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DETERMINING THE NUMBER OF ITERATIONS FOR MONTE CARLO SIMULATIONS OF WEAPON EFFECTIVENESS

by

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ABSTRACT

Many weapon effectiveness tools are implemented using a Monte Carlo simulation approach since closed form solutions are too mathematically intractable to compute. A question that usually arises in connection with such simulations is to ask how many iterations of a particular Monte Carlo simulation are needed. This report proposed the probability-based approach to computing effectiveness measures for better feedback to the user regarding the relationship between the number of iterations executed and confidence measures associated with the result.

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1. Introduction

Many weapon effectiveness tools are implemented using a Monte Carlo simulation approach since closed form solutions are too mathematically intractable to compute. A question that usually arises in connection with such simulations is to ask how many iterations of a particular Monte Carlo simulation are needed? This question may only be answered if it is qualified by some performance measure expected of the simulation. For example, a common output of such a simulation might be the average damage a target sustained when attacked by a specific weapon with a known accuracy of delivery. If the simulation were allowed to run for an extremely large number of iterations we would expect the average damage to be reasonably accurate whereas for a smaller number of iterations we would expect a different result. The question asked above may be re-phrased in the form; how many iterations need to be performed in order to obtain a specified accuracy in the result?

This issue is relevant to a number of Weaponeering tools currently used in JAWS (BAM, BAS, TARCOM, HTM), JWES (PVTM, FBAR) and IMEA.

2. Population and Sample

The discussion above illustrates the concept of population and sample. If a Monte Carlo simulation were run an infinite number of times each damage value computed would be one data point, x, of the population, and we could compute statistics such as the mean μ_x and standard deviation σ_x. Usually however we cannot let a simulation run an infinite number of trials and we are limited to just n trials, each providing one damage value. This limited collection of data is known as the sample. We can still compute the sample mean, although it is called the average and denoted by $\overline{x}$, and the sample standard deviation S_x. Clearly of interest is how close are the population and sample statistics?

3. Weaponeering example

In order to illustrate concepts to be discussed later, we will consider a specific example. This is the case of a unitary target attacked by a weapon described by a Carleton damage function with known delivery accuracy. It is the example given in the writers Weaponeering textbook chapter 7, and the issue is to compute the amount of damage caused by a single weapon (SSPD), where for simplicity only the range direction is considered. This example is dealt with in detail in appendix A and has the advantage that a closed form solution for the SSPD is mathematically tractable as shown in the appendix. Also shown in the appendix is a Monte Carlo simulation approach to solving for the SSPD, and therefore this allows us to compare both the closed form and simulation approaches to computing weapon effectiveness. This example should be studied carefully before proceeding.

The Monte Carlo results shown in the table illustrate one way to determine how many iterations are needed to compute the correct SSPD to three decimal places, at least for this model and the specific inputs used. Two issues of interest are:

(i) Can we determine the number of iterations needed without running the model many times?

(ii) Given a specified number if iterations, how close is the computed damage to the true answer?

Both of these questions are addressed by estimation theory in general, and confidence levels and limits in particular.

4. Mean, Variance and Standard Deviation

The **mean** of a sample comprising n numbers is defined by

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}(x_1 + x_2 + \ldots + x_n) \tag{1}$$

The **variance**, defined by

$$VAR = S_x{}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n-1}\left[(x_1-\overline{x})^2 + (x_2-\overline{x})^2 + ... + (x_n-\overline{x})^2\right] \quad (2)$$

The square root of the variance S_x is known as the standard deviation.

5. Confidence Level, Confidence Limits and Intervals

From tables of the cumulative distribution function for a normally distributed random variable it may be seen that about 68% of the impacts should lie in the range $\pm\sigma$, 95.5% will lie in the range $\pm 2\sigma$, and 99.7% in the range $\pm 3\sigma$. This allows us to illustrate the use of confidence limits and levels in the following manner:

(i) We can be 68% confident that a random sample of the variable x will lie within plus or minus one sigma of the mean.

(ii) We can be 95.5% confident that a random sample of the variable x will lie within plus or minus two sigma of the mean.

(iii) We can be 99.7% confident that a random sample of the variable x will lie within plus or minus three sigma of the mean.

The percentage value is the confidence level, and the interval within which the value of x is expected to fall is the confidence limit.

This range may be expressed in the form of an upper (U) and lower bound (L) where

$$U = \overline{x} + z_c\sigma \quad (3)$$

and

$$L = \overline{x} - z_c\sigma \quad (4)$$

Values of confidence coefficients z_c for different confidence levels are given in Table 1.

Table 1. Values of z_c for different confidence levels.

Confidence Level %	99.75	99	98	96	95.5	95	90	80	68	50
$\mathbf{z_c}$	3	2.58	2.33	2.05	2	1.96	1.645	1.28	1	0.6745

6. Confidence Intervals for the Mean

The confidence interval (for a given confidence level) described above is an estimation of the variable x, a member of the population. It is also possible to generate confidence intervals for the mean of a population.

Suppose we have a normally distributed random variable x where the population mean is μ_x and variance σ_x^2, and a sample size n is drawn from the population where the sample mean is $\bar{x}$ and variance S_x^2. If we do this a large number of times we have a distribution of $\bar{x}$ and S_X^2. What can we say about the distribution of means? It is possible to show that:

(i) The expected[1] mean of $\bar{x}$ is μ_x the population mean

(ii) The expected variance of $\bar{x}$ may be expressed in terms of the variance of x, and is given by

$$\sigma_{\bar{x}}^2 = \frac{\sigma_x^2}{n} \tag{5}$$

We are now able to define a confidence limit and level for the mean $\bar{x}$. If the confidence level is selected at say 95%, then the confidence interval of the mean is

$$(L,U)_{0.95} = \mu_{\bar{x}} \pm 1.96\,\sigma_{\bar{x}} \tag{6}$$

[1] The usual interpretation of the expected value is the mean, therefore the expected value of the mean reads " the mean of the means"

This is stated as: we are 95% confident that a sample of the mean $\overline{x}$ will be within (L,U) of the true mean, or stated another way, we are 95% confident that the true mean is within (L,U) of a sample of the mean $\overline{x}$.

However, using equation (6) we write

$$(L,U)_{0.95} = \mu_{\overline{x}} \pm 1.96\,\sigma_{\overline{x}} = \mu_{\overline{x}} \pm 1.96\frac{\sigma_x}{\sqrt{n}} \tag{7}$$

Since the population mean and variance are unknown, the unbiased estimators $\overline{x}$ and S_x are used giving

$$(L,U)_{0.95} = \overline{x} \pm 1.96\frac{S_x}{\sqrt{n}} \tag{8}$$

The more general expression is

$$(L,U) = \overline{x} \pm z_c \frac{S_x}{\sqrt{n}} \tag{9}$$

In terms of the Weaponeering test case in the appendix, the variable x is the value of damage probability for a single iteration, i.e. selecting a random weapon impact point and computing the damage to the unitary target. The population statistics μ_x and variance σ_x^2 would be obtained by repeating this process an infinite number of times, where μ_x is the SSPD. If the simulation is run for a finite number if iterations (n) the sample statistics $\overline{x}$ and variance S_x^2 are estimates of the population statistics and $\overline{x}$ is the best estimate of the SSPD. We can use equation (9) however to estimate the confidence interval associated with a particular confidence level for the SSPD.

Consider the following example. Suppose the Weaponeering example in the appendix is run once each for 100, 1000 and 100,000 iterations and we are interested in a 95% confidence level for the mean. The results may be summarized in the following table.

Table 2. Single run of weaponeering example.

n	$\overline{x}$	S_x	Half Interval
100	0.153	0.785	0.154
1,000	0.172	0.302	0.019
100,000	0.177	0.308	0.002

Taking the example for 1000 iterations, we may state that we are 95% confident that the true SSPD (mean) is within 0.172±0.019, i.e. in the range 0.153 to 0.191. Clearly as we take more iterations the half interval decreases and the sample mean approaches the population mean, and the sample variance approaches the population variance. Note that for a fixed number of iterations we will in general get different sample means and confidence limits for the same confidence level.

It may be concluded that for any Monte Carlo simulation of the type described above a confidence interval for a specified confidence limit may be calculated, but that these values apply only to the simulation completed and will be different if the simulation is run again for the same inputs.

7. Confidence Interval Estimates for Small Samples of the Mean

If the number of samples of the mean is small, typically less than about 25, then the confidence coefficients given in Table 1 cannot be used. Instead of using the normal distribution, the so-called **t distribution** has to be used. This distribution has a similar shape to the normal distribution, but is a function of an additional parameter called the **degrees of freedom, v**. The probability density function for different degrees of freedom is shown in Figure 1.

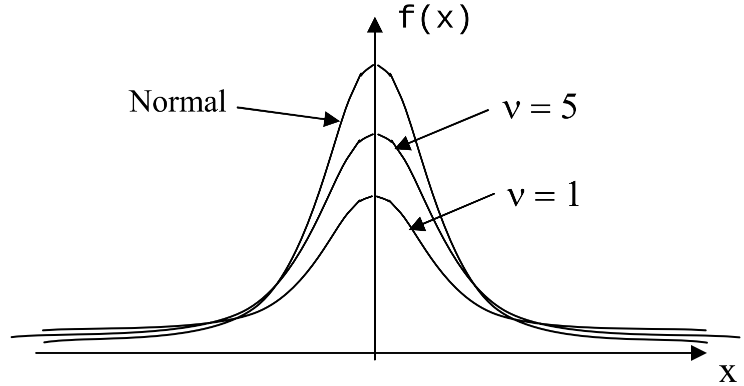

Figure 1. The normal and t-distributions.

As may be seen, the t-distribution tends to the normal distribution as the number of degrees of freedom increases. In fact, the number of degrees of freedom is determined as the number of samples minus one, so this why for small sample sizes the t distribution has to be used instead of the normal distribution. Just as the integral of the normal probability density function (the CDF) is tabulated, so is the integral of the t-distribution.

The confidence level and limits work the same way as for the normal distribution. Shown in Figure 2 is a t-distribution for a specific number of degrees of freedom together with a confidence limit (L,U) for a sample of the random variable x.

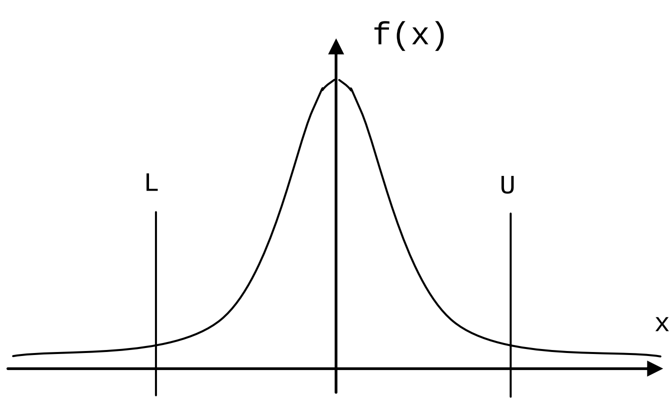

Figure 2. Confidence limits for a t-distribution.

For a normal distribution, the range (L,U) would be given by equation (9), but for a t-distribution it is given by

$$(L,U) = \bar{x} \pm t_c \frac{S_x}{\sqrt{n-1}} \tag{10}$$

Here the mean and variance are calculated from the small sample. As before the confidence coefficient t_c is obtained from the area beneath the PDF, but this time it will dependent on the number of degrees of freedom for which Figure 2 is drawn. It is usual to tabulate the area under the PDF for a single "tail" i.e. t_c is given only for the upper interval, U. What this means is that if we require for example the 95% confidence interval, then the sum of the area under the PDF less than L and grater than U must be 5% of the total area, hence the confidence coefficient needed in equation (10) is $t_{0.975}$. Table 3 shows values of the confidence coefficients for different confidence levels and degrees of freedom.

Table 3. Confidence coefficients t_c for t-distribution.

ν	**90%**	**95%**	**97.5%**	**99.5%**
1	3.08	6.31	12.71	63.66
2	1.89	2.92	4.30	9.93
3	1.64	2.35	3.18	5.84
4	1.53	21.3	2.78	4.60
5	1.48	2.01	2.57	4.03
10	1.37	1.81	2.23	3.17
30	1.31	1.70	2.04	2.75
∞	1.28	1.65	1.96	2.58

From this point on we will assume we are dealing with "large" samples so the normal distribution is reasonable, however this section provides the necessary tools to make the adjustments if the sample size is small.

8. Number of Iterations for a Specified Error Bound

Reconsider the Weaponeering example. The true SSPD is known from the closed form solution to be 0.176, so Table 2 could include a column that shows the percentage error between the estimated SSPD and the true value.

Table 4. Percentage error as a function of iterations.

n	$\overline{x}$	**% error**
100	0.153	13.1
1,000	0.172	2.3
100,000	0.177	0.1

It is also noted that the population standard deviation is not known, but may be accurately estimated by running the simulation a large number of times, say 10^7. The result of doing this is

$$\sigma_x \approx S_x = 0.3073 \tag{11}$$

The question suggested by **Error! Reference source not found.**Table 4 is whether we can specify a maximum acceptable percentage error for the mean and determine the required number of iterations.

By considering the confidence interval to represent twice this maximum error we can write

$$\text{error}_{\max} = \frac{z_c S_x}{\sqrt{n}} \tag{12}$$

The percentage error of the mean becomes

$$\mathrm{E} = \frac{100 \times z_c S_x}{\overline{x}\sqrt{n}} \tag{13}$$

Transforming for n yields

$$n = \left[\frac{100\, z_c S_x}{E\, \overline{x}}\right]^2 \tag{14}$$

So, for the example used where the confidence level is 95%, z_c=0.196, $\overline{x}$=0.176, S_x=0.3073 and E=5, the required number of iterations becomes 4684.

In words this reads: If the simulation is run for 4684 iterations, we are 95% confident that the calculated SSPD will not differ by more than 5% from the true SSPD.

Running the simulation ten times for 4684 trials produced percentage errors of 4.89, 4.99, 4.92, 4.92, 5.1, 5.0, 4.95, 4.98, 5.00 and 4.94, so it appears to work. The only problem is that the separately calculated population statistics μ_x and σ_x were used in equation (14) and these are not known.

9. Estimating the Number of Iterations Required

Consider an iterative Monte Carlo simulation that has no upper bound to the number of iterations to be performed. As the iterations proceed, the damage estimates accumulate into a sample of increasing size. As more iterations take place the sample approaches the population. Table 2 shows the estimates of the sample statistics approach that of the population. It is proposed to calculate the sample mean and variance and use these values in equation (14) to determine how many iterations are needed to achieve a specified maximum percentage error with a specified confidence level. It has been observed that this number of iterations converges quickly and even for sample sizes an order of magnitude lower than the number required, the calculation of that number is quite stable.

Consider the previous example where the exact number of iterations was determined to be 4684. We can calculate this number as a function of the number of iterations actually performed. This estimation is shown in Figure 3.

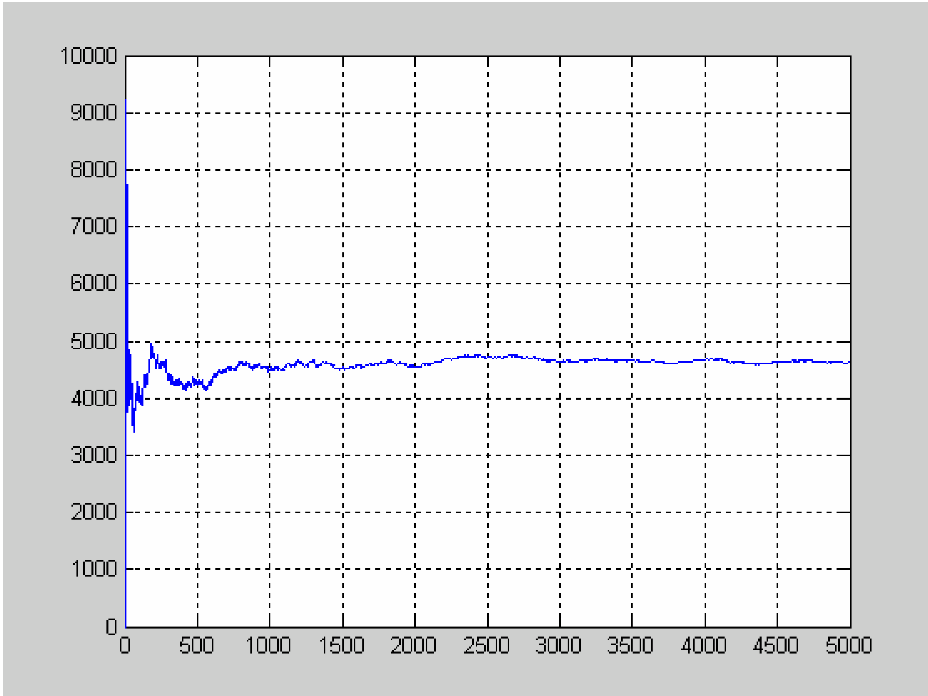

Figure 3. Number of iterations required vs. iteration number.

This shows how the number of iterations needed stabilizes within about 500-1000 iterations, and within about 100 iterations it is accurate to 20%.

10. Conclusions and Recommendations

It is proposed that Weaponeering programs that utilize Monte Carlo simulation approaches to computing effectiveness measures provide better feedback to the user regarding the relationship between the number of iterations executed and confidence measures associated with the result. Specifically, selected programs can use the methods outlined in this paper to provide the following user assistance.

(1.) For a given number of iterations give the confidence limits associated with a user supplied confidence level(s) at the end of the run.

(2.) While the run is executing, provide the user with the number of iterations needed to achieve a bounded error on the result subject to a user specified confidence level.

Neither of these features has a computational overload of any significance and should be easily incorporated into any simulation for which the source code or interface program is available. An example of a dialog box incorporating the features above is shown in Figure 4, and would be displayed during program execution.

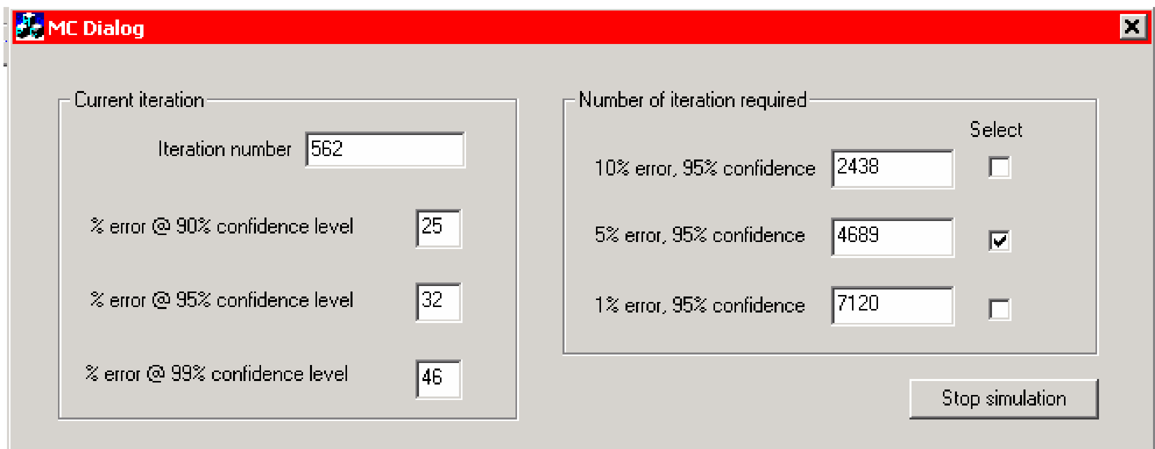

Figure 4. Suggested dialog box to display confidence levels.

If this dialog box were displayed during the Monte Carlo simulation it would enable the user to determine how accurate the simulation is so far, and give an estimate of how many iterations are required to achieve a specific error with known confidence levels. The user may then stop the simulation if sufficient accuracy has been achieved, or let it run to a specific terminal error criterion.

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