Theoretical and Experimental
Determination of Mechanical Properties
of Superconducting Composite Wire

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THEORETICAL AND EXPERIMENTAL DETERMINATION OF MECHANICAL 
PROPERTIES OF SUPERCONDUCTING COMPOSITE WIRE

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THEORETICAL AND EXPERIMENTAL DETERMINATION OF MECHANICAL
PROPERTIES OF SUPERCONDUCTING COMPOSITE WIRE*

W. H. Gray and C. T. Sun†

ABSTRACT

The object of this research is to characterize the mechanical
properties of a composite superconducting (NbTi/Cu) wire in terms
of the mechanical properties of each constituent material. For a
particular composite superconducting wire, five elastic material
constants were experimentally determined and theoretically calculated.

Since the Poisson's ratios for the fiber and the matrix material
were very close, there was essentially no (less than 1%) difference
among all the theoretical predictions for any individual mechanical
constant. Because of the expense and difficulty of producing elastic
constant data of 0.1% accuracy, and therefore conclusively determining
which theory is best, no further experiments were performed.

INTRODUCTION

The object of this research is to characterize the mechanical
properties of a composite superconducting (NbTi/Cu) wire in terms of
the mechanical properties of each constituent material. In 1973, the
Cryogenics Division of the National Bureau of Standards (NBS) published
an interim report¹ in which a preliminary investigation of the mechan-
ical properties of a solenoid coil composite was made. The coil inves-
tigated consisted of epoxy, fiberglass, and composite superconducting
wire. Both theoretical and experimental elastic constants were tabulated

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for a typical piece of coil cut out of a small solenoid. Our report differs from this NBS work in that we consider only the mechanical properties of an individual composite superconducting wire.

The theoretical predictions and the experimental procedures to determine the effective elastic constants of the composite wire are described in the next two sections of this report.

THEORETICAL INVESTIGATION

Most of the analytical work for predicting the mechanical and thermal properties of fiber-reinforced composites in terms of volumetric composition, geometrical arrangement of the fibers, and constituent material properties was done before 1970. There are five approaches to predict the micromechanical behavior of fiber-reinforced composites. The essential characteristic of each is described below.

1.1 SELF-CONSISTENT MODEL METHODS

This method was originally proposed by Hershey\textsuperscript{3} and Kroner\textsuperscript{4} for crystal aggregates, and was first employed by Hill\textsuperscript{5} to derive expressions for elastic constants. Hill modeled the composite as a single fiber embedded in an unbounded macroscopically homogeneous medium, subjected to a uniform loading at infinity. This uniform loading produces a uniform strain field in the filament which is then used to estimate the elastic constants. A similar model proposed by Frohlich and Sack\textsuperscript{6} for predicting the viscosity of a Newtonian fluid containing a dispersion of equal elastic spheres consists of three concentric cylinders, the outer one being unbounded. The innermost cylinder is assumed to have the elastic properties of the filaments; the middle one has the properties of the matrix; and the outermost has the properties of the composite. The solid is subjected to homogeneous stresses at infinity. The resulting elastic fields are determined, and then are

\* No attempt is made here to give a comprehensive literature survey regarding this subject. More references can be found in Ref. 2.
employed to predict the elastic constants of the composite. Applications of the self-consistent model methods can be found, for example, in Refs. 7-9.

1.2 VARIATIONAL METHODS

In this method, the energy theorems of classical elasticity are used to obtain bounds on the mechanical and physical properties of filamentary composites. The minimum complementary energy theorem yields a lower bound, while the minimum potential energy theorem yields the upper bound. Using this approach, bounds for the elastic and thermal properties of composites have been obtained by many investigators. 10-12

1.3 EXACT METHODS

By assuming that the fibers are arranged in a doubly periodic rectangular array, a fundamental or repeating element can be established. The resulting elasticity problem can then be solved either by introducing a stress function using a series development, or by numerical techniques such as finite difference or finite element methods. Once the problem is solved elastically, the resulting elastic fields can be averaged to get expressions for the desired elastic constants. Typical applications of this method can be found in Refs. 13-18.

1.4 MECHANICS OF MATERIALS METHOD

By making simplifying assumptions regarding the mechanical or thermal behavior of a composite material, the mechanics-of-materials expressions for the equivalent elastic or thermal constants of unidirectionally reinforced fibrous composite materials can be derived. For example, to determine the longitudinal Young's modulus, one assumes that the longitudinal strains in both the matrix and the fiber are the same; in order to determine the transverse Young's modulus, one assumes that transverse stresses in both materials are the same. This approach
usually is referred to as the "rule of mixtures." The "rule of mixtures" expressions for elastic moduli and thermal conductivities can be found in Refs. 19-21.

1.5 THE HALPIN-TSAI EQUATIONS

For designers, it is often necessary to have simple and rapid computational procedures for estimating the macromechanical properties of a fibrous composite. Such empirical formulas have been developed by Halpin and Tsai\textsuperscript{22} based upon modifications of the results discussed under approaches 1.1 and 1.3. By estimating the value of a factor which depends on the geometry of the inclusions, spacing geometry, and loading conditions, the composite elastic moduli can be approximated. Reliable estimates for this factor can be obtained by comparing the Halpin-Tsai equation with the numerical micromechanics solutions. If used appropriately, the Halpin-Tsai equation can yield very reliable results without elaborate calculations.

All the above methods make the following three basic assumptions:
(1) each constituent material behaves linearly elastically, (2) the fibers are straight (without twist), and (3) there are no residual stresses.
For this investigation, we use several of the available theoretical equations to predict the effective elastic mechanical properties of superconducting composite wire. These equations are listed in Appendix I.

In general, a superconducting composite wire may be twisted to minimize ac power losses in a superconducting magnet. Twisting of the wire violates an assumption implicit in the derivation of all the equations presented in Appendix I. However, we believe that this effect is small (see Appendix II), and for engineering purposes can be neglected. Other effects, such as inelastic behavior of the wire at higher loading levels, and residual stresses in the wire introduced during fabrication, may influence the results presented in this paper, and should be analyzed more thoroughly.
EXPERIMENTAL DETERMINATION OF ELASTIC CONSTANTS

Assuming the NbTi/Cu wire behaves like a transversely isotropic material, there are five elastic constants of significance. These constants are: (1) Young's modulus along the direction of the fiber, $E_L$ or $E_{11}$ (longitudinal Young's modulus), (2) major Poisson's ratio, $\nu_L$ or $\nu_{12}$, (3) Young's modulus along the direction normal to the fiber, $E_T$ or $E_{22}$ (transverse Young's modulus), (4) minor Poisson's ratio, $\nu_T$ or $\nu_{23}$, and (5) longitudinal shear modulus, $G_L$ or $G_{12}$. Since the material properties in a plane normal to the fiber direction are assumed to be isotropic, the transverse shear modulus $G_T$ or $G_{23}$ can be determined from the isotropic relation

$$G_T = \frac{E_T}{2(1 + \nu_T)} \quad (1)$$

The particular superconducting composite wire chosen for our experiment was KROY-210. This conductor has cross-sectional dimensions of 10.16 mm by 5.08 mm with a copper matrix containing 2640 Nb-45 wt % Ti superconducting filaments. The copper to superconductor ratio is 6. The elastic constants which were used for the comparisons are tabulated below. All elastic constant measurements were made at room temperature.

<table>
<thead>
<tr>
<th>Material</th>
<th>Young's Modulus (GPa)</th>
<th>Poisson's Ratio</th>
<th>Shear Modulus (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cu</td>
<td>123</td>
<td>0.345</td>
<td>45.7</td>
</tr>
<tr>
<td>NbTi</td>
<td>84</td>
<td>0.33</td>
<td>31.5</td>
</tr>
</tbody>
</table>

* Registered trademark of Magnetics Corporation of America (MCA).

† Tradenames of material are used in this report for clarity. In no case does such selection imply recommendations or endorsement by the authors, nor does it imply that the material is necessarily the best available for the purpose.
2.1 EXPERIMENTAL DETERMINATION OF $E_{11}$ AND $\nu_{12}$

$E_{11}$ and $\nu_{12}$ can be determined from a simple tension test (see Fig. 1). The direction of loading is parallel to the fibers of the conductor. The longitudinal Young's modulus is determined from a $\sigma_1$ vs $\varepsilon_1$ diagram where $\sigma_1$ is equal to the applied load divided by the cross sectional area of the specimen, and $\varepsilon_1$ is the strain along the fiber direction of the conductor. Figure 2 represents an experimentally determined plot of this diagram for KRYO-210 superconductor showing a value for $E_{11}$ of 119 GPa. A comparison of the theoretical prediction and the experimental data is shown in Fig. 3 in this graph, as well as the four normalized comparison graphs which follow; the legend refers to the theoretical equations presented in Appendix I. Both equations predict the same behavior, which deviates approximately 3% from the experimental data.

The major Poisson's ratio is determined from the slope of the $\varepsilon_2$ vs $\varepsilon_1$ diagram during the same experiment, where $\varepsilon_2$ is the strain in either transverse direction. The experimentally determined value was 0.347 (see Fig. 4). The normalized plot (see Fig. 5) showing the comparison between theoretical prediction and experimental value again demonstrates little difference between theories with the experimental data differing from the predictions by about 2%.

2.2 EXPERIMENTAL DETERMINATION OF $E_{22}$ AND $G_{23}$

$E_{22}$ and $G_{23}$ are determined from a test similar to that described in 2.1 (see Fig. 6). The direction of the applied load is normal to the fiber axis. The transverse Young's modulus is determined from the $\sigma_2$ vs $\varepsilon_2$ diagram where $\sigma_2$ is the stress, and $\varepsilon_2$ is the strain in the direction of the applied force. Figure 7 represents an experimentally determined plot of this diagram for KRYO-210 superconductor showing a value for $E_{22}$ of 122 GPa. The experimental data are compared to the theoretical predictions for $E_{22}$ in Figure 8. An error of approximately 5% is observed.

Analogously the minor Poisson's ratio, and therefore $G_{23}$, is determined from an $\varepsilon_3$ vs $\varepsilon_2$ diagram. The experimental value for $G_{23}$ is 43.1 GPa (see Fig. 9). Figure 10 compares this data point with the
theoretical predictions. An error of approximately 2% is observed.

2.3 EXPERIMENTAL DETERMINATION OF THE LONGITUDINAL SHEAR MODULUS $G_{12}$

The simplest way to determine the longitudinal shear modulus $G_{12}$ is to use a tensile specimen with the fibers oriented at a $45^\circ$ direction to the geometrical axis of the specimen. A realistic specimen of a composite superconductor, however, would be very difficult and expensive to fabricate. An alternate method to evaluate $G_{12}$ experimentally is outlined below.

From the two-dimensional anisotropic stress-strain relation, we have

$$
\begin{pmatrix}
\varepsilon_0 \\
\varepsilon_0 + \pi/2 \\
\gamma_0
\end{pmatrix} =
\begin{bmatrix}
\overline{s}_{11} & \overline{s}_{12} & \overline{s}_{16} \\
\overline{s}_{12} & \overline{s}_{22} & \overline{s}_{26} \\
\overline{s}_{16} & \overline{s}_{26} & \overline{s}_{66}
\end{bmatrix}
\begin{pmatrix}
\sigma_0 \\
\sigma_0 \\
\tau_0
\end{pmatrix}
$$

(2)

where $\varepsilon_0$ and $\varepsilon_0 + \pi/2$ are the axial strains of $x'$ and $z'$ axes. These lie in the $xz$ plane and make angles $\theta$ and $\theta + \pi/2$ with the $x$-axis respectively; $\gamma_0$ is the shear strain of $x'$ and $z'$ axes, and $\sigma_0$, $\sigma_0 + \pi/2$ and $\tau_0$ are the corresponding normal and shear stresses respectively (see Figure 11). The matrix $[\overline{s}]$ in Eq. (2) represents the compliance matrix of the composite material in $x'z'$ directions. In a simple tension test with the applied load and the fibers oriented along the $x$ direction, $\varepsilon_0$ and $\varepsilon_0 + \pi/2$ can be measured directly and $\gamma_0$ can be computed, using the data obtained from strain gage rosette readings. For axial tension, $\sigma_0$, $\sigma_0 + \pi/2$ and $\tau_0$ are given by

$$
\begin{align*}
\sigma_0 &= \sigma_x \cos^2 \theta \\
\sigma_0 + \pi/2 &= \sigma_x \sin^2 \theta \\
\tau_0 &= -\sigma_x \sin \theta \cos \theta
\end{align*}
$$

(3)
where \( \sigma_X \) is equal to the applied force divided by the cross sectional area of the specimen. The elastic compliances \( S_{11} \), \( S_{12} \), \( S_{22} \), \( S_{16} \), \( S_{26} \), and \( S_{66} \) are related to \( E_{11} \), \( E_{22} \), \( \nu_{12} \), and \( G_{12} \) by the following relations.\(^2\)

\[
S_{11} = \frac{\cos^4 \theta}{E_{11}} + 2\left( \frac{1}{G_{12}} - \frac{\nu_{12}}{E_{11}} \right) \sin^2 \theta \cos^2 \theta + \frac{\sin^4 \theta}{E_{22}} \tag{4}
\]

\[
S_{12} = \left( \frac{1}{E_{11}} + \frac{1}{E_{22}} - \frac{2}{G_{12}} \right) \sin^2 \theta \cos^2 \theta - \frac{\nu_{12}}{E_{11}} (\sin^4 \theta + \cos^4 \theta) \tag{5}
\]

\[
S_{22} = \frac{\sin^4 \theta}{E_{11}} + 2\left( \frac{1}{G_{12}} - \frac{\nu_{12}}{E_{11}} \right) \sin^2 \theta \cos^2 \theta + \frac{\cos^4 \theta}{E_{22}} \tag{6}
\]

\[
S_{16} = \left( \frac{1}{E_{11}} + \frac{\nu_{12}}{E_{11}} - \frac{1}{G_{12}} \right) \sin \theta \cos \theta + \left( \frac{1}{E_{11}} - \frac{\nu_{12}}{E_{11}} - \frac{1}{E_{22}} \right) \sin^3 \theta \cos \theta \tag{7}
\]

\[
S_{26} = \left( \frac{1}{E_{11}} + \frac{\nu_{12}}{E_{11}} - \frac{1}{G_{12}} \right) \sin^3 \theta \cos \theta + \left( \frac{1}{E_{11}} - \frac{\nu_{12}}{E_{11}} - \frac{1}{E_{22}} \right) \sin \theta \cos^3 \theta \tag{8}
\]

\[
S_{66} = \left( \frac{1}{E_{11}} + \frac{1}{E_{22}} + \frac{2\nu_{12}}{E_{11}} - \frac{1}{G_{12}} \right) \sin^2 \theta \cos^2 \theta + \frac{1}{2G_{12}} (\sin^4 \theta + \cos^4 \theta) \tag{9}
\]

Since \( E_{11} \), \( E_{22} \), and \( \nu_{12} \) are determined from tests (2.1) and (2.2), and \( \varepsilon_0 \), \( \varepsilon_0 + \pi/2 \), \( \gamma_0 \), \( \gamma_0 + \pi/2 \), and \( \tau_0 \) are obtained either from direct measurement by strain gages or from Eq. (3), the only unknown in Eq. (2) is \( G_{12} \). Thus \( G_{12} \) can be computed from any one of the three equations in Eq. (2). Its value is found to be 44.8 GPa which is within 5% of the theoretical predictions (see Fig. 12).

**CONCLUSIONS**

The goal of this experiment was to determine which theory of composites best predicted the elastic mechanical behavior of a superconducting (NbTi/Cu) composite wire. Examination of each elastic mechanical property reveals that all theories examined are capable of predicting experimental data to within 5%.
Since the Poisson's ratios for both the fiber and the matrix material were very close, there was essentially no (less than 1%) difference among all the theoretical predictions for any individual mechanical constant. Because of the expense and difficulty of producing elastic constant data within 0.1% accuracy, and therefore, conclusively determining which theory is best, no further experiments were performed.

In conclusion, for a superconducting composite wire, NbTi/Cu, a simple, fast, and reliable engineering estimate of its elastic mechanical behavior can be made by using the "rule of mixtures." It is unnecessary to use one of the more rigorous theories.
REFERENCES


NOMENCLATURE

Symbol Definition

E - Young's modulus
G - Shear modulus
ν - Poisson's ratio
K - Bulk modulus in plane strain
V - Volume fraction
f - Fiber material (NbTi)
m - Matrix material (Cu)
L - Composite material constants along the fiber direction
T - Composite material constants along a transverse direction
APPENDIX I

This appendix presents the various theoretical equations which were used to predict the elastic mechanical properties for the superconducting composite wire studied in this report. Their alphabetic index refers to the legend on the particular graph where a comparison with experimental data was performed. Also included in this appendix is the reference in which these equations may be found.

A program which numerically tabulates all of these equations is listed in Appendix III.
1. Longitudinal Young's modulus $E_L$

(a) $E_L = E_m V_m + E_f V_f$ \hspace{1cm} (10)

Source: Reference 2

(b) $E_L = \left( E_m V_m + E_f V_f \right) \phi$ \hspace{1cm} (11)

$$
\phi = \frac{E_m \left( D_1 - D_3 F_1 \right) + E_f \left( D_2 - D_4 F_2 \right)}{E_m \left( D_1 - D_3 \right) + E_f \left( D_2 - D_4 \right)}
$$

$$
D_1 = 1 - \nu_f \hspace{1cm} D_2 = \frac{1 + \nu_f}{\nu_m + \nu_f}
$$

$$
D_3 = 2 \nu_f^2 \hspace{1cm} D_4 = 2 \nu_f^2 \frac{\nu_f}{\nu_m}
$$

$$
F_1 = \frac{V_m E_m}{\nu_f E_f} \frac{\nu_f}{\nu_f + \nu_m} \frac{E_f}{E_m}
$$

$$
F_2 = \nu_f F_1 \frac{V_m}{\nu_m}
$$

Source: Reference 10
2. Major Poisson's ratio $\nu_L$

(a) $\nu_L = \frac{V_{EF}L_1 + V_{m}m_m L_2}{V_{f}f_1 L_3 + V_{m}m_m L_2}$

\[ L_1 = 2D_f (1 - D_m^2) V_f + D_m (1 + D_m) V_m \]
\[ L_2 = (1 - D_f + 2D_m^2) V_f \]
\[ L_3 = 2(1 - D_m^2) V_f + (1 + D_m) V_m \]

Source: Reference 10

(b) $\nu_L = \nu_m m_m + \nu_f f_f$

Source: Reference 2
3. **Transverse Young's modulus** $E_T$

(a) \[ E_T = \frac{E_f E_m}{E_f V_f + E_m V_f} \]  

\[ (14) \]

Source: Reference 23

(b) \[ E_T = \frac{1 + \phi \eta V_f}{1 - \eta V_f} E_m \]  

\[ (15) \]

\[ \eta = \frac{E_f}{E_m} + \phi \]

\[ \phi = 2 \text{ for circular fiber} \]

\[ \phi = 2 \left( \frac{d}{b} \right) \text{ for rectangular fiber} \]

\[ y \]

\[ x \]

\[ E_T = E_x \]

Source: Reference 2

(c) \[ E_T = \frac{E_f}{2(1 - V_f)} \left[ 1 - V_f + \left( V_f - V_m \right) V_m \right] \frac{H_f (2H_m + G_m) - G_m (H_f - H_m) V_m}{2H_m + C_m + 2(H_f - H_m) V_m} \]

\[ (16) \]

\[ H_f = \frac{E_f}{2(1 - V_f)} \]

\[ H_m = \frac{E_m}{2(1 - V_m)} \]

Source: Reference 23
(d) \( \frac{E_T}{E_m} = \left( 1 - 2 \frac{\sqrt{V_f}}{\pi} \right) + \frac{1}{\alpha} \left\{ \pi - \frac{4}{\sqrt{1 - \alpha^2 V_f/\pi}} \tan^{-1} \left[ \sqrt{\frac{1 - (\alpha^2 V_f/\pi)}{1 + \alpha^2 V_f/\pi}} \right] \right\} \) \hspace{1cm} (17)

\[ \alpha = 2 \left( \frac{E_m}{E_f} - 1 \right) \]

Source: Reference 23

(e) \( \frac{1}{E_T} = \frac{V_m}{E_m} + \frac{V_f}{E_f} \left( \frac{E_f}{E_m} \frac{V_m - V_f}{V_f E_f} \right)^2 \) \hspace{1cm} (18)

Source: Reference 23
4. Transverse shear modulus $G_T$

(a) \[ G_T (\text{lower bound}) = G_m \left[ 1 - \frac{2(1 - \nu_m)}{1 - 2\nu_m} V_f A_4^c \right] \quad (19) \]

where $A_4^c$ is obtained by solving the following equations

\[
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4^c \\
B_1 \\
B_2
\end{bmatrix} =
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

where

\[
[P] =
\begin{bmatrix}
1 & V_f^{-1} & V_f^2 & V_f & 0 & 0 \\
0 & \frac{4\nu_m - 3}{V_f^3} & -2V_f^2 & \frac{V_f}{1 - 2\nu_m} & 0 & 0 \\
1 & 1 & 1 & -1 & -1 \\
0 & \frac{4\nu_m - 3}{3 - 2\nu_m} & -2 & \frac{1}{1 - 2\nu_m} & 0 & \frac{3 - 4V_f}{3 - 2V_f} \\
1 & \frac{3}{3 - 2\nu_m} & -3 & \frac{1}{1 - 2\nu_m} & -\frac{G_f}{G} & \frac{3 G_f/G_m}{2V_f - 3} \\
0 & -\frac{1}{3 - 2\nu_m} & 2 & \frac{-i}{1 - 2\nu_m} & 0 & \frac{G_f/G_m}{3 - 2V_f}
\end{bmatrix}
\]

Source: Reference 10
(b) $G_T$ (upper bound) = $G \left[ \frac{1}{1 + \frac{2(1 - \nu_m)}{1 - 2\nu_m} \nu_m A_4^\sigma} \right]$  \hspace{2cm} (20)

and $A_4^\sigma$ is obtained by solving

$$
\begin{bmatrix}
A_1 \\
A_2 \\
A_3^\sigma \\
A_4^\sigma \\
B_1 \\
B_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
$$

where

$$
[R] = \begin{bmatrix}
1 & \frac{3}{3 - 2\nu_m} & \frac{1}{V_f} & -3V_f^2 & \frac{V_f}{1 - 2\nu_m} & 0 & 0 \\
0 & \frac{-1}{3 - 2\nu_m} & \frac{1}{V_f} & 2V_f^2 & \frac{-V_f}{1 - 2\nu_m} & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & -1 \\
0 & \frac{4\nu_m - 3}{3 - 2\nu_m} & -2 & \frac{1}{1 - 2\nu_m} & 0 & \frac{3 - 4\nu_f}{3 - 2\nu_m} \\
1 & \frac{3}{3 - 2\nu_m} & -3 & \frac{1}{1 - 2\nu_m} & -G_f & \frac{G_f / G_m}{3 - 2\nu_m} \\
0 & \frac{-1}{3 - 2\nu_m} & 2 & \frac{-1}{1 - 2\nu_m} & 0 & \frac{G_f / G_m}{3 - 2\nu_f}
\end{bmatrix}
$$

Source: Reference 10
\[
G_T = G_m \frac{2G_f (K_m + G_m) V_f}{2G_f (K_m + G_m) V_f + 2G_f G_m V_m + \ldots Y_m (G_m + G_f)}
\]

Source: Reference 7

\[
G_T \text{ (lower bound)} = G_m + \frac{V_f}{G_f - G_m + \frac{2G_m (K_m + G_m)}{2G_f (K_f + G_f)}}
\]

Source: Reference 11

\[
G_T \text{ (upper bound)} = G_f + \frac{V_m}{G_m - G_f + \frac{2G_f (K_f + G_f)}{2G_f (K_f + G_f)}}
\]

Source: Reference 11
5. Longitudinal shear modulus $G_L$

(a) $G_L = \frac{G_f (1 + V_f) + V_m}{\frac{V_m G_f}{G_m} + 1 + V_f}$

Source: Reference 10

(b) $G_L = \frac{G_f G_m}{V_m G_f + V_f G_m}$

Source: Reference 23

(c) $G_L = \frac{1 + \eta V_f}{1 - \eta V_f} G_m$

$\eta = \left(\frac{G_f}{G_m} - 1\right) \left(\frac{G_f}{G_m} + \phi\right)$

$\phi = \left(\frac{a}{b}\right)^{2/3}$ for rectangular fiber

$\phi = 1$ for circular fiber

Source: Reference 2
(d) \[ G_L = \frac{2G_f - (G_f - G_m)V_m}{2G_m + (G_f - G_m)V_m} \] \tag{27}

Source: Reference 10

(e) \[ G_L \text{ (lower bound)} = G_m + \frac{V_f}{\frac{1}{G_f - G_m} + \frac{V_m}{2G_m}} \] \tag{28}

Source: Reference 11

(f) \[ G_L \text{ (upper bound)} = G_f + \frac{V_m}{\frac{1}{G_m - G_f} + \frac{V_f}{2G_f}} \] \tag{29}

Source: Reference 11

(g) \[ G_L = \frac{G_m}{2} \left[ \frac{4 - \frac{\pi}{4} + \frac{\phi}{2(4 - \frac{\pi}{4})}}{4} + \frac{\phi}{2(4 - \frac{\pi}{4}) + \pi} \right] \] \tag{30}

\[ \phi = \frac{G_f (\pi + 4V_f) + (G_m \pi - 4V_f)}{G_f (\pi - 4V_f) + (G_m \pi + 4V_f)} \]

Source: Reference 23
APPENDIX II

Whitney\textsuperscript{25} has investigated the influence of twist on graphite fibers in an epoxy matrix. He derived an equation for the reduction of the longitudinal elastic Young's modulus for graphite as a function of the geometry of the fibers. This equation is directly applicable in estimating how twisting affects a superconducting wire.

If the initial and reduced moduli are \( E_I \) and \( E_R \), respectively, then their ratio can be expressed as:\textsuperscript{25}

\[
\frac{E_R}{E_I} = \frac{1}{1 + 4\pi^2 N_0^2 R^2},
\]

where \( N_0 \) is the number of twists per centimeter and \( R \) is the radius of the fiber in centimeters.

For the superconducting wire analyzed in this report,

\[
N_0 = 0.132 \text{ cm}^{-1}
\]

\[
R = 3.17 \times 10^{-3} \text{ cm},
\]

which yields

\[
\frac{1}{1 + 4\pi^2 N_0^2 R^2} = \frac{1}{1 + 6.87 \times 10^{-6}} \approx 1,
\]

Clearly, the effect of twist for this superconducting wire can be neglected.
PROGRAM FIBER VIA
LAST UPDATE: 1APR76

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LANGUAGE: DECSYSTEM-10, FORTRAN-10

SUBROUTINES REQUIRED:

MINV - STANDARD IBM SCIENTIFIC SUBROUTINE PACKAGE.

IMPLICIT REAL (K)
REAL MUP, MUPV
DIMENSION UG, G(i), L(G), M(G), H(G)
DIMENSION UMG, G, E(2, I), NCUT(2)

DATA PI/3.1415926/
DATA IGRED, IGRED+/5.5/

BEGIN PROGRAM FIBER VIA

QUERY THE USER FOR THE MATERIAL PROPERTIES
WRITE(10, 11)
11 FORMAT('FIBER VIA DOCUMENTED IN ORNL/TH-5331'."

1 'INPUT THE FOLLOWING ORDER':/
2 'EF(FIBER YOUNG'S MODULUS)':/
3 'EM(MATRIX YOUNG'S MODULUS)':/
4 'MUF(FIBER POISSON RATIO)':/
5 'MMUF(MATRIX POISSON RATIO)':/
6 'VF(FIBER VOLUME FRACTION)':/
7 'ORDER IS EM, EF, UG, MUF, MMUF, VF FORMAT(SF)'/
8 'REMEMBER A SPACE DELIMITS THE INPUT VARIABLES'/
9 READ(IRED, 10) EF, EM, MUF, MMUF, VF
10 FORMAT(SF)
999 CONTINUE

BEGIN PROGRAM FIBER VIA

CALCULATE THE SHEAR MODULII
GF=EF/(2.(1.+MUP))
GM=MV/(2.(1.+MUP))

CALCULATE THE MATRIX VOLUME FRACTION
VM=1.-VF

CALCULATE THE BULK MODULII
KF=EF/(3.-(1.+MUP))
KM=EM/(3.-(1.+MUP))

CALCULATE THE FIBER-MATRIX RATIOS.
CMGF=GF/GM
CMGM=GM/GM
EFEM=EM/EF
GMEM=EM/GM
KMGK=KM/GM
KFGF=KF/GF
C ECHO INPUT DATA
WRITE(OUT,29) EF,EM,NMF,NUM,VF,GF,GM,EFEM,KF,KM,GFEM

29 FORMAT(102) EF = 'TIB.IPE11.4/',
      1' EM = 'TIB.IPE11.4/',
      2' NMF = 'TIB.IPE11.4/',
      3' NUM = 'TIB.IPE11.4/',
      4' VF = 'TIB.IPE11.4/',
      5' GF = 'TIB.IPE11.4/',
      6' GM = 'TIB.IPE11.4/',
      7' EFEM = 'TIB.IPE11.4/',
      8' KF = 'TIB.IPE11.4/',
      9' KM = 'TIB.IPE11.4/',
      *.GF/GM = 'TIB.IPE11.4')

C THE EQUATION NUMBERS GIVEN IN THE REST OF THIS CODE REFER TO ORNL/IM-5331

C CALCULATE THE LONGITUDINAL YOUNG'S MODULUS

C EQUATION 18
C EQUATION 11
IF(VM,MI,EM) GO TO 21
ALPHA=1.
GO TO 20
21 CONTINUE
F1=NUM*VF*EFEM/NMF
F2=NUM*VF/NMF
F3=F2*VF/NMF
D1=1.-VF
D2=1.*VF
D3=2.*VF/NMF
D4=(1.-VF)*VF
ALPHA=F1+D1*EFEM/(D2-D4*F2)
ALPHA=ALPHA/D1+D3*EFEM/(D2-D4)

20 ELEM(2)=ALPHA
WRITE(OUT,25) ELEM(2)
25 FORMAT(102) ELEM(2) = ALPHA

C OUTPUT DATA ON DATA FILE
DO 801 J=1,2
801 ELEM(J)=ELEM(J)*EM
DO 30 J=1,2
JJ=J+2
WRITE(OUT,24) JJ,ELEM(J)
30 CONTINUE

C CALCULATE THE MAJOR POISSON RATIO

C EQUATION 12
XL1=NUM*(1.-NUM*VF)*VF/NMF
XL2=NUM*(1.-VF)*VF/NMF
XL3=NUM*(1.-VF)*VF/NMF
XMULT1=VF*EFEM/(VF*EFEM+NUM)*1.4
XMULT2=VF*EFEM/(VF*EFEM+NUM)*2
XMULT1=XMULT1+XMULT2

C EQUATION 13
XMULT2=VF*NUM
WRITE(OUT,25)
225 FORMAT(102) XMULT2 = VF*NUM
C OUTPUT DATA ON DATA FILE
DO 31 J=1,2
J=J+1
WRITE(COUT,24) JJ,NMULT(J)
CONTINUE

31 CALCULATE THE TRANSVERSE YOUNG'S MODULUS

C EQUATION 14
ETEM(1)=1.0/((1.-EMF)/EMF)

C EQUATION 15
ETEM(2)=(1.02*EMF)/((1.-EMF))

C EQUATION 16
XDF=EDEM(2,1,-1)/EFEM(2,1,-1)
YDF=EDEM(2,1,-2)/EFEM(2,1,-2)
PART=2.0-(1.-1.0H+1.0M)*MD
PART=PART*(2.0-(1.0M)+1.0M)*MD
ETEM(1)=PART/(2.0H*EMF)/EMF

C EQUATION 17
ALPHA=2.*EMF

C CHECK FOR A POSITIVE RADICAL FOR SORT
IF(1,-1*ALPHA/ALPHA/ALPHA/ALPHA/ALPHA/ALPHA,0.) GO TO 160
XDM=SORT(1,-1*ALPHA/ALPHA/ALPHA/ALPHA/ALPHA/ALPHA)
TANST=2.0H*EMF=EMF
SD+4.*SORT(1,-1*ALPHA/ALPHA/ALPHA/ALPHA)
ETEM(4)=-2.*SORT(1,-1*ALPHA/ALPHA/ALPHA/ALPHA)
ALPHA=ALPHA/ALPHA/ALPHA/ALPHA/ALPHA/ALPHA
GO TO 161

160 CONTINUE
ETEM(4)=1.

161 CONTINUE

C EQUATION 18
SD=(EFEM2+EFEM1)*EFEM2*EFEM1
ETEM(5)=V/V*EMF/EF-SD/(V*EMF*V*EMF)
ETEM(5)=1./ETEM(5)*EMF
WRITE(COUT, 625)
FORMAT(4/, "TRANSVERSE YOUNG'S MODULUS")

625 C OUTPUT DATA ON DATA FILE
ENG-MAXI(EMF,EMF)
ELW-MIN1(EMF,EMF)
DO 805 J=1,4
805 ETEM(J)+ETEM(J)=EMF
DO 92 J=1,5
J=J+3
IF(ETEM(J),GT.ENG,OR.ETEM(J),LT.EW) GO TO 82
WRITE(COUT,24) JJ,ETEM(J)
GO TO 92
82 WRITE(COUT,63) JJ
83 FORMAT("", 62, " NOT APPLICABLE")
92 CONTINUE

C CALCULATE THE TRANSVERSE SHEAR MODULUS

C EQUATION 19
C BEGIN TO SET UP THE COEFFICIENT MATRIX
  U(1,1)=1.
  U(2,1)=0.
  U(3,1)=1.
  U(4,1)=0.
  U(5,1)=1.
  U(6,1)=0.
  U(1,2)=0.
  U(2,2)=1.6/F.
  U(3,2)=U(4,2)/F.
  U(5,2)=1./U(6,2).
  U(1,3)=U(2,3)=U(1,3).
  U(1,4)=U(2,4).
  U(3,4)=U(4,4).
  U(5,4)=U(6,4).
  U(1,5)=U(2,5).
  U(3,5)=U(4,5).
  U(5,5)=U(6,5).

C STORE THE COEFFICIENT MATRIX INTO UINV
  DO 71 IH=1,6
  UINV(IH,JH)=U(IH,JH).
  71

C FORMULATE RIGHT HAND SIDE
  A(1)=1.
  A(2)=0.
  A(3)=0.
  A(4)=0.
  A(5)=0.
  A(6)=0.

C INVERT THE COEFFICIENT MATRIX. SUBROUTINE MINV IS THE STANDARD
C IBM SCIENTIFIC SUBROUTINE MATRIX INVERTER.
C CALL MINV(UINV,E,D,L,M)

C OBTAIN THE SOLUTION VECTOR
  DO 69 MK=1,6
  RES(MK)=0.
  DO 69 ML=1,6
  RES(MK)=RES(MK)+UINV(MK,ML)*A(ML).
  69
EQUATION 20

BEGIN TO SET UP THE COEFFICIENT MATRIX

\[ \begin{array}{c}
\text{U}(1) = 1 \\
\text{U}(2) = 2 \\
\text{U}(3) = 3 \\
\text{U}(4) = 4 \\
\text{U}(5) = 5 \\
\text{U}(6) = 6 \\
\text{U}(7) = 7 \\
\text{U}(8) = 8 \\
\text{U}(9) = 9 \\
\text{U}(10) = 10 \\
\end{array} \]

STORE THE COEFFICIENT MATRIX INTO UINV.

DO 81 JH = 1,6
81 UINV(JH) = U(JH)

FOR每一EH AND SIDE.

A(1) = 0
A(2) = 0
A(3) = 0
A(4) = 0
A(5) = 0
A(6) = 0

INVERT THE MATRIX.

CALL MINV(UINV,E,D,L,TD)

OBTAIN THE SOLUTION VECTOR.

DO 89 K = 1,6
RES(MK)=.0,
DO 89 ML=1,6
RES(MK) = RES(MK) * INV(MK,ML) * RES(MK)
89 CONTINUE

GTRH(2)=1.+2.*((1.-NUM)/(1.-2.*NUM))
C EQUATION 21
FKGC1=KMDH1.*VF2,
FKGC2=KMDH1.*VF2,
GTRH(3)=GTRH(2)+.5*(GTRH(2)*FKGC1)
C EQUATION 22
GTRH(4)=1.*VF/(1.-(GTRH-1.)+(KMDH2.*VF(2.-(KMDH1.)))
C EQUATION 23
FRAC=((KFG1+2.2+VF2)/(KFG1+2.))-1.0
GTRH(3)=GTRH(2)+.5*(GTRH(2)*GTRH(4))
WRITE(10,425)
FORMAT(1, 'TRANSVERSE SHEAR MODULUS')
C OUTPUT DATA ON DATA FILE
DO 893 JJ=1,5
893 GTRH(J)=GTRH(J)*CH
DO 33 JJ=1,5
WRITE(10,24) JJ,GTRH(J)
CONTINUE
C CALCULATE THE MINOR POISSON RATIO USING THE RULE OF MIXTURES
C TRANSVERSE YOUNG'S MODULUS AND EACH OF THE PREVIOUSLY CALCULATED C TRANSVERSE SHEAR MODULI.
E22=ETEM(1)
WRITE(10,79)
DO 75 JJ=1,5
WRITE(10,79) JJ,E22
75 CONTINUE
C EQUATION 24
POI23=(E22/(2.*GTRH(J))-.1,
IFPOI23.LT.0.0,6,POI23,GT.0.5) GO TO 76
WRITE(10,77) JJ,POI23
76 CONTINUE
WRITE(10,78) JJ
77 CONTINUE
WRITE(10,78) JJ
78 CONTINUE
WRITE(10,78) JJ
79 CONTINUE
C CALCULATE THE LONGITUDINAL SHEAR MODULUS
C EQUATION 25
GLGH(1)=GFGM(1.+VF)/(GFGM+1.+VF)
C EQUATION 26
GLHGM(2)=1./(GFGM+1.+VF)
C EQUATION 27
C
GLCM=2.*GFGM-(GFGM-1.)*XH
GLGM=4.*GLGM/(2.*GFGM-1.)*XH

C EQUATION 28
FRAC=1./(1.-GFGM)
GLGM5=1.*XH/(FRAC+XH/2.)

C EQUATION 29
FRAC=1./(1.-GFGM)*XH/(2.*GFGM)
GLGM6=0.5*GFGM*XH/FRAC

C EQUATION 30
ALPHA=GFGM*(A.XH+4.*PI)+PI-4.*XH
ALPHA=ALPHA/(GFGM*(PI-4.*XH)+PI4.*XH)
GLGM7=3./(4.*PI+PI4.*ALPHA4.*ALPHA*(ALPHA*(4.*PI)+PI))
WRITE(OUTRT, 325)

325 FORMAT(12, 'LONGITUDINAL SHEAR MODULUS')

C OUTPUT DATA ON DATA FILE
GKE=RM(X1,GF.GM)
GKE=RM11(GF.GM)
DO 882 J=1,7
882 GLGM3=GLGM(J)*XH
DO 34 J=1,7
J=J+33
IF(GLGM(J).GT.GLH.OR.GLGM(J).LT.GLH) GO TO 85
WRITE(OUTRT, 24) JJ, GLGM(J)
GO TO 34
85 WRITE(OUTRT, 83) JJ
34 CONTINUE

C MORE DATA
WRITE(OUTRT, 998)

998 FORMAT(12, 'MORE DATA?')
1 * INPUT EF,EN,HUF,NUM.VF FORMAT(5F)
2 * NEGATIVE EF STOPS THE PROGRAM
READ(9,19)EF,EN,HUF,NUM.VF
IF(EF.GT.0.) GO TO 999
STOP
END
ACKNOWLEDGMENTS

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FIGURE CAPTIONS

Fig. 1. Direction of loading for the determination of the longitudinal Young's modulus and the major Poisson's ratio.

Fig. 2. \( \sigma_1 \) vs \( \varepsilon_1 \) diagram for Kryo-210 superconductor.

Fig. 3. Normalized plot of the longitudinal Young's Modulus versus volume fraction of the fiber (NbTi): In this graph, as well as the other normalized comparison graphs, the legend refers to the theoretical equations presented in Appendix I.

Fig. 4. \( \varepsilon_2 \) vs \( \varepsilon_1 \) diagram for Kryo-210 superconductor.

Fig. 5. Normalized plot of the major Poisson's ratio vs volume fraction of the fiber (NbTi).

Fig. 6. Direction of loading for the determination of the transverse Young's Modulus and the minor Poisson's ratio.

Fig. 7. \( \sigma_2 \) vs \( \varepsilon_2 \) diagram for Kryo-210 superconductor.

Fig. 8. Normalized plot of the transverse Young's Modulus versus volume fraction of the fiber (NbTi).

Fig. 9. \( \varepsilon_3 \) vs \( \varepsilon_2 \) diagram for Kryo-210 superconductor.

Fig. 10. Normalized plot of \( C_{23} \) vs the volume fraction of the fiber (NbTi).

Fig. 11. Coordinate system for the determination of the longitudinal shear modulus.

Fig. 12. Normalized plot of the longitudinal shear modulus vs volume fraction of the fiber (NbTi).
Figure 1

SUPERCONDUCTOR (NbTi) FILIMENT
Figure 2

\[ \sigma \sim \text{MPa} \]

\[ \epsilon \sim \mu \text{m/m} \]

SLOPE = $E \approx 119 \text{GPa}$
Figure 4

SLOPE = \frac{1}{12} \times 0.347
Figure 5
Figure 6

SUPERCONDUCTING (NbTi) FILAMENT
Figure 9

\[ \varepsilon_3 = \mu \text{m/m} \]

\[ \varepsilon_2 = \mu \text{m/m} \]

SLOPE = \( \varepsilon_3 = 0.42 \)

\[ G_{23} \frac{E_{22}}{2(1-\nu_{23})} = 43.1 \text{ GPa} \]
Figure 10
Figure 11
Figure 12