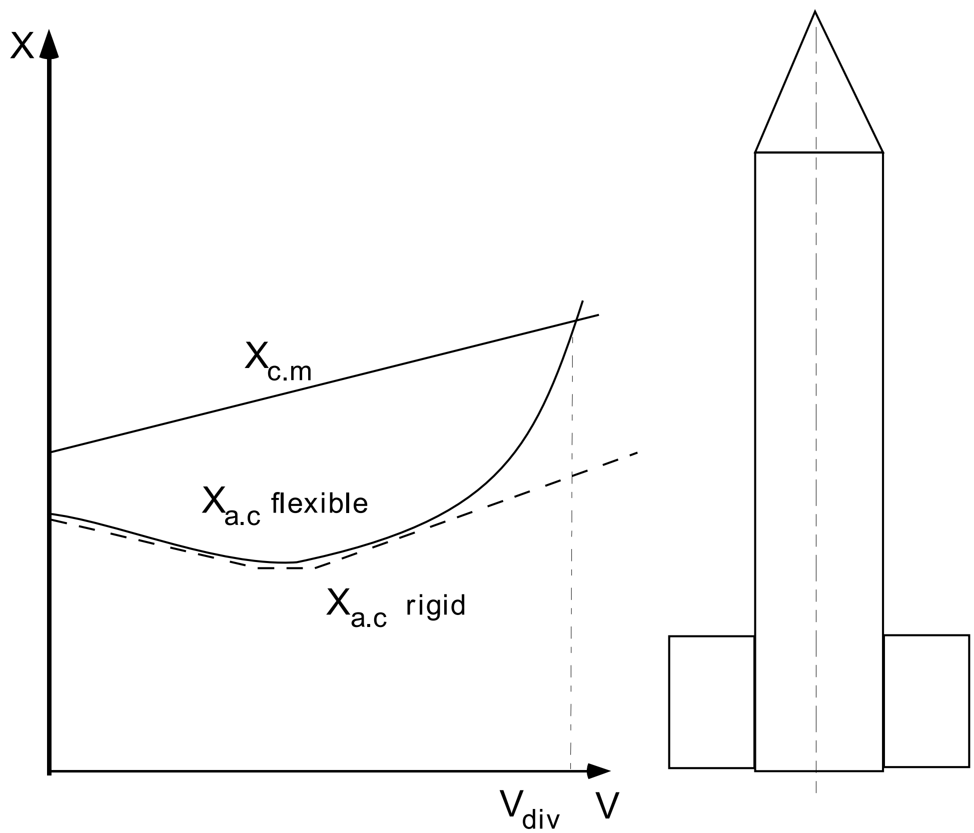

Figure 2: Typical movement of the center of mass and aerodynamic center

here are the undeformed beam cross-sectional frame (the b basis), the deformed cross-sectional frame (the B basis), and the inertial frame (the i basis). Here we follow the same rule for the variable notation as shown by Hodges (1990), except that the subscript o represents the missile reference point for taking care of rigid-body motion. The variables with subscript b and o are measured in the b frame, except for u_o, the basis for which is the inertial frame. The variational formulation starts with extended Hamilton's principle

$$\int_{t_1}^{t_2}\int_0^l \left[\delta(K-U)+\overline{\delta W}\right]dx_1\,dt=\overline{\delta \mathrm{A}} \tag{1}$$

where t_1 and t_2 specify the time interval over which the solution is sought; K and U are the kinetic and strain energy densities per unit length, respectively; and $\overline{\delta \mathrm{A}}$ is the virtual action at the ends of the beam and at the ends of the time interval. The contribution of all gravitational forces is handled by means of its potential energy, which is written as

$$G=\int_0^\ell mge_3^T\left[u_o+C_o^T(r_b+u_b+C^T\xi_B)\right]dx_1 \tag{2}$$

where the superscript T indicates the transpose of a matrix, $e_3=\lfloor 0\ \ 0\ \ 1\rfloor^T$, r_b is the position from the missile body reference point, u_o is the displacement of missile reference point in the i frame, u_b is the displacement of the points on missile reference line in the b frame, ξ_B is the mass offset from the missile reference line, m is mass per unit length, C_o is the rotation matrix from i frame to b frame, and C is the rotation matrix from b frame to B frame. The kinematic relationships and the expressions for the velocities and the generalized strains can be written as

$$v_o=C_o\dot{u}_o \tag{3}$$

$$\widetilde{\omega}_o=-\dot{C}_oC_o^T \tag{4}$$

$$V_B=C\left[v_o+\dot{u}_b+\widetilde{\omega}_o(r_b+u_b)\right] \tag{5}$$

$$\Omega_B=\left(\frac{\Delta-\frac{\widetilde{\theta}}{2}}{1+\frac{\theta^T\theta}{4}}\right)\dot{\theta}+C\omega_o \tag{6}$$

$$\gamma=C(e_1+u_b')-e_1 \tag{7}$$

$$\kappa=\left(\frac{\Delta-\frac{\widetilde{\theta}}{2}}{1+\frac{\theta^T\theta}{4}}\right)\theta' \tag{8}$$

where the $\widetilde{(\ )}$ operator converts a 3×1 column matrix, say $v=\lfloor v_1\ \ v_2\ \ v_3\rfloor^T$, to its 3×3 antisymmetric dual matrix

$$\widetilde{v}=\begin{bmatrix}0 & -v_3 & v_2\\ v_3 & 0 & -v_1\\ -v_2 & v_1 & 0\end{bmatrix} \tag{9}$$

$e_1=\lfloor 1\ \ 0\ \ 0\rfloor^T$, $\dot{(\ )}$ and $(\ )'$ are differentiations with respect to time and x_1, respectively. The orientation of the B frame with respect to the b frame is represented using Rodrigues parameters, which have been applied to nonlinear beam problems with success. The rotation matrix relating the B frame to the b frame is written as

$$C = \frac{\left(1 - \frac{1}{4}\theta^T\theta\right)\Delta - \widetilde{\theta} + \frac{1}{2}\theta\theta^T}{1 + \frac{1}{4}\theta^T\theta} \tag{10}$$

For the orientation of the missile body frame (i.e. the b frame), however, the regular use of the Rodrigues parameters is insufficient because of their well-known singularity at a rotation angle of $180°$. Thus, the direction cosines of b in i are used as rotational variables for the rigid-body motion of the missile. The strain and force measures, along with velocity and momentum measures, are related through the constitutive laws in the form

$$\begin{aligned} \begin{Bmatrix} P \\ H \end{Bmatrix} &= \begin{bmatrix} m\Delta & -m\widetilde{\xi} \\ m\widetilde{\xi} & I \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix} \\ \begin{Bmatrix} F \\ M \end{Bmatrix} &= [S] \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \end{aligned} \tag{11}$$

All the elastic virtual variations are the same as the expressions in Hodges (1990) except for the virtual quantities related to rigid-body variables. Here the details of the rigid-body part are described. After some manipulations, the virtual variations of rigid-body variables v_o and ω_o in Eqs. (3) and (4) may be expressed as

$$\delta v_o = \delta\dot{q}_o + \widetilde{\omega}_o\delta q_o + \widetilde{v}_o\delta\psi_o \tag{12}$$

$$\delta\omega_o = \delta\dot{\psi}_o + \widetilde{\omega}_o\delta\psi_o \tag{13}$$

where δq_o and $\delta\psi_o$ are virtual quantities defined in the b frame, $i.e.$ $\delta q_o = C_o\delta u_o$. Also, Eq. (2) can be expressed as

$$\begin{aligned} \delta G =& \delta q_o^T C_o \int_0^\ell mge_3 dx_1 - \delta\psi_o^T \widetilde{C_o e_3} \int_0^\ell mg(r_b + u_b + C^T\xi_B)dx_1 \\ &+ \int_0^\ell \overline{\delta q}_B^T CC_o e_3 mg dx_1 + \int_0^\ell \overline{\delta\psi}_B^T \widetilde{\xi_B} CC_o e_3 mg dx_1 \end{aligned} \tag{14}$$

Adding the varying action and virtual work terms contributed by the rigid-body variables, one finds that

$$\overline{\delta \mathrm{A}}_o = \left(\delta q_o^T \hat{P}_o + \delta\psi_o^T \hat{H}_o\right)\Big|_{t_1}^{t_2} \tag{15}$$

$$\overline{\delta W}_o = \delta q_o^T f_o + \delta\psi_o^T m_o \tag{16}$$

where f_o and m_o are column matrices containing the measure numbers of force and moment vectors acting on the reference point in the b frame; $\hat{P}_o$ and $\hat{H}_o$ are linear and angular momenta of reference point at the ends of specified time interval in the b frame. Additional terms of elastic virtual quantities stemming from rigid-body variations are

$$\delta V_B^T : \delta v_o^T C^T + \delta\omega_o^T(\widetilde{r}_b + \widetilde{u}_b)C^T \tag{17}$$

$$\delta\Omega_B^T : \delta\omega_o^T C^T \tag{18}$$

For the variations of individual energies, and virtual work done on the system, we have

$$\delta K = \delta v_o^T P_o + \delta\omega_o^T H_o + \int_0^l (\delta V_B^T P_B + \delta\Omega_B^T H_B) dx_1 \tag{19}$$

$$\delta U = \delta G + \int_0^l (\delta\gamma^T F_B + \delta\kappa^T M_B) dx_1 \tag{20}$$

$$\overline{\delta W} = \delta q_o^T f_o + \delta\psi_o^T m_o + \int_0^l (\overline{\delta q}_B^T f_B + \overline{\delta\psi}_B^T m_B) dx_1 \tag{21}$$

where the unknowns are F_B and M_B, the sectional force and moment measures in the B basis, respectively; P_B and H_B, the sectional linear and angular momentum measures in the B basis, respectively; γ and κ, the force and moment strains, respectively; V_B and Ω_B, the linear and angular velocity measures of the beam reference line in the B basis, respectively; and f_B and m_B, the external force and moment, respectively.

The expressions for various virtual quantities such as δV_B, $\delta\Omega_B$, $\delta\gamma$, and $\delta\kappa$ are substituted into the energy equations. In the mixed variational formulation, the appropriate kinematical and constitutive relations are enforced as additional constraints using Lagrange multipliers and are then adjoined to Hamilton's weak principle expressed in terms of given energies.

The modified weak form from the original mixed variational formulation including the rigid-body part in the proper way then can be written as

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_0^{\ell}\Bigg\{\left[\overline{\delta q}_B'^T - \overline{\delta q}_B^T\widetilde{\kappa} - \overline{\delta\psi}_B^T(\widetilde{e}_1+\widetilde{\gamma})\right]F_B + \left(\overline{\delta\psi}_B'^T - \overline{\delta\psi}_B^T\widetilde{\kappa}\right)M_B \\
&\quad - \Bigg[\dot{\overline{\delta q}}_B^T - \overline{\delta q}_B^T\widetilde{\Omega}_B - \overline{\delta\psi}^T\widetilde{V}_B + \delta v_o^T C^T \\
&\quad + \delta\omega_o^T(\widetilde{r}_b+\widetilde{u}_b)C^T\Bigg]P_B - \left(\dot{\overline{\delta\psi}}_B^T - \overline{\delta\psi}\widetilde{\Omega}_B + \delta\omega_o^T C^T\right)H_B \\
&\quad + \overline{\delta F}^T\left[e_1 - C^T\left(e_1+\gamma\right)\right] - \overline{\delta F}'^T u_b \\
&\quad - \overline{\delta M}^T\left(\Delta + \frac{1}{2}\widetilde{\theta} + \frac{1}{4}\theta\theta^T\right)\kappa - \overline{\delta M}'^T\theta \\
&\quad - \overline{\delta P}^T\left[v_o + \widetilde{\omega}_o(r_o+u_b) - C^T V_B\right] + \dot{\overline{\delta P}}^T u_b \\
&\quad - \overline{\delta H}^T\left(\Delta + \frac{1}{2}\widetilde{\theta} + \frac{1}{4}\theta\theta^T\right)(C\omega - \Omega_B) \\
&\quad + \dot{\overline{\delta H}}^T\theta - \overline{\delta q}_B^T f_B - \overline{\delta\psi}_B^T m_B\Bigg\}dx_1 dt \\
&\quad + \int_{t_1}^{t_2}\left(\delta G^* - \delta v_o^{*T}P_o - \delta\omega_o^{*T}H_o - \delta q_o^T f_o - \delta\psi_o^T m_o\right)dt \\
&= -\int_0^{\ell}\left(\overline{\delta q}_B^T\hat{P}_B + \overline{\delta\psi}_B\hat{H}_B - \overline{\delta P}^T\hat{u}_b - \overline{\delta H}^T\hat{\theta}\right)\Bigg|_{t_1}^{t_2}dx_1 \\
&\quad + \int_{t_1}^{t_2}\left(\overline{\delta q}^T\hat{F} + \overline{\delta\psi}^T\hat{M} - \overline{\delta F}^T\hat{u} - \overline{\delta M}^T\hat{\theta}\right)\Bigg|_0^{\ell}dt - \left(\delta q_o^T\hat{P}_o + \delta\psi_o^T\hat{H}_o\right)\Bigg|_{t_1}^{t_2}
\end{aligned}
\tag{22}
$$

where algebraic expressions defining certain variables in terms of displacement and rotation variables are denoted by $(\ )^*$ and $(\hat{\ })$ represents discrete boundary values either at the ends of beam or at the ends of time interval. In addition to the above formulation, Poisson equation ($\dot{C}_o+\widetilde{\omega}_o C_o = 0$) of direction cosine matrix is adjoined using Lagrange multipliers, which is not included here for the sake of brevity.

2.2 Aerodynamics

Available aerodynamics tools have been evaluated for computation of loads on missiles. Missile loads are very dependent on the flight condition and missile geometry. Several technical methods are extensively described in Moore *et al.* (1998), Nielsen (1988), and Moore (2000). The validity of slender-body theory, which is based on potential flow, has been well established by comparison with experimental data in Allen and Perkins (1951) for a wide range of Mach numbers. An extended slender-body theory is discussed by Adams and Sears (1952). An unsteady version of slender-body theory for aeroelasticity was presented in Bisplinghoff *et al.* (1955). For our purposes, the aerodynamic loads on a missile body can be calculated with sufficient accuracy for the sort of interdisciplinary tradeoff studies we anticipate doing by using slender-body theory augmented by a viscous cross-flow theory; see Allen (1949). There are parts of the missile for which these methods are not suitable, and for these other methods are used. For example, the loads on the missile fins and tail are calculated by thin-airfoil theory in low-speed flight and by piston theory Lighthill (1953) in hypersonic flight. With the combination of the viscous cross-flow theory of Allen (1949) and the potential flow slender-body theory in Bisplinghoff *et al.* (1955), we can take into account the bending deformation and unsteadiness of the flow. The resulting equation then reduces to

$$\begin{aligned}\frac{dN}{dx} =& -\rho_\infty \frac{dS}{dx}\left(U^2\frac{\partial\lambda}{\partial x} + U\frac{\partial\lambda}{\partial t}\right) + \eta c_d d \frac{\rho_\infty U^2}{2}\alpha^2 \\ & -\rho_\infty S\left(U^2\frac{\partial^2\lambda}{\partial x^2} + 2U\frac{\partial^2\lambda}{\partial x\partial t} + \frac{\partial^2\lambda}{\partial t^2}\right)\end{aligned} \tag{23}$$

where $\lambda = u_b + \alpha(x - x_o)$; N is the normal force column matrix; U and ρ_∞ are the freestream velocity and air density, respectively; α is the angle of attack and sideslip angle column matrix at the reference point; x_o is the location of the reference point; η is the ratio of the drag coefficient of a circular cylinder of finite length to that of a circular cylinder of infinite length; c_d is the drag coefficient of a circular cylinder and d is the missile diameter. Since the aerodynamic forces involve higher derivatives, which do not allow one to use low order shape functions, the weak form including the aerodynamic forces in Eq. (23) needs to be integrated by parts to reduce the order of differentiation. Unfortunately, even after this integration by parts, there are still some derivatives of variables. Therefore, in accordance with the lowest order of differentiation for the variables in the expressions for aerodynamic forces, kinematic expressions such as

$$\dot{u}_b = (\Delta + \tilde{\theta})V_B - v_o - \tilde{\omega}_o(r_b + u_b) \tag{24}$$

$$u_b' = (\Delta - \tilde{\theta})(e_1 + \gamma) - e_1 \tag{25}$$

$$\dot{u}_2' = e_3^T(\Omega_B - C\omega_o) \tag{26}$$

$$\dot{u}_3' = -e_2^T(\Omega_B - C\omega_o) \tag{27}$$

$$U = \sqrt{v_o^T v_o} \tag{28}$$

are used to reduce the order of the derivatives. So, for differentiated λ,

$$\frac{\partial\lambda}{\partial x} = \begin{Bmatrix} 0 \\ u_2' + \beta \\ u_3' + \alpha \end{Bmatrix}$$

$$\frac{\partial\lambda}{\partial t} = \begin{Bmatrix} 0 \\ \dot{u}_2 + \dot{\beta}(x - x_0) \\ \dot{u}_3 + \dot{\alpha}(x - x_0) \end{Bmatrix}$$

In order to completely determine the angle of attack and sideslip angle quantities in terms of other kinematic quantities, we need to obtain the rotation matrix from the inertial frame to the wind frame, C^{WI}. From the frame definitions,

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} C_{11}^{WI} \\ C_{12}^{WI} \\ C_{13}^{WI} \end{Bmatrix} = \frac{\Delta}{U} C_o^T v_o \tag{29}$$

If θ_w is defined as a column matrix of Rodrigues parameters, we can obtain

$$\theta_w = \theta_{w1} e_1 + \frac{1}{1+q_1}[2\widetilde{e_1} + \theta_{w1}(\Delta - e_1 e_1^T)]C^{IW} e_1 \tag{30}$$

as given by Hodges (1990). After a holonomic constraint, θ_{w1}=0 is imposed, we obtain

$$\theta_w = \frac{2}{1+q_1} \begin{Bmatrix} 0 \\ -q_3 \\ q_2 \end{Bmatrix} \tag{31}$$

so that

$$\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \frac{\Delta}{U^2}[\{C_o^T \tilde{\omega}_o v_o + C_o^T a\}U - \frac{C_o^T v_o v_o^T a}{U}] \tag{32}$$

and

$$\dot{\theta}_w = \frac{2}{(1+q_1)^2} \begin{bmatrix} 0 \\ -\dot{q}_3(1+q_1) + q_3\dot{q}_1 \\ \dot{q}_2(1+q_1) - q_2\dot{q}_1 \end{bmatrix} \tag{33}$$

Thus, if one can get the one row components of rotation matrix C^{WI} from Eq. (29), rotation matrix C^{WI} and angular velocity of wind frame with respect to inertial frame can be determined as follows

$$C^{WI} = \frac{\left(1 - \frac{1}{4}\theta_w^T \theta_w\right)\Delta - \widetilde{\theta_w} + \frac{1}{2}\theta_w \theta_w^T}{1 + \frac{1}{4}\theta_w^T \theta_w} \tag{34}$$

$$\omega^{WI} = \frac{(\Delta - \frac{\tilde{\theta}_w}{2})}{1 + \frac{\theta_w^T \theta_w}{4}} \dot{\theta}_w \tag{35}$$

Since $C^{bW} = C_o C^{IW}$ and $\dot{C}^{IW} = C^{IW} \widetilde{\omega}^{WI}$, it then follows that

$$\dot{C}^{bW} = -\tilde{\omega}_o C_o C^{IW} + C_o C^{IW} \tilde{\omega}^{WI} \tag{36}$$

Then, we can find α, β, $\dot{\alpha}$ and $\dot{\beta}$ in terms of different variables. It should be noted that as the definition of λ implies, we still have local angles of attack varying along the missile even when the rigid-body angle of attack at the reference point is zero. That leads to the idea that in simple rectilinear flight, a missile can still experience aeroelastic deformation in various speed ranges.

The above slender-body aerodynamics is thought to be relatively useful at the flight range below Mach 5 since above 5, more advanced and complicated aerodynamics caused by aerodynamic heating will be needed. Also, most full-scale missiles operates at below Mach 4, so the current aerodynamics will be used in flight speed range between Mach 2 and Mach 5.

Results according to above formulation are in good agreement with existing experimental data. Fig. 4 represents the comparison between slender-body theory and experiments from Lordon *et al.* (1990) for steady flow when the angle of attack is 10°. The average normal force and pitching moment are in excellent agreement and the distributed force shows sufficiently good agreement for the purposes of our current research.

Drag is very dependent on the configuration and flight condition. Body, wings and tails all make contributions to the drag, and the body drag is dominant especially in the supersonic flight regime. For the calculation of skin friction drag, a turbulent skin friction coefficient and laminar skin friction coefficient should be obtained. For most flight conditions laminar flow prevails over the extreme forward portion of the missile body, followed by completely turbulent flow over the remaining portion of the missile Chin (1961). The difficulty lies in determining the transition point from laminar flow to turbulent flow. Since no theoretical methods are known to accurately determine the transition point, the point on the missile body where the Reynolds number reaches 10^6 is generally taken from experience and test data Moore *et al.* (1998). A reasonable assumption for a missile body with normal roughness it to take the nose tangency point at the end of the nose or forebody section as transition point Chin (1961). Fig. 3 shows a typical drag distribution over missile body at supersonic flight. One engineering method for calculating skin-friction drag is to compute skin friction for an equivalent flat plate of the same surface area, length, and Reynolds number as the original body. The axial force is then corrected for body shape by the use of a three-dimensional shape factor. The method of Van Driest (1951) is used for mean skin friction coefficient of compressible flow on a two-dimensional flat plate, and a modified Blasius theory including compressibility effects, as discussed by Moore *et al.* (1995), is used for laminar skin friction drag. To get the wave drag over the range of Mach number from 2 – 5, the second-order shock expansion method (see De Jarnette *et al.* (1979)) or a modified Newton method is usually applied to the entire body. At the missile base, the pressure goes down below the freestream pressure due to the external flow. This base drag is highly dependent on Mach number and the presence of a boat tail or flare. For the purposes of the current research, methods based on approximate, closed-form solutions, or that at least require the least computational effort have been employed, such as the modified Newtonian method and tangent cone method. Also, in case of spin stabilized missiles, additional lift should be considered due to the effect of spin, which is called the Magnus effect; see Jumper *et al.* (1991). All the available methods for missile aerodynamics are well documented in Moore (2000); Mendenhall (1991).

The rigid-body force and moment on the reference point due to distributed force are explained in the appendix. The above discussed aerodynamics along with structural dynamics formulation will lead to a complete solution for aeroelastic stability problems for various missiles and projectiles. Some additional variables such as acceleration, angular velocity and linear velocity at the final time of time interval, and direction cosine matrix will appear and they should be embed-

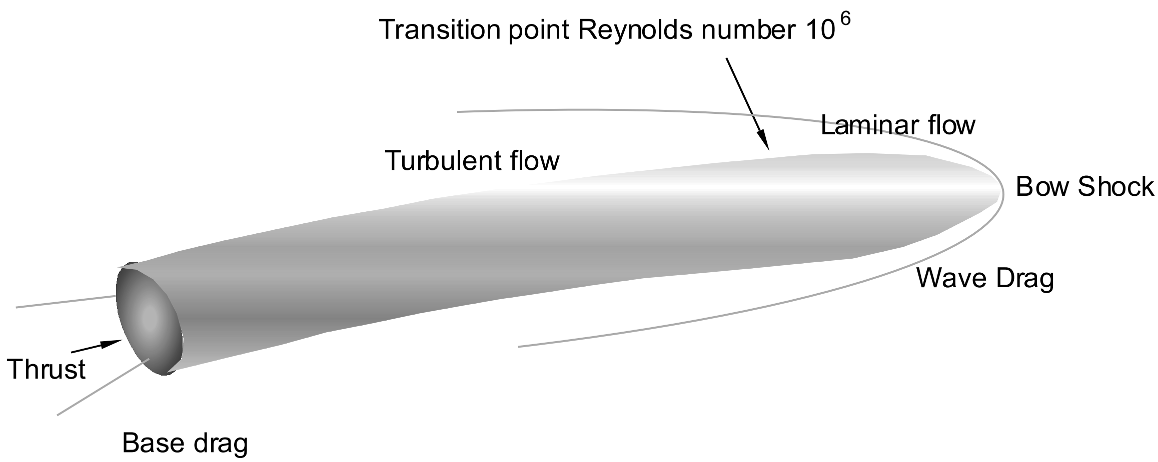

Figure 3: Body drag distribution at supersonic flow

ded properly in the system equations. This combined aeroelastic formulation will yield iterative solutions over time which affect both aerodynamic loads and structural loads.

2.3 Solution Methodology

Now space-time finite elements are used to obtain the time history of the missile motion, which is needed to investigate the nonlinear dynamics of the missile in flight. This kind of space-time finite element approach is useful in finding the amplitude of the limit cycle oscillations and checking the nonlinear system response. To use this space-time finite element, the formulation should be converted into its weakest form in space as well as time. After integration by parts of the additional energy expression due to rigid-body motion, the unknowns are neither differentiated with respect time nor space from henceforth, so that constant shape functions may be used for them. Since the weak form is linear in the virtual quantities and they may be differentiated with respect to both space and time, and linear/bilinear shape functions are used for them, and element numerical

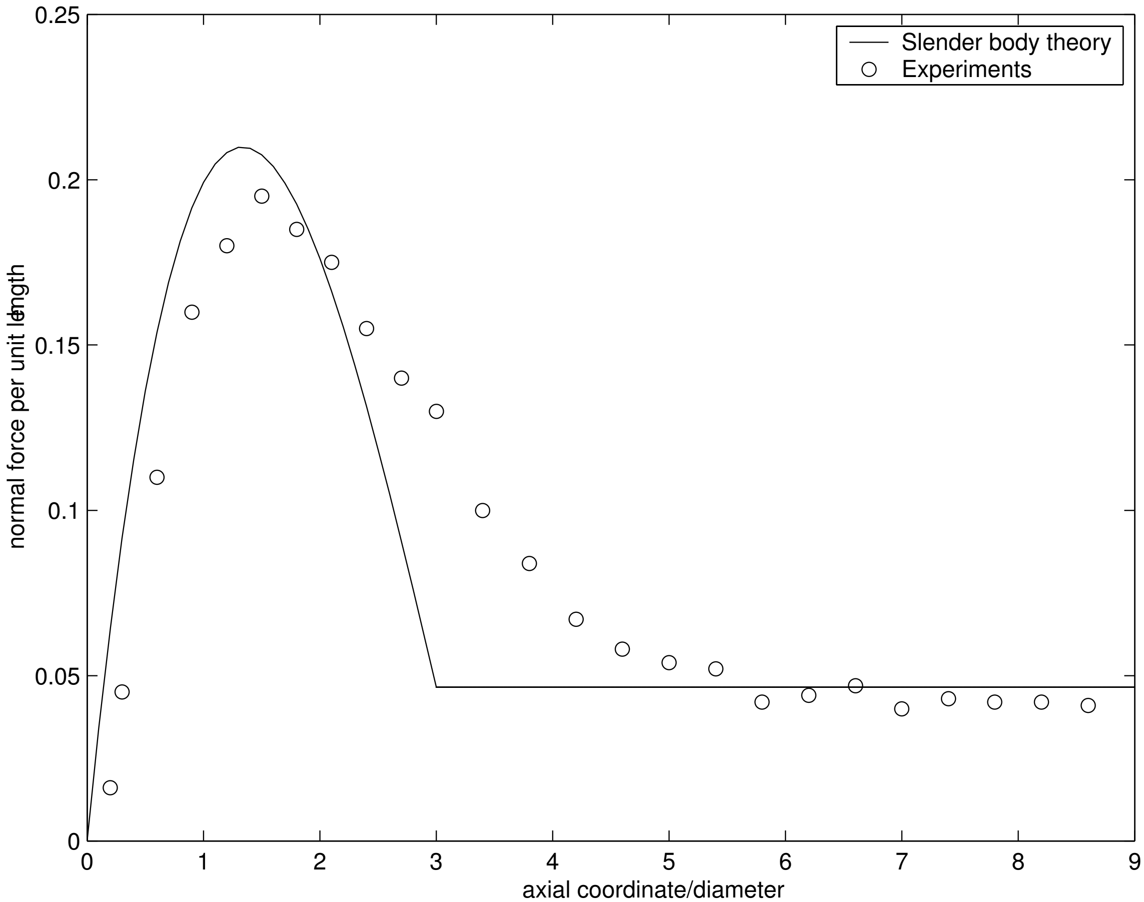

Figure 4: Comparison of slender-body theory with experiments

quadrature is not needed. Thus,

$$
\begin{aligned}
\overline{\delta q}_B &= \overline{\delta q}_{i_s}(1-\xi)(1-\tau) + \overline{\delta q}_{i_f}(1-\xi)\tau \\
&\quad + \overline{\delta q}_{i+1_s}\xi(1-\tau) + \overline{\delta q}_{i+1_f}\xi\tau && u = u_i \\
\overline{\delta \psi}_B &= \overline{\delta \psi}_{i_s}(1-\xi)(1-\tau) + \overline{\delta \psi}_{i_f}(1-\xi)\tau \\
&\quad + \overline{\delta \psi}_{i+1_s}\xi(1-\tau) + \overline{\delta \psi}_{i+1_f}\xi\tau && \theta = \theta_i \\
\overline{\delta F} &= \overline{\delta F}_i(1-\xi) + \overline{\delta F}_{i+1}\xi && F = F_i \\
\overline{\delta M} &= \overline{\delta M}_i(1-\xi) + \overline{\delta M}_{i+1}\xi && M = M_i \\
\overline{\delta P} &= \overline{\delta P}_{i_s}(1-\tau) + \overline{\delta P}_{i_f}\tau && P = P_i \\
\overline{\delta H} &= \overline{\delta H}_{i_s}(1-\tau) + \overline{\delta H}_{i_f}\tau && H = H_i \\
\delta q_o &= \delta q_{os}(1-\tau) + \delta q_{of}\tau \\
\delta \psi_o &= \delta \psi_{os}(1-\tau) + \delta \psi_{of}\tau
\end{aligned}
$$

where subscripts s and f denote the variable values at the starting and final time of time interval. After some manipulations it can be shown that some of the resulting discretized equations are linear combinations of the others, leaving us free to discard the excess equations. For illustrative purposes, we consider only the structural part for the time being. Then,

$$
\begin{aligned}
&\overline{\delta q}_{if} && \overline{P}_i = \frac{\hat{P}_{if} + \hat{P}_{is}}{2} \\
&\overline{\delta \psi}_{if} && \overline{H}_i = \frac{\hat{H}_{if} + \hat{H}_{is}}{2} \\
&\overline{\delta P}_{if} && \overline{u}_i = \frac{\hat{u}_{if} + \hat{u}_{is}}{2} \\
&\overline{\delta H}_{if} && \overline{\theta}_i = \frac{\hat{\theta}_{if} + \hat{\theta}_{is}}{2} \\
&\overline{\delta q}_{n+1} && \overline{F}_i = \frac{\hat{F}_i + \hat{F}_{i+1}}{2} \\
&\overline{\delta \psi}_{n+1} && \overline{M}_i = \frac{\hat{M}_i + \hat{M}_{i+1}}{2} \\
&\overline{\delta F}_{n+1} && \overline{u}_i = \frac{\hat{u}_i + \hat{u}_{i+1}}{2} \\
&\overline{\delta M}_{n+1} && \overline{\theta}_i = \frac{\hat{\theta}_i + \hat{\theta}_{i+1}}{2} \\
&\delta q_{of} && \overline{P}_o = \frac{\hat{P}_{of} + \hat{P}_{os}}{2} \\
&\delta \psi_{of} && \overline{H}_o = \frac{\hat{H}_{of} + \hat{H}_{os}}{2}
\end{aligned}
$$

By virtue of these relations, the number of unknowns corresponding to each virtual quantity is reduced. Then, the total number of equations related to elastic variables; $\overline{\delta q}_{is}$, $\overline{\delta \psi}_{is}$, $\overline{\delta P}_{is}$, $\overline{\delta H}_{is}$, $\overline{\delta F}_i$, $\overline{\delta M}_i$ (i=1 to n), is 18n. The total number of equations defining rigid-body motion related to δq_{os}(3), $\delta \psi_{os}$(3), δv_o(3), $\delta \omega_o$(3), is 12 if we do not consider direction cosine and acceleration variables. Unknown variables are $\hat{F}_i$, $\hat{M}_i$, $\hat{P}_{if}$, $\hat{H}_{if}$, $\hat{u}_{if}$, $\hat{\theta}_{if}$, $\hat{P}_{of}$, $\hat{H}_{of}$, $\hat{v}_o$, $\hat{\omega}_o$ after specifying boundary conditions ($\hat{F}_{n+1}$, $\hat{M}_{n+1}$ and $\hat{u}_1$, $\hat{\theta}_1$) and initial conditions ($\hat{P}_{is}$, $\hat{H}_{is}$, $\hat{u}_{is}$, $\hat{\theta}_{is}$, $\hat{P}_{os}$, $\hat{H}_{os}$) for each element, therefore in total 18n +12. The above discussion shows that the total number of equations and the total number of unknowns are equal.

With these system equations and unknown variables, if we just consider structural dynamics, the mixed variational formulation takes the form

$$F(X_s, X_f, X) = 0 \tag{37}$$

where X is a column matrix of all structural variables and X_s and X_f are its initial and final values. This nonlinear algebraic equation can be solved by Newton-Raphson. The Jacobian matrix of the above set of nonlinear equations can be obtained analytically or numerically and is found to be extremely sparse due to the formulation's weakest form. This helps to obtain the high computational efficiency. So, if the initial conditions and boundary conditions are specified, the final values after one time step can be found very efficiently using the damped Newton-Raphson method, and time history is obtained by doing time marching iteration. The structural part of the above formulation has been well validated against the stability subject to thrust. Fig. 5 shows more specific time-marching scheme.

Apart from the above discussion, several issues on computational stability and efficiency should be addressed. First, the kinematic quantities for initial conditions should satisfy certain kinematical relations since they are not independent. So if one variable is perturbed, other variables are affected; that is, all the kinematic quantities which are related to it should have modified values. This is an important aspect of the formulation, since it predominantly affects the sensitivity and convergence of the solution for the time-marching scheme. Second, depending on the type of problems proposed, some variables can be added or removed for computational efficiency. For example, for rectilinear flight, direction cosine variables would not be needed. Missile aerodynamics discussed will need additional variables and equations such as acceleration and direction cosines and related equations.

2.4 Nonlinear Stability Analysis Without Aerodynamics

Based on the methodology set forth here, a computer code for investigation of the nonlinear dynamics of a missile has been developed. The various stability problems due to thrust which appear in the literature can be examined in terms of their time history. First, for validation purpose of the current work, the case without directional control considered by Beal (1965) is addressed. Since it is known that the mass distribution also contributes to the critical load for thrust, constant mass will be considered for a comparison purpose. When a small perturbation of the transverse deflection is imposed at the initial time and the thrust level is below Beal's critical value, the deflection indeed

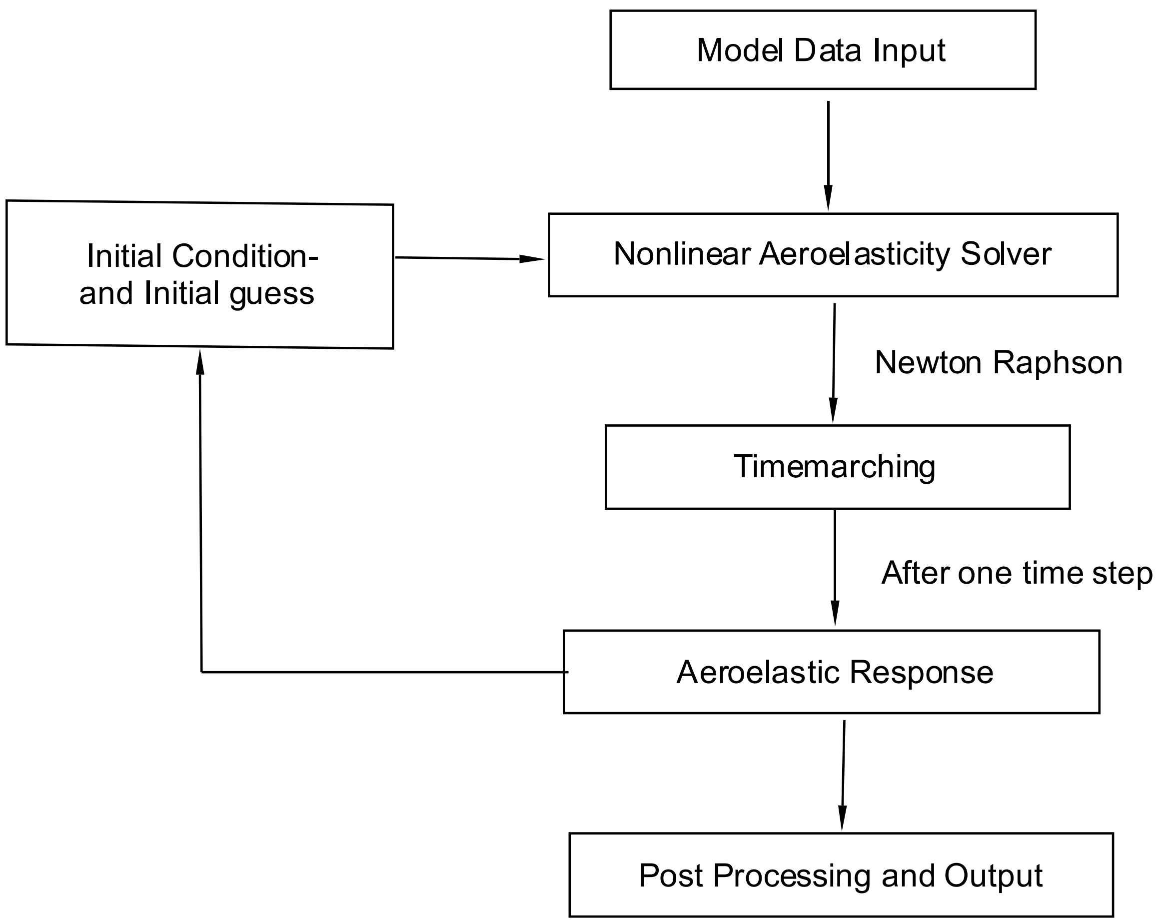

Figure 5: Time-marching scheme

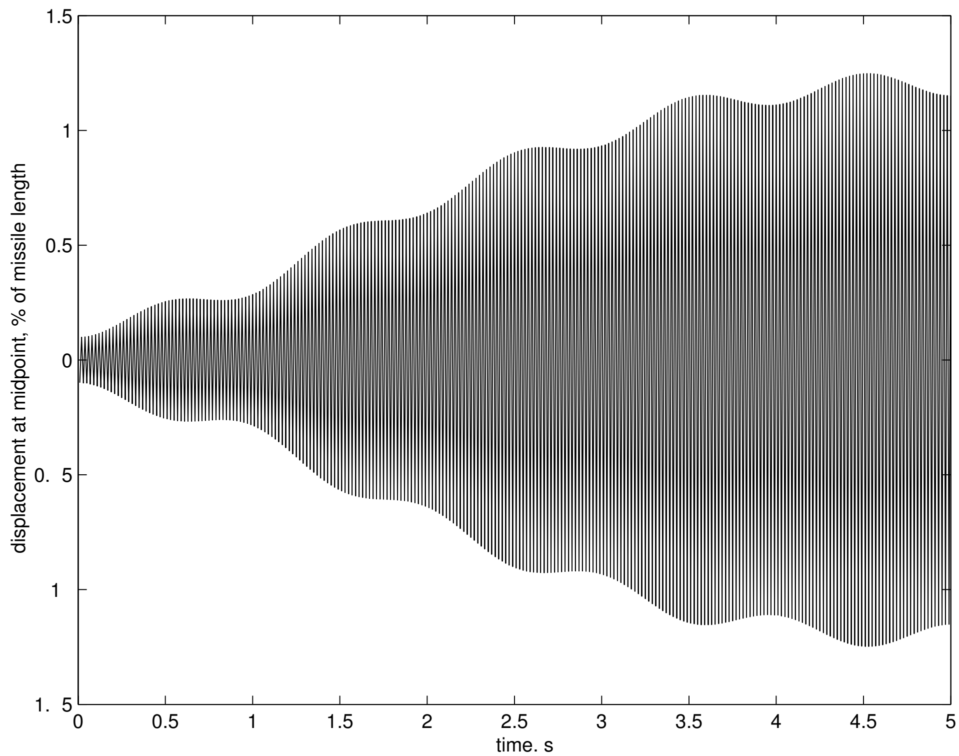

Figure 6: Time history above critical thrust

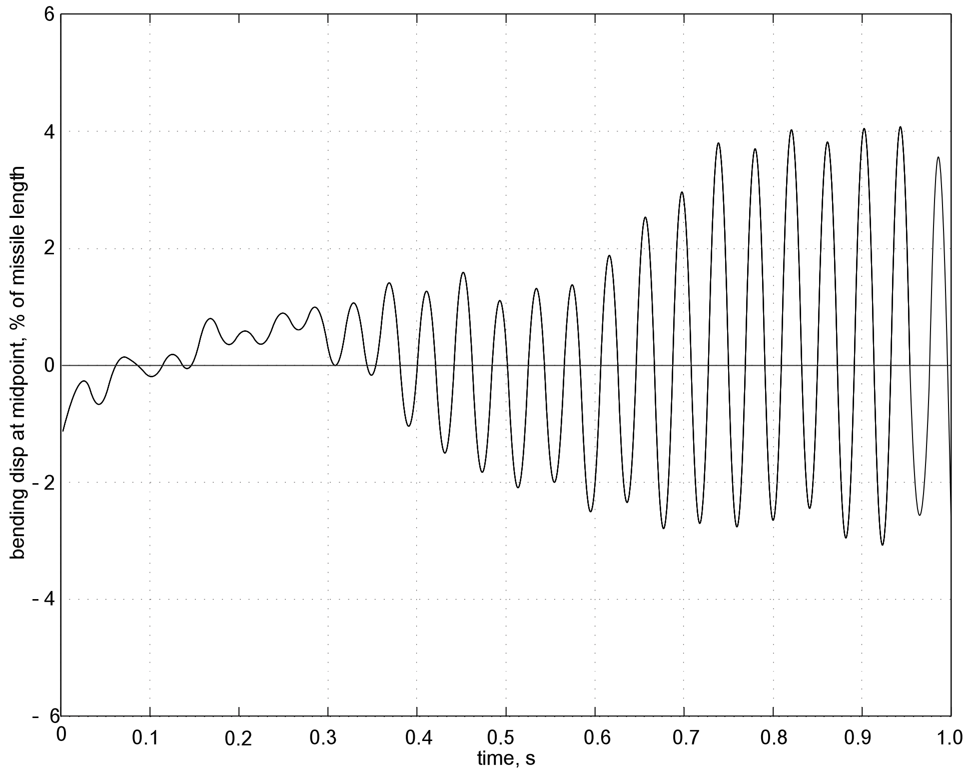

Figure 7: Time history below critical thrust

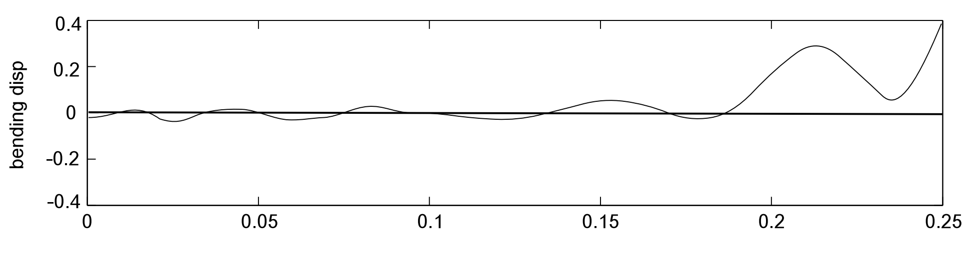

Figure 8: Time history well above critical thrust

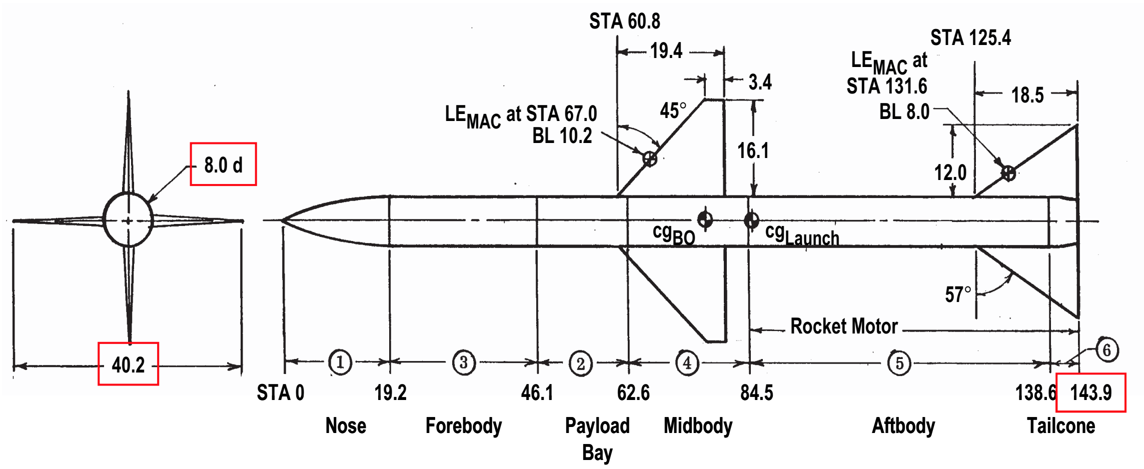

Figure 9: Baseline missile configuration

Table 1: Test case data, ballistic flight

Bending Stiffness:	uniform 2×10^6N-m^2
Altitude:	20 km
Density:	0.0889 kg/m^3
Temperature:	216.7K

dies out in time. However, as expected, when the thrust level is a little larger than the critical value, the deflections grow until they reach an oscillatory motion with bounded amplitude, suggesting a limit cycle. A typical result is shown in Fig. 6. Results for several cases show that limit cycles can develop from disturbances with thrust values that are either just below the critical value suggested by Beal (1965) (Fig. 7) or just above it. However, the motion is divergent well above the critical value as shown in Fig. 8. This observation serves to partially validate the current approach.

2.5 Ballistic Flight

To see if there are any aeroelastic effects on the stability as a function of thrust or on accelerated flight, Fig. 9 is used as a baseline missile configuration; see Bithell and Stoner (1982). Both movable and fixed tail fins with a cruciform pattern have two sets of wedge-shaped panels. Around the nose, the large missile body drag is applied. The fins and tail are both under the influence of wave drag and skin friction drag. At the nozzle base, base drag and thrust are both applied. The total drag force is distributed along the body. Here it should be noted that missile fins are not a consideration for stability because we are interested in only missile body bending modes. The loads on the fins are applied as concentrated forces. In reality, missile fin flutter is more common

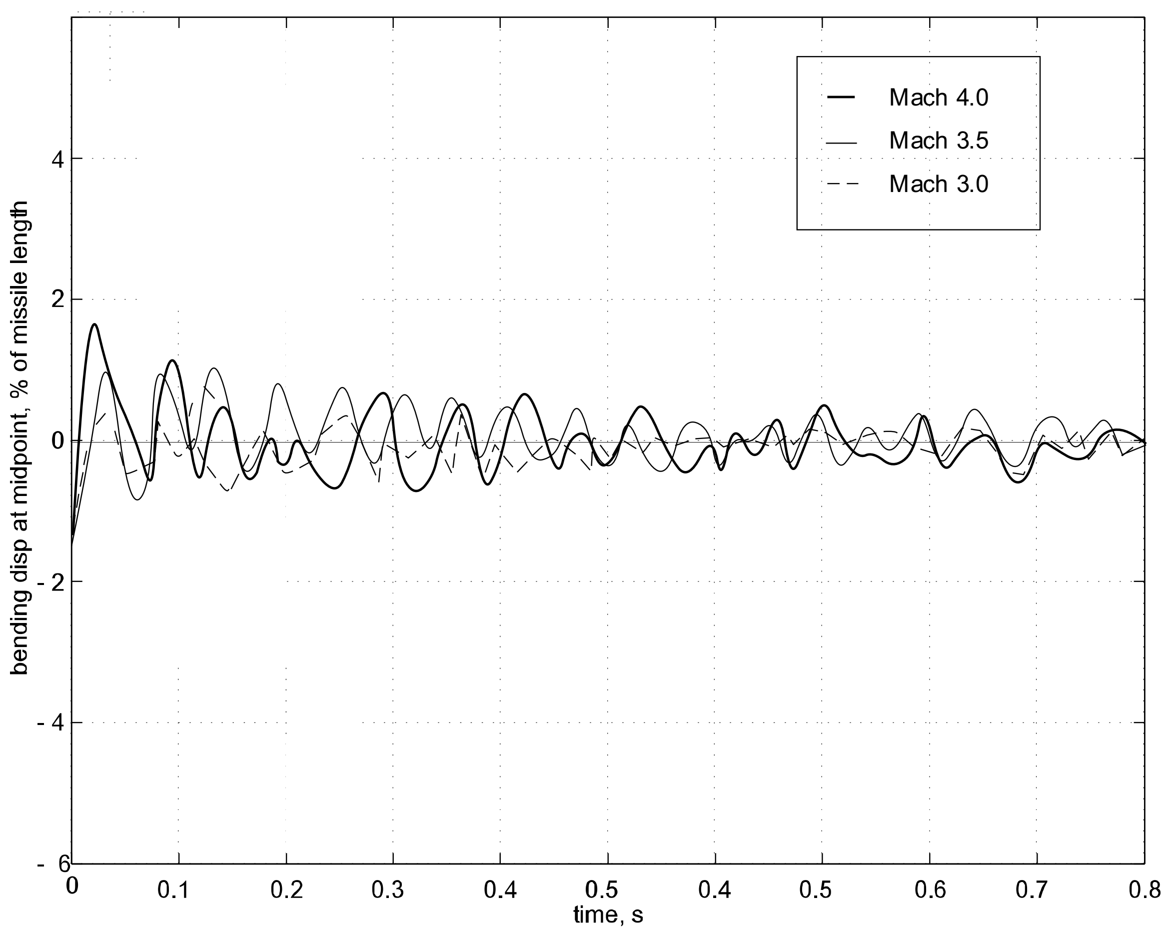

Figure 10: Stability for ballistic flight case

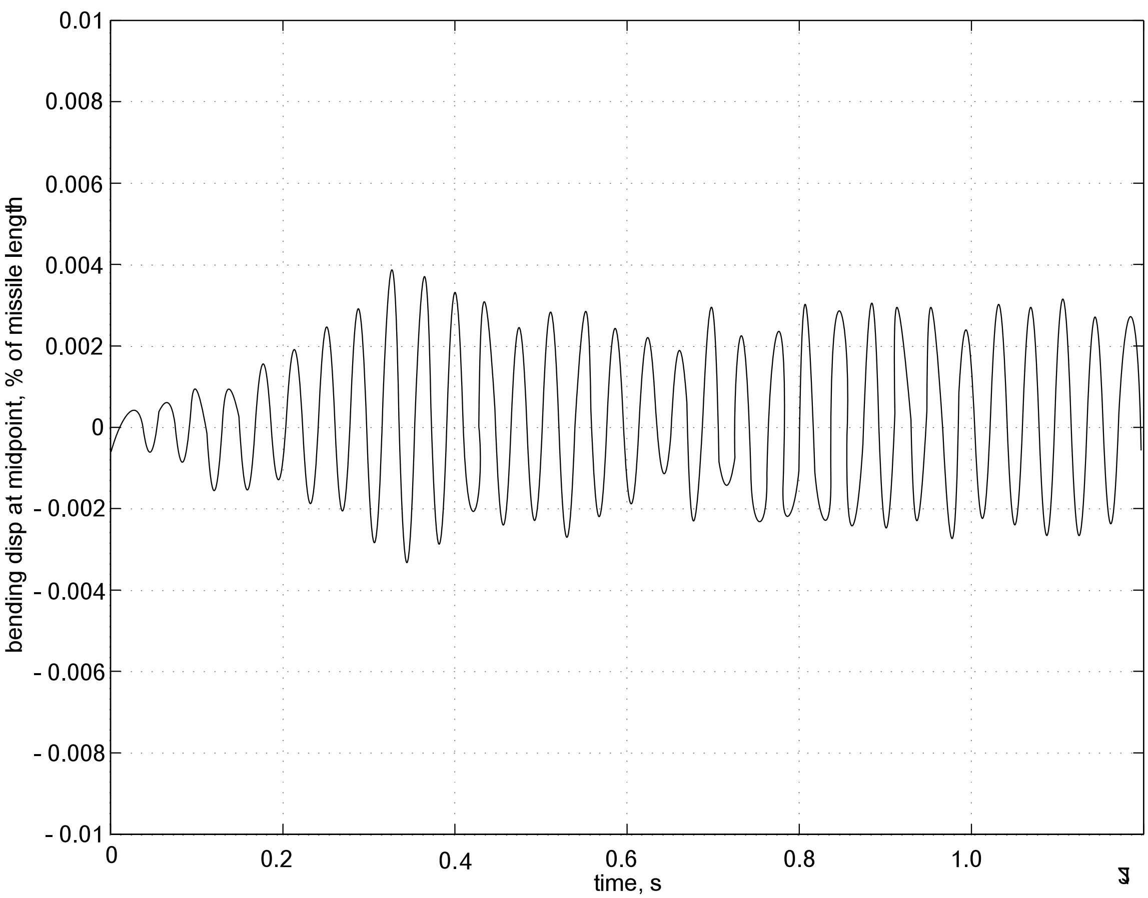

Figure 11: Effect of reduced bending stiffness on stability in ballistic flight

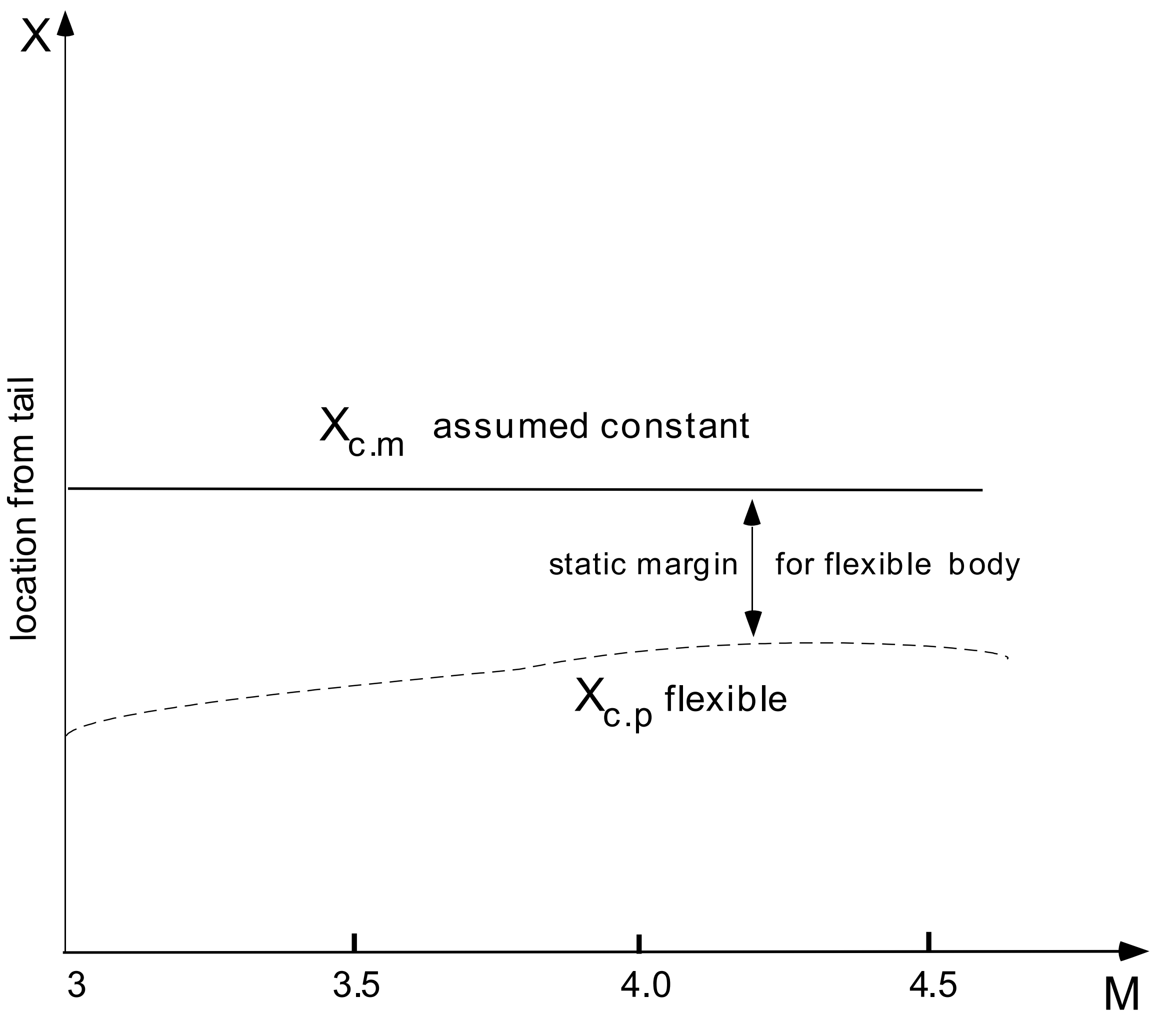

Figure 12: Static stability showing static margin

phenomena since fins are more flexible compared to the missile body. But the purpose of this research is to identify how bending of the missile body affects aeroelastic stability. Therefore, this assumption will be maintained under the current research. The basic idea about how the aeroelastic phenomena occur is that missile bending will change the local angle of attack on the body and the changed local of attack will in turn give different aerodynamic loads on the missile, which will further deform the missile. This iterative process of yielding new aerodynamic loads and deforming the missile will result in stable or unstable flight depending on the various flight conditions and missile characteristics.

When the thrust force is balanced with the total drag, the missile maintains equilibrium by flying at constant speed. But depending on whether the thrust force magnitude is bigger or smaller than the total drag, the missile either accelerates or decelerates. First the aeroelastic stability of missile flying at high supersonic velocity will be addressed. Once velocity is specified, the missile drag is determined from unsteady aerodynamics for missiles. The initial flight condition satisfying kinematic relations and initial deflection for bending are given to run this case. Fig. 10 shows the bending response in rectilinear flight for the test case from Table 1. The case represents deceleration from steady-state flight. The responses showed that there was no aeroelastic instability for the uniform bending stiffness in this test case. With a very small time interval, less than one second was good enough to identify the decay. The velocity increase noticeably affects the amplitudes of the response after small lateral disturbances are given. The flexural stiffnesses are relatively large, but the distributed drag forces appear to play the role of reducing the effective stiffness. To see the effect of bending stiffness on aeroelastic stability in ballistic flight, the size of bending stiffness was reduced to about 1/100 of the original value. Fig. 11 shows limit cycle at the same Mach 4.0. One can see the conspicuous effect of bending stiffness on the aeroelastic stability at ballistic flight.

Under the current formulation, the total mass of the missile does not change. Thus, the center of mass location along the missile axis is assumed to be constant, and the center of pressure of missile can be calculated from running the code. From Fig. 12 one can see that this flexible missile body model is statically stable. In reality the missile center of mass moves closer to missile nose as the fuel is consumed. Therefore, the body will have a little larger static margin.

2.6 Aeroelastic Effects of Thrust

As shown in previous sections, thrust and aerodynamic forces themselves have a destabilizing effect on the missile stability. Aeroelastic interactions between structural load, aerodynamic load, and inertial load are a continuous iterative process between each load. That is, the missile bending brought on by aerodynamic normal forces will change the local angles of attack along the missile body. Altered angles of attack will, in turn, change the aerodynamic forces on the missile body. That will yield additional inertial loading over the missile body. And these inertial loading further deforms the missile body. During this iteration, the missile will reach an equilibrium state where all the forces are balanced.

Besides all this, thrust will also influence the results. It is natural to ask how thrust interacts

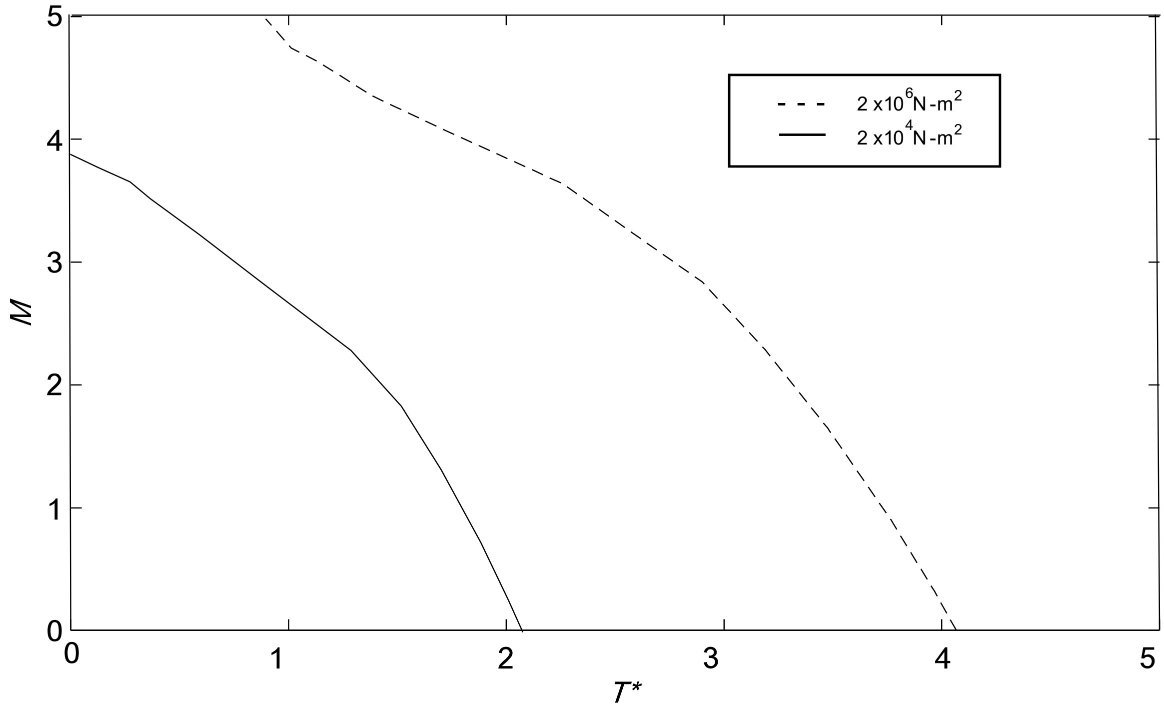

Figure 13: Variation of flexibility

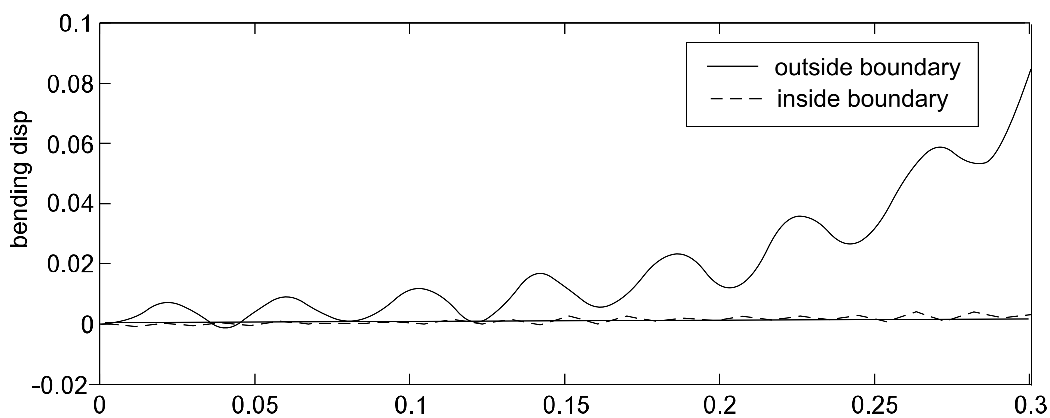

Figure 14: Responses well above and below flutter point

with aerodynamics. To answer that question, in this section several parametric studies will be presented. First, to see the aeroelastic effects of thrust, the flutter boundary is found for several different bending stiffnesses. To locate flutter points, more than 2 seconds of time-marching was needed. The reason for this is because, in the case of a limit cycle, the code had to be run iteratively to find a decay. Once the decay is found by reducing the flight speed from high values at a thrust level with a given bending stiffness, the flutter point is determined. Such a process is repeated with a different thrust level for a complete curve.

Fig. 13 shows the stability boundary for two different bending stiffnesses at the same altitude. Here $T^* = \log_{10}(T/mg)$. According to these results, it seems that thrust is a little bit more influential than aerodynamic force near thrust equal to zero, $i.e.$, ballistic flight. Also it can be seen that when thrust is a dominant factor on stability, aerodynamic forces have less effect than thrust. It appears that the curve close to the thrust abscissa has acceleration dominant stability and the curve close to ballistic flight has deceleration dominant. From the limitation of the current slender-body aerodynamics, some caution should be taken: It is meaningless to run a case at a higher flight speed than Mach 5. Considering the realistic value for bending stiffnesses, at this altitude, the dashed line results are more likely to occur. When obtaining each flutter point, limit cycles appeared either a little below or above the flutter speed. That means there is a certain mechanism, which is inherently nonlinear, to prevent from immediate structural failure. Fig. 14 shows bending deflection responses by changing thrust level at Mach 3 for the bending stiffness $2{\times}10^4$N-m^2. The solid line was obtained about thrust level outside the flutter curve and dashed line about thrust level inside. Well above flutter speed and thrust level, the response becomes unbounded within a very small time.

3 Conclusions and Recommendations

3.1 Conclusions from Present Work

The effects of follower forces on the aeroelastic behavior of flexible missiles have been investigated. Follower forces on their own have been found to greatly affect the instability of flexible structures. Indeed, the well known Beck's problem, a cantilever beam excited by an axially compressive force, is a commonly analyzed problem in the literature. However, free-free beams with follower forces have not received as much attention. In aerospace structures, missile thrust is a typical example of a follower force. A missile may become unstable under the action of thrust. Also missiles can experience aeroelastic instabilities only caused by aerodynamics during flight. The goal of this research was to investigate the interaction of follower forces with aeroelastic loads for missiles. The missile body is modeled in terms of geometrically-exact, nonlinear, beam finite elements. This methodology allows for use of simple shape functions and facilitates time-marching and eigen-analyses. The original mixed variational formulation has been modified to include rigid-body dynamics so that velocity or acceleration can be either specified or left as free to vary.

The structural formulation has been transformed into its weakest form in space and time so that variables are not differentiated. It was shown that by removing redundant equations related to some virtual quantities, the total number of system equations and unknowns was reduced a lot. Unless this process is performed, one redundant equation has to be chosen and thrown out. In addition, it gives the relations between interior values. Aerodynamics loads are based on unsteady slender-body theory and several closed form aerodynamic theories. The aerodynamic loads include second partial derivatives of certain unknowns, and thus to obtain the weakest form requires integrations by parts. After one time integration by parts, there are still variables with derivatives, which are removed using inverse kinematical relations. The aeroelastic code is based on combining the aerodynamics and structural formulations. Unlike most eigenanalyses, this time-marching scheme is useful in finding solutions over time. However, if initial conditions are not exactly satisfied among kinematical quantities, the convergence will exhibit sensitivity problems.

The present code has been validated against several cases, especially the critical load under thrust without directional control. The response is divergent for values of thrust far above the critical value given in several literature. However, near the critical value whether the thrust is high or low, limit cycles were observed. As with general aircraft, velocity is an important aerodynamic element related with missile aeroelastic instability. As the flight speed increases, the amplitude of perturbed deflection does as well. This result indicates that the aerodynamics serve to decrease the effective stiffness of the missile. Missile stiffness along with velocity greatly affects missile stability. Rigid-body stability analysis does not necessarily guarantee the stability of highly flexible structures like missiles with a high fineness ratio. With velocity increased, the center of pressure of missile was shown to approach the center of mass. Depending on the missile and flight conditions, it is possible that the flexible static margin would be negative. Thus, a designer would have to find such a flexible static margin at the early design phase to avoid structural failure. The stability boundaries were obtained for given bending stiffnesses. The response, at the point far outside the stability boundary, was shown to be divergent but inside boundary, as expected, it was bounded or convergent over time. It can be seen from the current results that highly flexible missiles such as ballistic missiles should be carefully designed to avoid aeroelastic instability.

Finally, while the current analysis is capable of analyzing anisotropic beams, our investigation did not reveal any potential advantages for aeroelastic stability from use of the various types of couplings (extension-twist, bending-twist, etc.).

3.2 Recommendations for Future Work

The structural and aerodynamic formulations have been combined to give a complete dynamic and aeroelastic analysis of flying missiles. With some additions to this tool, investigations of a wide variety of dynamic and aeroelastic stability phenomena can be undertaken over a wide range of steady flight conditions, including spin and thrust with present analysis. As one of the ways to stabilize the directional control, spinning is used. When spinning speed increases, the stabilization effect gets larger based on the rigid-body model. But in the flexible model, the stability region is known to vary. The aeroelastic effects of spin are not well understood.

The work performed on the missile aeroelasticity with a follower force are related with a rectilinear flight with zero rigid-body angle of attack. The result verified the interaction of thrust and aerodynamic force. Generally, missiles experience small angle of attack during powered flight and high angle of attack during ballistic flight. A high angle-of-attack analysis would require a much more refined aerodynamic theory.

The current formulation does not include mass variation effect. To see more clearly the dynamic response and stability issues during the powered flight, mass variation according to fuel consumption will be needed. In addition, mass distribution along with bending stiffness is known to significantly change the critical load associated with a follower force. A lot of research on the effect of concentrated mass and its location on the stability has been performed for the flexible system subjected to a compressive follower force. However, there is currently no closed form or analytical optimization method. Thus, it will be of interest to investigate the mass effect on stability problems.

Finally, possible coupling between the flight mechanics (i.e., trajectory optimization, constraints, etc.) and the aeroelasticity (including internal loads and stability) has yet to be approached. For example, turning ability can be quantified in terms of internal loads, and the applicability of the corresponding simplistic constraint (the so-called q-α constraint) imposed in trajectory optimization can be examined in this broader context. The present analysis is not sufficiently computationally efficient to undertake such a study at this time. However, with additional attention devoted to efficiency and with faster computers in the future, such a study should become more feasible.

4 Miscellaneous Information

4.1 Publications Under this Grant

1. Hodges, D. H.; Patil, M. J.; and Chae, S.: “Effect of Thrust on Bending-Torsion Flutter of Wings,” *Journal of Aircraft*, vol. 39, no. 2, Mar.-Apr. 2002, pp. 371 – 376.

2. Chae, S.; and Hodges D. H.: “Dynamics and Aeroelastic Analysis of Missiles,” *Proceedings of the 44th Structures, Structural Dynamics, and Materials Conference*, Norfolk, Virginia, Apr. 7 – 10, 2003, Paper AIAA-2003-1968.

3. Chae, S.; and Hodges D. H.: “Dynamics and Aeroelastic Analysis of Missiles,” based on paper number 2 and subsequent work, currently in preparation for submission to an archive journal.

4.2 Participating Scientific Personnel

Prof. Dewey H. Hodges, Principal Investigator

Seungmook Chae, Graduate Research Assistant, Ph.D. expected July 30, 2004.

4.3 Inventions

None.

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