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A Study of Adaptive Detection of Range-Distributed Targets

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A Modified Generalized Likelihood Ratio Test (MGLRT) for the adaptive detection of a target or targets that are distributed in range is derived. The unknown parameters associated with the hypothesis test are the complex amplitudes in range of the desired target and the unknown covariance matrix of the additive interference, which is assumed to be characterized as complex zero-mean correlated Gaussian random variables. The target's or targets' complex amplitudes are assumed to be distributed across the entire input data block (sensor × range). Results on probabilities of false alarm and detection are derived and a Constant False Alarm Rate (CFAR) detector is developed. Simulation results are presented. It is shown that the derived MGLRT of range-distributed targets is much more effective in detecting targets distributed in range than an M out of K detector that is cascaded with a single-point target Kelly detector. In addition, the MGLRT associated with detecting a single point target with signal-contaminated secondary data is also derived. It is shown, surprisingly, that this detector is totally ineffective in solving this problem.

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A STUDY OF ADAPTIVE DETECTION OF RANGE-DISTRIBUTED TARGETS

I. INTRODUCTION

A recent paper by Bose and Steinhardt [1] significantly advanced the theory of space-time adaptive detection of a desired signal in the presence of correlated Gaussian noise. Space-time adaptive detection is concerned with detecting a signal of partially known direction and doppler and unknown strengths across time (or range) in the presence of Gaussian noise whose covariance matrix is unknown. Their work generalized much of the adaptive detection research [2-6] of the last twenty years. A variety of application problems can be solved using their formalism. They did this by generalizing the desired signal waveform to be a rank one waveform (in matrix form) that is uncertain; i.e. only partially specified. More precisely, they modeled the space-time steering vector as a Kronecker product of two vectors, each of which is unknown but is known to lie in a known subspace.

One limitation of their approach is that the dimensions of these subspaces are constrained with respect to the dimensions of the space-time array and the number of snapshots (data vectors) associated with the space-time array. These constraints were imposed because in the derivation of the Generalized Likelihood Ratio Test (GLRT) detector it is possible that a certain estimated covariance matrix could become singular which would significantly complicate the derivation. Hence constraints were imposed to eliminate the possibility of generating singular matrices. We introduce what we term a Modified Generalized Likelihood Ratio Test (MGLRT) detector in order to generate a useful detection structure for the case when the estimated covariance matrix is singular.

In this paper, we solve a problem of practical importance which cannot be solved by the formalism of [1]. Consider the problem of adaptively detecting a range distributed target or targets. Suppose the desired target or targets could be spatially distributed across the entire range interval whose returns are to be used as the input data block (sensor $\times$ range) from which the adaptive detector is derived. Assume that the scatterers remain in their respective range cells during a coherent processing interval (CPI); i.e. the scatterers do not range walk during a CPI. The scatterers returns may be uniformly doppler shifted due to target(s) motion. If one uses a single point target adaptive detector (such as Kelly's [2] or the Adaptive Match Filter Detector [4,6]) to detect individual point scatterers across the range interval of the data block, significant detection performance degradation could occur in each range cell where a target scatterer is present due to the desired signal contamination of the secondary data (data where it is incorrectly assumed a desired signal is not present) [7]. Hence the total adaptive detection performance (detecting the presence of any target or targets across the range interval of interest) could be significantly degraded.

We derive the MGLRT associated with adaptively detecting a range distributed target or targets. It is pointed out that our methodology is not restricted to detecting a single target that is range distributed (resulting from using a high-range resolution (HRR) radar waveform) but could also be used to detect a formation of targets with the same velocity that are spatially distributed in range (say from a low range resolution radar waveform). In fact the problem of signal cancellation that occurs when one performs single point target adaptive detection can be significantly abated by using the proposed methodology: instead of trying to detect individual targets and suffering the potentially large detection losses due to signal contamination, one should try to detect the presence of the target set within the input data block. A benefit of doing this is that there will be an effective integration gain of target returns which enhances the target set's detectability. Hence what was thought to be a source of performance degradation (signal contamination) can actually be a source of performance enhancement. Obviously, the range resolution of the target could suffer via this methodology but at least a detection may not be lost.

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The outline of this paper is as follows. The MGLRT for a range distributed target(s) is defined and derived in Section II. Expressions for false alarm and detection probabilities are derived in Section III. Unfortunately, the MGLRT for a single data block is not a constant false alarm rate (CFAR) detector, i.e. the test statistic is not independent of the external noise environment. In Section IV, it is shown with the use of multiple data blocks how to construct a CFAR detector. In the preceding development, it has been assumed that the desired steering vector was known. In Section V, we derive the MGLRT for range distributed targets when it is assumed that the desired steering vector is unknown but lies on a known subspace. Some simulation results are presented in Section VI. In Section VII, the single point target MGLRT with assumed contamination of the secondary data is derived. However the resultant MGLRT is useless since it is shown that none of the secondary data is used. This was an attempt to solve the long standing problem of how to detect a point target in contaminated secondary data. The MGLRT formalism turned out not to be an effective solution.

II. MGLRT FOR RANGE DISTRIBUTED TARGETS

A. Preliminaries

Data is collected from N sensors for a radar that has a transmitted waveform with a fixed pulse repetition interval (PRI). For example, the N sensor inputs could consist of an array of antennas and time taps (with the time delay equal to the PRI) of these antennas. A data vector consisting of K elements is collected from each sensor. The data associated with the k'th element of each sensor data vector is assumed to represent data coming from the k'th range cell. Assume that the target (or targets) is spatially distributed across all of the K contiguous range cells in some fashion (some range cells could have no target scatterers). The returns for the K contiguous cells on each of the N sensors could consist of the target returns plus interference or interference by itself. The interference, assumed to be strict-sense stationary, could come from a variety of sources: e.g. clutter, jamming, and system noise. Inputs are complex valued.

It is pointed out that if we had assumed that only a specified m contiguous range cells out of the K range cells could contain target returns and $K - m \geq N$, then the methodology of [1] could be applied. However, for our development, it is reasonable to assume that we do not know which of the K range cells contain the target or targets. Thus we assume that the target returns could occur anywhere in the K range cells.

We desire to construct a hypothesis test which distinguishes between the signal plus interference hypothesis (H_1) and the interference only hypothesis (H_0). Consider the two hypotheses:

$$\begin{aligned} H_0 &: \quad \mathbf{z}_k = \mathbf{x}_k \\ H_1 &: \quad \mathbf{z}_k = \mathbf{x}_k + a_k^* \mathbf{s} \quad , k = 1, 2, \ldots, K \end{aligned} \tag{1}$$

where $\mathbf{x}_k$, $\mathbf{z}_k$, $\mathbf{s}$ are the interference vector, received vector, and desired signal steering vector, respectively, all of length N associated with the k-th range cell of the N sensors. The desired steering vector, $\mathbf{s}$, is normalized so that $\mathbf{s}^H\mathbf{s} = 1$ where H denotes conjugate transpose. For examples, the steering vector for the space/time adaptive processor (STAP) is the Kronecker product of desired angular (direction of arrival) steering vector and the desired doppler shift steering vector. The complex scalar a_k^* where * denotes complex conjugation is the unknown complex signal amplitude of the desired target's scatterer in the k'th range cell (we conjugated a_k for notational purposes). Eq. (1) can be rewritten more succinctly as

$$
\begin{aligned}
H_0 &: \; Z = X \\
H_1 &: \; Z = X + \mathbf{s}\mathbf{a}^{\mathrm{H}},
\end{aligned}
\tag{2}
$$

where Z and X are denoted as the $N \times K$ input data matrix and interference matrix, respectively, $\mathbf{a} = (a_1, a_2, \ldots, a_K)^{\mathrm{T}}$, and T denotes transpose. The input interference vectors, $\mathbf{x}_k$ ($k = 1, 2, \ldots, K$) are assumed to be independent and identically distributed (i.i.d.) zero-mean complex circular Gaussian vectors. The covariance matrix of each vector is given by an $N \times N$ complex Hermitian matrix, $\bar{R}$, which is assumed unknown. Assume $K \geq N$. For $K \geq N$ and $\bar{R}$ positive definite, the rank of Z equals N with probability 1 [10].

It can be shown (see e.g. [8]) that the probability density function (pdf) for Z under each hypothesis can be written as

$$
\begin{aligned}
H_0 &: \; p_0(Z; \bar{R}) = \frac{c}{\|\bar{R}\|_p^K} \exp[-\, Tr\{\bar{R}^{+} Z Z^{\mathrm{H}}\}]\; \bar{\delta}(\Phi_2^{\mathrm{H}} Z) \\
H_1 &: \; p_1(Z; \bar{R}, \mathbf{a}) = \frac{c}{\|\bar{R}\|_p^K} \exp[-\, Tr\{\bar{R}^{+}(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})^{\mathrm{H}}\}]\; \bar{\delta}(\Phi_2^{\mathrm{H}}(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})),
\end{aligned}
\tag{3}
$$

where c is the pdf normalization constant, $\|\bar{R}\|_p$ denotes the positive determinant of R (the product of the positive eigenvalues of $\bar{R}$), $Tr\{\ \}$ is the trace of the argument, $\bar{R}^{+}$ denotes the Moore-Penrose generalized inverse of $\bar{R}$, $\bar{\delta}$ is the product of Dirac delta functions of the matrix elements of the argument, and Φ_2 will be defined shortly. The generalized inverse is defined as follows. We can write $\bar{R} = \Phi\Lambda\Phi^{\mathrm{H}}$ where Φ is the $N \times N$ matrix of eigenvectors of $\bar{R}$ and Λ is the diagonal matrix of eigenvalues of $\bar{R}$. Write $\Lambda = \mathrm{diag}\ \{\lambda_1, \lambda_2, \ldots, \lambda_M, 0, \ldots, 0\}$ where $\lambda_m > 0$, $m = 1, 2, \ldots, M$ and $M \leq N$ and $\Lambda^{+} = \mathrm{diag}\ \{\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_M^{-1}, 0, \ldots, 0\}$. Then $\bar{R}^{+} = \Phi\Lambda^{+}\Phi^{\mathrm{H}}$ [9]. Set $\Phi = (\Phi_1\ \Phi_2)^{\mathrm{H}}$ where the $(N - M) \times N$ matrix, Φ_2, is associated with the null space of the columns of $\bar{R}$.

In the adaptive detection literature, it is almost always assumed that $\bar{R}$ is nonsingular so that $\bar{R}^{+} = \bar{R}^{-1}$ and $\|\bar{R}\|_p = \|\bar{R}\|$ where $\|\cdot\|$ denotes the determinant of $\bar{R}$. However, we will find in our formulation that this assumption need not hold. We will find that when we formulate a modified maximum likelihood (MML) solution, R_{MML}, for $\bar{R}$ under H_1 that R_{MML} can be made singular for a certain subspace of $\mathbf{a}$.

The GLRT is formulated as follows.

$$
\frac{\sup\limits_{R,\mathbf{a}} p_1(Z; R, \mathbf{a})}{\sup\limits_{R} p_0(Z; R)} \underset{H_0}{\overset{H_1}{\gtrless}} t. \tag{4}
$$

where R is the estimate of $\bar{R}$ under each hypothesis. It will be found for our problem (where p_0 and p_1 are defined by (3)) that $\sup\limits_{R,\mathbf{a}} p_1(Z; R, \mathbf{a})$ is unbounded. The pdf p_1 is infinity as the range space associated with the columns of R overlaps the null space associated with the columns of $Z - \mathbf{s}\mathbf{a}^{\mathrm{H}}$. The null space of $Z - \mathbf{s}\mathbf{a}^{\mathrm{H}}$ is non-empty if we can choose an $\mathbf{a}$ such that $Z - \mathbf{s}\mathbf{a}^{\mathrm{H}}$ has rank less than N. If Z has full

rank, we can always find an $\mathbf{a}$ that drops the rank of $Z - \mathbf{s}\mathbf{a}^{\mathrm{H}}$ to $N - 1$. For example, set $\mathbf{a} = Z^{\mathrm{H}}\mathbf{s}$. Equivalently, p_1 approaches infinity as $(R, \mathbf{a})$ approaches (with respect to some distance measure) a member of the following set, S

$$S = \{R, \mathbf{a} \mid \rho(R) = N - 1,\ R = (Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})\ (Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})^{\mathrm{H}}\}$$

where $\rho(\cdot)$ denotes the rank of the argument. Here, we see S consists of the subspace of $\mathbf{a}$ where $(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})\ (Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})^{\mathrm{H}}$ is a singular matrix. Thus it would seem that a useful GLRT for this problem does not exist since $\sup_{R,\mathbf{a}} p_1$ is unbounded. As Grenander [13] points out, in certain ML problems the parameter space is too large. He proposes salvaging the method by performing the maximization over a non-trivial, constrained set. This method is called the "Method of Sieves." Using the Method of Sieves [13], we can restrict $(R, \mathbf{a})$ to a reasonably subspace such that a maximum likelihood solution exists. Intuitively it would seem that if there were a useful estimate of $(R, \mathbf{a})$, it should come from set S since there exists $(R, \mathbf{a})$ in the neighborhood of a given member of S which yield an arbitrarily large value for p_1. Thusly, it will be found that a reasonable estimator and detector can be found for this problem when we restrict $(R, \mathbf{a})$ to be a member of S. However, on this subspace we maximize with respect to R the multipliers on the δ functions seen in Eq. (3). Set $m_0(Z;\bar{R})$ and $m_1(Z;\ R,\ \mathbf{a})$ equal to the multipliers of the δ functions for p_0 and p_1, respectively seen in Eq. (3). On this subspace for cases of interest, $\sup m_1\ (Z;\ R,\ \mathbf{a}) < \infty$ since m_1 is defined by (3) for all R with rank greater than or equal to one. It will be seen in the following subsection that we will have to further restrict the form of R in order to obtain a unique ML solution. Hence this suprema will be used in the likelihood ratio test expression and a finite test statistic results (see [15] for other examples of where the Method of Sieves is used for estimating covariances matrices). We note for the case when $N = 1$ that $\sup_{R,a \in S} m_1(Z;\ R,\ \mathbf{a})$ is unbounded. This case was presented in [14] as a counterexample to the statement that a maximum likelihood estimator always exists.

When we restrict the subspace of the unknown parameters and find the maximum of the multiplier of the delta function for each pdf of the received data (under either hypothesis) with respect to the unknown parameters, we call this the modified maximum likelihood (MML) estimate. When we use the ratio of the m_1 and m_0 estimates we call this the modified GLRL (MGLRT). As in the case of the GLRT, there is no claim to optimality as a detector for the MGLRT. However, as in the case of the GLRT, the MGLRT seems like a reasonable approach for finding a good detector. This property of "goodness" must be checked out via simulation and comparison with alternate approaches. We will see that the MGLRT for this particular problem is a useful detector. However, as we will show in Section VII for a different detection problem, a useful solution does not always result.

B. Modified Maximum Likelihood Solution for a Singular Sampled Covariance Matrix

Let Z have the following pdf:

$$p_0(Z;\ \bar{R}) = \frac{c}{\|\bar{R}\|_p^K} \exp[-\ Tr\{\bar{R}^{+}\ \hat{R}\}]\ \delta(\Phi_2^{\mathrm{H}} Z), \tag{5}$$

where $N \geq 2$, $\hat{R} = ZZ^{\mathrm{H}}$ can be singular with at least rank 1 and $\bar{R}$ has at least rank 1. If $\rho(\hat{R}) = M < N$ then it can be shown that the ML solution for $\bar{R}$ does not exist. However, we can find a unique MML estimate for $\bar{R}$ by restricting R as follows. Define the set S_0 of $N \times N$ matrices:

$$S_0 = \left\{ R | R = \hat{\Phi} \cdot \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \hat{\Phi}^{\mathrm{H}} \right\} \tag{6}$$

where $\hat{R} = \hat{\Phi}\hat{\Lambda}\hat{\Phi}^{\mathrm{H}}$ is the eigenvalue decomposition of $\hat{R}$ and A is any $\rho(\hat{R}) \times \rho(\hat{R})$ matrix. Then we will show, $R_{\mathrm{MML}} = \arg \max_{R \in S_0} m_0(Z, R)$; i.e., a unique MML solution exists on S_0.

We will need the following lemma for our development of the MGLRT.

Lemma 1: If we restrict $R \in S_0$, the MML estimate solution for $\bar{R}$ is $R_{\mathrm{MML}} = \hat{R}$.

proof: We desire to solve

$$R_{\mathrm{MML}} = \arg \max_{R \in S_0} m_0(Z; R). \tag{7}$$

This is equivalent to solving

$$R_{\mathrm{MML}} = \arg \min_{R \in S_0} [K \ln \|R\|_p + Tr\{R^+\hat{R}\}]. \tag{8}$$

Rewrite (8) as

$$R_{\mathrm{MML}} = \arg \min_{R \in S_0} [K \ln \|R\|_p + Tr\{R^+\hat{\Phi}\hat{\Lambda}\hat{\Phi}^{\mathrm{H}}\}]. \tag{9}$$

This can be shown to be equivalent to finding a matrix R_0 such that

$$\bar{R}_0 = \arg \min_{R_0 \in S_1} [K \ln \|R_0\|_p + Tr\{R_0^+\hat{\Lambda}\}], \tag{10}$$

where

$$R_0 = \hat{\Phi}^{\mathrm{H}} R \hat{\Phi}, \tag{11}$$

S_1 is defined as

$$S_1 = \left\{ R_0 | R_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right\} \tag{12}$$

and A is any $\rho(\hat{R}) \times \rho(\hat{R})$ matrix. Let $\hat{\Lambda}_r$ be the $\rho(\hat{R}) \times \rho(\hat{R})$ diagonal matrix of the positive diagonal elements of $\hat{\Lambda}$. Then (10) is equivalent to finding

$$A_{\min} = \arg \min_{\mathbf{A}} [K \ln \|A\|_p + Tr\{A^+\hat{\Lambda}_r\}]. \tag{13}$$

It is well-known that $A_{\min} = \hat{\Lambda}_r$ and the lemma follows. □

C. MGLRT Derivation

Using Lemma 1, we find that

$$\sup_R m_0(Z;\ R) = \frac{c}{\|ZZ^{\mathrm{H}}\|_p^K} \exp - Tr\{(ZZ^{\mathrm{H}})^+ ZZ^{\mathrm{H}}\}. \tag{14}$$

We assume that $K \geq N$ and the underlying covariance matrix for the N-length $\mathbf{z}$ vectors is positive definite, so that ZZ^{H} has rank N with probability 1, [10]. Thus

$$\sup_R m_0(Z;\ R) = \frac{c'}{\|ZZ^{\mathrm{H}}\|^K}, \tag{15}$$

where c' is a constant.

Again using Lemma 1:

$$\sup_R m_1(Z;\ R,\ \mathbf{a}) = \frac{c}{\|R_{\mathrm{MML}}\|_p^K} \exp - Tr\{R_{\mathrm{MML}}^+ \hat{R}_a\}, \tag{16}$$

where

$$R_{\mathrm{MML}} = \hat{R}_a \equiv (Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})^{\mathrm{H}}. \tag{17}$$

We note that if $\hat{R}_a$ has rank $N - 1$, then $Tr\{\hat{R}_a^+ \hat{R}_a\} = N - 1$. Thus

$$\sup_R m_1(Z;\ R,\ \mathbf{a}) = \frac{c''}{\|(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})^{\mathrm{H}}\|_p^K}, \tag{18}$$

where c'' is a constant. Thus the MGLRT takes the equivalent form

$$\frac{\|ZZ^{\mathrm{H}}\|}{\min\limits_{\rho(\hat{R}_a) \leq N-1} \|(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})^{\mathrm{H}}\|_p} \underset{H_0}{\overset{H_1}{\gtrless}} t. \tag{19}$$

Henceforth for convenience, we shorten our notation for expressing the domain over which the minimization of the positive determinant is taken. It will be assumed that the domain is restricted such that the matrix in the argument of $\|\cdot\|_p$ is singular. For example, we write $\min\limits_{\mathbf{a}} \|\cdot\|_p$ instead of $\min\limits_{\rho(\hat{R}_a) \leq N-1} \|\cdot\|_p$. The optimal solution for $\mathbf{a}$ is given by the following theorem.

Theorem 1: The solution for $\mathbf{a}_{\mathrm{MML}} = \arg\min_{\mathbf{a}} \|(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})^{\mathrm{H}}\|_p$ is given by $\mathbf{a}_{\mathrm{MML}} = Z^{\mathrm{H}}\mathbf{s}$.

proof: Let A_s be a unitary $N \times N$ matrix such that $A_s\mathbf{s} = \mathbf{1}_0$ where $\mathbf{1}_0 = (1\ 0\ 0\ \ldots\ 0)^{\mathrm{T}}$. Let the Q - R decomposition of $A_sZ = [L : 0_{N,K-N}]Q$ where L is an $N \times N$ lower triangular matrix (LTM), Q is a $K \times K$ unitary matrix and $0_{N,K-N}$ is a $N \times (K - N)$ matrix of zeros. Now

$$\begin{aligned} \|(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})^{\mathrm{H}}\|_p &= \|A_s(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})Q^{\mathrm{H}}Q(Z - \mathbf{s}\mathbf{a}^{\mathrm{H}})^{\mathrm{H}}A_s^{\mathrm{H}}\|_p \\ &= \|[(L : 0_{N,K-N}) - \mathbf{1}_0\mathbf{a}_0^{\mathrm{H}}][(L : 0_{N,K-N}) - \mathbf{1}_0\mathbf{a}_0^{\mathrm{H}}]^{\mathrm{H}}\|_p, \end{aligned} \tag{20}$$

where $\mathbf{a}_0 = Q\mathbf{a}$. Thus an equivalent problem to finding the optimal $\mathbf{a}$ is to find the $\bar{\mathbf{a}}_0$ that minimizes the far right hand side (RHS) of Eq. (20). Equation (20) can be further simplified to yield

$$\bar{\mathbf{a}}_0 = \arg\min_{\mathbf{a}_0} \|(L - \mathbf{1}_0\mathbf{a}_1^{\mathrm{H}})(L - \mathbf{1}_0\mathbf{a}_1^{\mathrm{H}})^{\mathrm{H}} + \alpha\mathbf{1}_0\mathbf{1}_0^{\mathrm{T}}\|_p \tag{21}$$

where $\mathbf{a}_1$ is an N-length vector of the first N elements of $\mathbf{a}_0$ and α is the sum of the magnitude squares of the last $K - N$ elements of $\mathbf{a}_0$. Using Weyl's Monotonicity Theorem [9], the n'th ordered eigenvalue of $(L - \mathbf{1}_0\mathbf{a}_1^{\mathrm{H}})(L - \mathbf{1}_0\mathbf{a}_1^{\mathrm{H}})^{\mathrm{H}} + \alpha\mathbf{1}_0\mathbf{1}_0^{\mathrm{T}}$ is greater than or equal to the n'th ordered eigenvalue of $(L - \mathbf{1}_0\mathbf{a}_1^{\mathrm{H}})\ (L - \mathbf{1}_0\mathbf{a}_1^{\mathrm{H}})^{\mathrm{H}}$. This implies that (21) is minimized with respect to α when $\alpha = 0$. Thus our problem is reduced to finding an N-length vector $\bar{\mathbf{a}}_1$ such that

$$\bar{\mathbf{a}}_1 = \arg\min_{\mathbf{a}_1} \|(L - \mathbf{1}_0\mathbf{a}_1^{\mathrm{H}})(L - \mathbf{1}_0\mathbf{a}_1^{\mathrm{H}})^{\mathrm{H}}\|_p. \tag{22}$$

If $L = (l_{mn})$, we will show that $\bar{\mathbf{a}}_1 = (l_{11}^*\ 0\ 0\ \ldots\ 0)^{\mathrm{T}}$.

Set $\mathbf{a}_1^{\mathrm{H}} = \mathbf{a}_2^{\mathrm{H}}L$. Thus our problem is equivalent to finding an N-length vector $\bar{\mathbf{a}}_2$ such that

$$\bar{\mathbf{a}}_2 = \arg\min_{\mathbf{a}_2} \|(I_N - \mathbf{1}_0\mathbf{a}_2^{\mathrm{H}})LL^{\mathrm{H}}(I_N - \mathbf{1}_0\mathbf{a}_2^{\mathrm{H}})^{\mathrm{H}}\|_p \tag{23}$$

where I_N is the $N \times N$ identity matrix. In order for the matrix in the argument of $\|\cdot\|_p$ to be singular, it is required that $I_N - \mathbf{1}_0\mathbf{a}_2^{\mathrm{H}}$ is singular which will be if and only if

$$\mathbf{a}_2^{\mathrm{H}}\mathbf{1}_0 = 1. \tag{24}$$

Using singular value decomposition (svd), it can be shown that

$$I_N - \mathbf{1}_0\mathbf{a}_2^{\mathrm{H}} = P_1\Lambda_1Q_1^{\mathrm{H}}, \tag{25}$$

where $Q_1 = [Q_{\perp 1_0}, \mathbf{1}_0]$ is an $N \times N$ unitary matrix, $Q_{\perp 1_0}$ is an $N \times (N - 1)$ matrix consisting of

columns that are orthogonal to $\mathbf{1}_0$; $P_1 = \left[P_{\perp a_2}, \frac{\mathbf{a}_2}{|\mathbf{a}_2|} \right]$ is an $N \times N$ unitary matrix, $P_{\perp a_2}$ is an $N \times (N-1)$ matrix consisting of columns that orthogonal to $\mathbf{a}_2$, and Λ_1 is an $N \times N$ diagonal matrix with diagonal elements: $\lambda_{11} = (\mathbf{a}_2^{\mathrm{H}}\mathbf{a}_2)^{1/2}$, $\lambda_{12} = \lambda_{13} = \ldots = \lambda_{1,N-1} = 1$, and $\lambda_{1N} = 0$.

We can show

$$\begin{aligned} \| (I_N - \mathbf{1}_0\mathbf{a}_2^{\mathrm{H}}) LL^{\mathrm{H}} (I_N - \mathbf{1}_0\mathbf{a}_2^{\mathrm{H}})^{\mathrm{H}} \|_p &= \| \Lambda_1 Q_1^{\mathrm{H}} LL^{\mathrm{H}} Q_1 \Lambda_1 \|_p \\ &= (\mathbf{a}_2^{\mathrm{H}}\mathbf{a}_2) \| Q_{\perp 1_0}^{\mathrm{H}} LL^{\mathrm{H}} Q_{\perp 1_0} \| . \end{aligned} \tag{26}$$

It is proved in Appendix A (*Lemma* 1A) that if $Q_{\perp 1_0}^{(1)}$ and $Q_{\perp 1_0}^{(2)}$ are any two $N \times (N-1)$ matrices whose columns span the space that is orthogonal to $\mathbf{1}_0$ then

$$\| Q_{\perp 1_0}^{(1)\mathrm{H}} LL^{\mathrm{H}} Q_{\perp 1_0}^{(1)} \|_p = \| Q_{\perp 1_0}^{(2)\mathrm{H}} LL^{\mathrm{H}} Q_{\perp 1_0}^{(2)} \|_p . \tag{27}$$

Hence we can always find a $Q_{\perp 1_0}$ that is independent of $\mathbf{a}_2$ and the minimization of the far RHS of (26) is independent of the positive determinant term. Hence, we desire to find $\bar{\mathbf{a}}_2 = \arg \min_{\mathbf{a}_2} \mathbf{a}_2^{\mathrm{H}}\mathbf{a}_2$ with the constraint $\mathbf{a}_2^{\mathrm{H}}\mathbf{1}_0 = 1$. Obviously $\bar{\mathbf{a}}_2 = (1\ 0\ 0\ \ldots\ 0)^{\mathrm{T}}$. Working backwards, it is straightforward to show $\bar{\mathbf{a}}_1 = L^{\mathrm{H}}\mathbf{1}_0$, $\bar{\mathbf{a}}_0 = (L : 0_{N,K-N})^{\mathrm{H}}\mathbf{1}_0$, and $\mathbf{a} = Q^{\mathrm{H}}\bar{\mathbf{a}}_0 = Q^{\mathrm{H}}(L : 0_{N,K-N})^{\mathrm{H}} A_s A_s^{\mathrm{H}} \mathbf{1}_0 = Z^{\mathrm{H}}\mathbf{s} = \mathbf{a}_{\mathrm{MML}}$. □

Collecting our results, we have the following result.

Theorem 2: The MGLRT for detection of a target (or targets) that are range distributed is given by

$$\beta \equiv \frac{\| ZZ^{\mathrm{H}} \|}{\| (I_N - \mathbf{s}\mathbf{s}^{\mathrm{H}}) ZZ^{\mathrm{H}} (I_N - \mathbf{s}\mathbf{s}^{\mathrm{H}}) \|_p} \underset{H_0}{\overset{H_1}{\gtrless}} t . \tag{28}$$

We see from the form of β that the denominator term which was derived under H_1 (signal plus noise) will be without the additive desired signal vector because $I_N - \mathbf{s}\mathbf{s}^{\mathrm{H}}$ is the null projection matrix of $\mathbf{s}$. Thus $(I_N - \mathbf{s}\mathbf{s}^{\mathrm{H}})Z$ will contain no desired signal.

We now present a simple procedure for finding β. A unitary matrix B_s exists such that

$$B_s\mathbf{s} = (0\ 0\ \ldots\ 0\ 1)^{\mathrm{T}} \equiv \bar{\mathbf{1}}_0 . \tag{29}$$

Let the Q - R decomposition of $B_s Z = [L : 0_{N,K-N}]Q$ where L is an $N \times N$ LTM. It is straightforward to show that the test statistic β defined by (28) reduces to

$$\beta = \frac{\| LL^{\mathrm{H}} \|}{\| (I_N - \bar{\mathbf{1}}_0 \bar{\mathbf{1}}_0^{\mathrm{T}}) LL^{\mathrm{H}} (I_N - \bar{\mathbf{1}}_0 \bar{\mathbf{1}}_0^{\mathrm{T}}) \|_p} . \tag{30}$$

If $L = (l_{nn})$, the above reduces to

$$\beta = l_{NN}^2 . \tag{31}$$

Thus the procedure for finding the detection statistic is straightforward:

1. Perform the Q - R decomposition of the $N \times K$ data matrix, Z.

2. The NN'th element of the $N \times N$ triangular matrix associated with the Q - R decomposition is the test statistic.

III. FALSE ALARM AND DETECTION PROBABILITIES

A. Probability of False Alarm

We can derive the pdf of test statistic, β, under the H_0 hypothesis by using the equivalent form

$$\beta = \frac{\| \tilde{Z}\tilde{Z}^{\mathrm{H}} \|}{\| (I_N - \bar{\mathbf{1}}_0\bar{\mathbf{1}}_0^{\mathrm{T}}) \tilde{Z}\tilde{Z}^{\mathrm{H}} (I_N - \bar{\mathbf{1}}_0\bar{\mathbf{1}}_0^{\mathrm{T}}) \|_p} , \tag{32}$$

where $\tilde{Z} = B_s Z$ and $B_s \mathbf{s} = \bar{\mathbf{1}}_0$. Let $\tilde{R}$ be the true covariance matrix associated with any column of $\tilde{Z}$. Let $\tilde{R} = \tilde{L}\tilde{L}^{\mathrm{H}}$ be the Cholesky decomposition of $\tilde{R}$ where $\tilde{L}$ is an $N \times N$ LTM. Thus we can construct the transformed data matrix as $\tilde{Z} = \tilde{L}V$ where V is a $N \times K$ matrix of i.i.d. zero-mean complex circular Gaussian random variables with variance equal to one. Hence

$$\| \tilde{Z}\tilde{Z}^{\mathrm{H}} \| = \| \tilde{L}VV^{\mathrm{H}}\tilde{L}^{\mathrm{H}} \| = \| \tilde{L} \|^2 \| VV^{\mathrm{H}} \| . \tag{33}$$

Now

$$(I_N - \bar{\mathbf{1}}_0\bar{\mathbf{1}}_0^{\mathrm{T}})\tilde{Z} = \begin{bmatrix} L_1 & \mathbf{0}_{N-1}^{\mathrm{T}} \\ \mathbf{0}_{N-1} & 0 \end{bmatrix} \begin{bmatrix} V_{11} \\ v_N \end{bmatrix} = \begin{bmatrix} L_1 V_{11} \\ \mathbf{0}_N^{\mathrm{T}} \end{bmatrix} , \tag{34}$$

where $L_1 = (N-1) \times (N-1)$ matrix of the first $N-1$ rows and columns of L, $V_{11} = (N-1) \times N$ matrix of the first $N-1$ rows of V, v_N = last row of V, and $\mathbf{0}_n$ denotes a column vector of n zeroes. Thus

$$\| (I_N - \bar{\mathbf{1}}_0\bar{\mathbf{1}}_0^{\mathrm{T}}) \tilde{Z}\tilde{Z}^{\mathrm{H}} (I_N - \bar{\mathbf{1}}_0\bar{\mathbf{1}}_0^{\mathrm{T}}) \|_p = \| L_1 V_{11} V_{11}^{\mathrm{H}} L_1^{\mathrm{H}} \| = \| \tilde{L}_1 \|^2 \| V_{11} V_{11}^{\mathrm{H}} \| , \tag{35}$$

and

$$\beta = \frac{\|\tilde{L}\|^2}{\|\tilde{L}_1\|^2} \cdot \frac{\|VV^{\mathrm{H}}\|}{\|V_{11}V_{11}^{\mathrm{H}}\|}. \tag{36}$$

If $\tilde{L}_0 = (\tilde{l}_{nm})$, then $\|\tilde{L}\|^2 / \|\tilde{L}_1\|^2 = \tilde{l}_{NN}^2$. Let the Q - R decomposition of $V = (L_v : 0_{N,K-N})Q_v$ where L_v is an $N \times N$ LTM and $L_v = (l_{nm}^{(v)})$. It is straightforward to show

$$\frac{\|VV^{\mathrm{H}}\|}{\|V_{11}V_{11}^{\mathrm{H}}\|} = [l_{NN}^{(v)}]^2, \tag{37}$$

where $[l_{NN}^{(v)}]^2$ is the sum of the squared magnitudes of $K - N + 1$ zero-mean complex circular Gaussian r.v.'s with variance equal one [11]. This results in $2[l_{NN}^{(v)}]^2$ having a chi-square pdf of order $2(K - N + 1)$. Set

$$\beta = \tilde{l}_{NN}^2 \cdot l_{NN}^{(v)2}. \tag{38}$$

Thus β has the following pdf

$$p_\beta(\beta \mid H_0) = \frac{1}{(K - N)!} \frac{1}{\tilde{l}_{NN}^2} \left[\frac{\beta}{\tilde{l}_{NN}^2}\right]^{K-N} \exp\left[-\frac{\beta}{\tilde{l}_{NN}^2}\right] \tag{39}$$

and

$$P_F = \int_t^\infty p_\beta(\beta \mid H_0)\, d\beta. \tag{40}$$

We observe that the MGLRT for a range distributed target(s) is not a CFAR detector with respect to the external noise environment; i.e. the test statistic and hence the false alarm probability, P_F, is a function of a parameter of the true covariance matrix of external noise environment. From (39), we see that the test statistic's pdf is a function only of $\tilde{l}_{NN}$ which depends on the true covariance matrix $\tilde{R}$ which in turn depends on the true covariance matrix $\bar{R}$.

Because the P_F is a function of a parameter of the external noise environment, there would seem to be no way of *a priori* specifying a threshold, T, in order to attain a given P_F. A methodology to overcome this deficiency and make the derived adaptive detection scheme practical is as follows. Let the first N_{DOF} columns of $\tilde{L}$ approximately span the vector space associated with the external jamming where N_{DOF} represents the number of degrees of freedom (*DOF*) necessary to span this space and $N_{DOF} < N$. The amplitude levels of the elements of the first N_{DOF} columns tend to be above the average internal noise amplitude level which we assume without loss of generality equals one. The last $N - N_{DOF}$ columns of $\tilde{L}$ approximately span the null space of jamming or equivalently the internal noise space. The amplitude levels of elements of the last $N - N_{DOF}$ columns are approximately at the averge internal noise amplitude level. Hence $\tilde{l}_{NN} \approx 1$. For the jamming scenarios when $N_{DOF} < N$, it will be found that the external jamming can suppressed at or below the internal noise by the non-adaptive matched filter (i.e. form $\mathbf{s}^{\mathrm{H}}\bar{R}^{-1}Z$).

A methodology for insuring that a given P_F is not exceeded would be to first find the maximum of $\tilde{l}_{NN}$ over all the expected external noise scenarios (in a systems design, this is normally known). If the non-adaptive matched filter is working well (i.e. suppressing the external noise below internal noise, which in most cases is the design goal and drives the design specification), then max $\tilde{l}_{NN}$ should be at or about the internal noise level (which is usually normalized to one). We form the following test statistic:

$$\beta = \frac{l_{NN}^2}{\max \tilde{l}_{NN}^2} \leq l_{NN}^{(v)2} \equiv \bar{\beta}. \tag{41}$$

Thus

$$P_F \equiv \text{Prob}\ \{\beta > t \mid H_0\} \leq \text{Prob}\ \{\bar{\beta} > t \mid H_0\} \equiv \bar{P}_F. \tag{42}$$

If we choose t based on obtaining a desirable $\bar{P}_F$, then the actual P_F is always upper bounded by $\bar{P}_F$.

Finally, in Section IV we introduce a technique which is CFAR but will require more than one block of input data.

B. Probability of Detection

In most cases of interest the probability of detection P_D will be computed using Monte Carlo methods. One example of where we can find the P_D explicitly is the following. Assume that range distributed complex amplitudes of the desired target are i.i.d. zero-mean complex circular Gaussian r.v.'s with variance equal to σ_s^2. In this case we find that $[l_{NN}^{(v)}]^2$ (see Eq. (37)) is the sum of the squared magnitudes of $K - N + 1$ zero-mean complex circular Gaussian r.v.'s with variance equal to $\sigma_s^2(\mathrm{s}^{\mathrm{H}}R^{-1}\mathrm{s}) + 1$, [11]. Set $\sigma_{ss}^2 = \sigma_s^2(\mathrm{s}^{\mathrm{H}}R^{-1}\mathrm{s})$. We find that for the test statistic given by (38) that $2(\sigma_{ss}^2 + 1)l_{NN}^{(v)2}$ has the chi-square pdf of order $2(K - N + 1)$. Hence we can write the pdf of β under the H_1 hypothesis as

$$p_\beta(\beta \mid H_1) = \frac{1}{(K - N)!} \frac{1}{\tilde{l}_{NN}^2(\sigma_{ss}^2 + 1)} \left[\frac{\beta}{\tilde{l}_{NN}^2(\sigma_{ss}^2 + 1)}\right]^{K-N} \exp\left[-\frac{\beta}{\tilde{l}_{NN}^2(\sigma_{ss}^2 + 1)}\right], \tag{43}$$

and

$$P_D = \int_t^\infty p_\beta(\beta \mid H_1)\, d\beta. \tag{44}$$

IV. A CFAR DETECTOR FOR RANGE DISTRIBUTED TARGETS

From the test statistic form given by (38), it is seen that for a given block of data, Z, that the test statistic under H_0 is the product of a term that is a function of the noise environment, $\tilde{l}_{NN}^2$ and one that is not, $l_{NN}^{(v)2}$. We can create a CFAR detector by using the MGLRT test statistics over multiple blocks of data where each block of data is associated with non-overlapping range intervals. We assume that the

interferences from the various range intervals are independent but that any snap shot of data (or N-length vector) share the same $N \times N$ covariance matrix, R.

We define Z to be the primary $N \times K$ data matrix and Z_m, $m = 1, 2, \ldots, M_0$ to be M_0 secondary $N \times K$ data matrices where we assume the noise vector component of the columns of Z and $Z_m (m = 1, 2, \ldots, M_0)$ are i.i.d zero-mean complex circular Gaussian r.v.'s. Let β, β_1, β_2, $\ldots$ β_{M_0} be the GLRT test statistics for the primary and secondary data, respectively. We form the test statistic

$$\eta = \frac{\beta}{\sum_{m=1}^{M_0} \beta_m}. \tag{45}$$

If we assume that only the primary data can contain the desired signal, then (45) takes on the appearance of a cell-averaging CFAR. Based on the form of the individual MGLRT test statistics given by (28) it is seen that under H_0 that the common factor of $\tilde{l}_{NN}^2$ cancels from the numerator and denominator of (45) and that η is independent of the external noise environment. Hence (45) defines a CFAR detector for a range distributed target.

We point out that we could have also formed a CFAR detector under these assumptions using the single point target Kelly detector. For this we would perform single point adaptive target detection for each range cell of the primary data and use the secondary data (i.e., the KM_0 data vectors) as the secondary data vectors of the Kelly detector. After doing this, a 1 out of K detector could be used to determine if a target is present. A disadvantage of using this detector term versus the proposed CFAR test (Eq. (45)) is the computational cost: a detection statistic must be computed for each range cell using the single point Kelly detection scheme (this includes all possible ranges) whereas our proposed CFAR test processes the ranges in blocks of K ranges.

Since β, $\beta_m (m = 1, 2, \ldots, M_0)$ are all chi-square distributed under H_0, it is clear that η will have an F distribution under H_0. In fact, if we set

$$\eta = \frac{\beta / [2(K - N + 1)]}{\sum_{m=1}^{M_0} \beta_m / [2M_0(K - N + 1)]}, \tag{46}$$

then $p_\eta(\eta \mid H_0) = f(\eta;\ 2(K - N + 1),\ 2M_0(K - N + 1))$, where f is the pdf associated with the central F distribution with parameters $p = 2(K - N + 1)$, $q = 2M_0(K - N + 1)$. Thus

$$P_F = \int_t^\infty f(\eta;\ 2(K - N + 1),\ 2M_0(K - N + 1))d\eta. \tag{47}$$

We can obtain the P_D for the special case where the range distributed complex amplitudes of the desired target are i.i.d. zero-mean complex circular Gaussian r.v.'s with variance equal to σ_s^2. It will be found that

$$p_\eta(\eta \mid H_1) = \frac{1}{\sigma_{ss}^2 + 1} f\left[\frac{\eta}{\sigma_{ss}^2 + 1};\ 2(K - N + 1),\ 2M_0(K - N + 1)\right], \tag{48}$$

and

$$P_D = \frac{1}{\sigma_{ss}^2 + 1} \int_t^\infty f\left[\frac{\eta}{\sigma_{ss}^2 + 1};\ 2(K - N + 1),\ 2M_0(K - N + 1)\right] d\eta, \tag{49}$$

where $\sigma_{ss}^2 = \sigma_s^2(\mathbf{s}^H R^{-1}\mathbf{s})$.

V. MGLRT FOR SIGNALS ON A SUBSPACE

In this section, we derive the MGLRT for a range distributed target or targets when the desired steering vector, $\mathbf{s}$, is not known explicitly but is defined to be on a known subspace spanned by the columns of the $N \times N_s$ matrix, P_s where the columns of P_s are orthonormal and $2 \le N_s \le N$. Thus $P_s^H P_s = I_{N_s}$ and $\mathbf{s} = P_s\tilde{\mathbf{s}}$ where $\tilde{\mathbf{s}}$ is an $N_s \times 1$ vector of unknown complex values with the constraint $\mathbf{s}^H\mathbf{s} = \tilde{\mathbf{s}}^H\tilde{\mathbf{s}} = 1$.

The MGLRT that results can be derived in similar fashion as the one derived in Section II when $\mathbf{s}$ is assumed known. The test statistic is given by

$$\beta = \frac{\|ZZ^H\|}{\min\limits_{\substack{\tilde{\mathbf{s}},\mathbf{a} \\ \mathbf{s}=P_s\tilde{\mathbf{s}}}} \|(Z - \mathbf{s}\mathbf{a}^H)(Z - \mathbf{s}\mathbf{a}^H)^H\|_p} \tag{50}$$

with constraint $\tilde{\mathbf{s}}^H\tilde{\mathbf{s}} = 1$. From Section II, we know that

$$Z^H\mathbf{s} = \arg\min_{\mathbf{a}} \|(Z - \mathbf{s}\mathbf{a}^H)(Z - \mathbf{s}\mathbf{a}^H)^H\|_p. \tag{51}$$

Thus we desire to find

$$\tilde{\mathbf{s}} = \arg\min_{\tilde{\mathbf{s}}} \|(I_N - P_s\tilde{\mathbf{s}}\tilde{\mathbf{s}}^H P_s^H)ZZ^H(I_N - P_s\tilde{\mathbf{s}}\tilde{\mathbf{s}}^H P_s^H)^H\|_p. \tag{52}$$

For notational convenience, we define $\|A\|_{psv}$ to equal the product of the positive singular values of A and recognize that $\|A\|_{psv}^2 = \|AA^H\|_p$. Thus (52) can be rewritten as the equivalent problem.

$$\tilde{\mathbf{s}} = \arg\min_{\tilde{\mathbf{s}}} \|(I_N - P_s\tilde{\mathbf{s}}\tilde{\mathbf{s}}^H P_s^H)Z\|_{psv}. \tag{53}$$

Let L_z be the $N \times N$ LTM associated with the Q - R decomposition of Z. Then

$$\tilde{\mathbf{s}} = \arg\min_{\tilde{\mathbf{s}}} \|(I_N - P_s\tilde{\mathbf{s}}\tilde{\mathbf{s}}^H P_s^H)L_z\|_{psv}. \tag{54}$$

We set $\tilde{P}_s = [P_{\perp P_s} \vdots P_s]$ where $P_{\perp P_s}$ is an $N \times (N - N_s)$ matrix whose columns are the orthonormal vectors that span the null space of P_s. Thus $\tilde{P}_s$ is an $N \times N$ unitary matrix and $\tilde{P}_s^{\mathrm{H}}\tilde{P}_s = I_N$. Now

$$\begin{aligned}
\|(I_N - P_s\tilde{\mathrm{s}}\tilde{\mathrm{s}}^{\mathrm{H}}P_s^{\mathrm{H}})L_z\|_{psv} &= \|\tilde{P}_s\tilde{P}_s^{\mathrm{H}}(I_N - P_s\tilde{\mathrm{s}}\tilde{\mathrm{s}}^{\mathrm{H}}P_s^{\mathrm{H}})\tilde{P}_s\tilde{P}_s^{\mathrm{H}}L_z\|_{psv} \\
&= \|(I_N - \tilde{P}_s^{\mathrm{H}}P_s\tilde{\mathrm{s}}\tilde{\mathrm{s}}^{\mathrm{H}}P_s^{\mathrm{H}}\tilde{P}_s)\tilde{P}_s^{\mathrm{H}}L_z\|_{psv} \\
&= \left\| \left[I_N - \begin{bmatrix} 0_{N-N_s} \\ \tilde{\mathrm{s}} \end{bmatrix} \begin{bmatrix} 0_{N-N_s} \\ \tilde{\mathrm{s}} \end{bmatrix}^{\mathrm{H}} \right] L_{sz} \right\|_{psv},
\end{aligned} \tag{55}$$

where L_{sz} is the LTM associated with the Q - R decomposition of $\tilde{P}_s^{\mathrm{H}}L_z$. Also

$$\left[I_N - \begin{bmatrix} \mathbf{0}_{N-N_s} \\ \tilde{\mathrm{s}} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{N-N_s} \\ \tilde{\mathrm{s}} \end{bmatrix}^{\mathrm{H}} \right] L_{sz} = \begin{bmatrix} L_{sz}^{(11)} & 0_{N-N_s,N_s} \\ L_{sz}^{(21)} & (I_{N_s} - \tilde{\mathrm{s}}\tilde{\mathrm{s}}^{\mathrm{H}})L_{sz}^{(22)} \end{bmatrix}, \tag{56}$$

where we have written L_{sz} as

$$L_{sz} = \begin{bmatrix} L_{sz}^{(11)} & 0_{N-N_s,N_s} \\ L_{sz}^{(21)} & L_{sz}^{(22)} \end{bmatrix}, \tag{57}$$

$L_{sz}^{(11)}$ is an $(N - N_s) \times (N - N_s)$ matrix, $L_{sz}^{(21)}$ is a $N_s \times (N - N_s)$ matrix, and $L_{sz}^{(22)}$ is a $N_s \times N_s$ matrix.

It is seen that the product of the positive singular values of the matrix given by (56) is equal to the product of the diagonal elements of $L_{sz}^{(11)}$ (these elements are positive with probability one) and the product of the positive singular values of $(I_{N_s} - \tilde{\mathrm{s}}\tilde{\mathrm{s}}^{\mathrm{H}})L_{sz}^{(22)}$. Hence the minimization of the far RHS of (55) is equivalent to

$$\tilde{\mathrm{s}} = \arg \min_{\tilde{\mathrm{s}}} \|(I_{N_s} - \tilde{\mathrm{s}}\tilde{\mathrm{s}}^{\mathrm{H}})L_{sz}^{(22)}\|_{psv}. \tag{58}$$

Write the SVD of $L_{sz}^{(22)}$ as

$$L_{sz}^{(22)} = P_{sz}^{(22)}\Lambda_{sz}^{(22)}Q_{sz}^{(22)\mathrm{H}}, \tag{59}$$

where $P_{sz}^{(22)}$ and $Q_{sz}^{(22)}$ are $N_s \times N_s$ unitary matrices and $\Lambda_{sz}^{(22)}$ is an $N_s \times N_s$ diagonal matrix whose elements are the N singular values of $L_{sz}^{(22)}$. It is known [1] that (58) is minimized when $\tilde{\mathrm{s}}$ equals the column of $P_{sz}^{(22)}$ associated with the largest singular value of $L_{sz}^{(22)}$. Call this solution, $\tilde{\mathrm{s}}_{\min}$.

We can rewrite the MGLRT test statistic as

$$\beta = \frac{\|L_{sz}\|}{\left\| \begin{bmatrix} L_{sz}^{(11)} & 0_{N-N_s,N} \\ L_{sz}^{(21)} & (I_{N_s} - \tilde{s}_{\min}\tilde{s}_{\min}^{H})L_{sz}^{(22)} \end{bmatrix} \right\|_{psv}}. \tag{60}$$

Let σ_n, $n = 1, \ldots, N_s$ be the singular values of $L_{sz}^{(22)}$ where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{N_s}$ and set $L_{sz} = (l_{mn}^{(sz)})$. Then

$$\beta = \frac{\prod_{m=N-N_s+1}^{N} l_{mm}^{(sz)}}{\prod_{m=2}^{N_s} \sigma_m}. \tag{61}$$

The procedure for finding the MGLRT test statistic of a range distributed target or targets with s on a subspace is as follows.

1. Set $\tilde{P}_s = [P_{\perp s}, P_s]$

2. Find the $N \times N$ LTM, L_z, associated with Q - R decomposition of Z.

3. Find the $N \times N$ LTM, L_{sz}, associated with the Q - R decomposition of $\tilde{P}_s^{H} L_z$.

4. Find the singular values of $L_{sz}^{(22)}$, defined by (57).

5. Form the test statistic as given by (61).

We point out that the distribution of β under H_0 for $N_s \geq 2$ is a function of the covariance matrix and 2-thus the detector is not CFAR. Furthermore, we know of no methodology at this time similar for the $N_s = 1$ case presented in Section III which results in a detector which is CFAR for $N_s \geq 2$.

VI. RESULTS

In this section, we present some simulation results demonstrating the effectiveness of the derived MGLRT in detecting a range distributed target or targets. We compare the performance of the new detector with that of what we call a M out of K Kelly detector which we now describe. The M out of K Kelly detector consists of two levels of detection. The first level is a single point scatterer Kelly detector [2]: a given range cell in the $N \times K$ data block is considered the candidate or primary data. This primary data is an $N \times 1$ vector. The other $K - 1$ range cell data are used as the secondary data which are assumed in the derivation of the Kelly detector not to contain any of the desired signal. This secondary data is represented as a $N \times (K - 1)$ secondary data matrix. The standard Kelly detection statistic is computed for each candidate range cell (a total of K) where the candidate or primary range cell is varied over all possible K range cells. The output from each of the K hypothesis tests is either a zero or one.

The second level of detection consists of counting the number of detections out of K from the first level detector and comparing this integer, m, to the integer threshold, M, of the second level detector. If $m \geq M$, a detection of a target in the $N \times K$ data block is declared. A disadvantage of the M out K Kelly detector is that in practice we must set $M = 1$. Otherwise for scattering scenarios where the number of scatterers in the K range cells is less than M, the probability of detecting the target on targets would be almost zero. For a more complete description and analysis of the M out of K Kelly detector see [12].

We present results for a single representative case which will illustrate the performance advantage that the MGLRT of a distributed target(s) has over the M out of K Kelly detector when there are multiple scatterers (or targets) in range. For our example we assume the inputs to the adaptive processor are the received returns for an N element linear array (half-wavelength spacing) with no time taps. We use the narrowband assumption so that the desired signal vector and jamming vectors can be represented as $(1/\sqrt{N})\ (1\ e^{j\psi}\ e^{j2\psi}\ \ldots\ e^{j(N-1)\psi})^{\mathrm{T}}$. Assume a single jammer is the only source of external interference so that the $N \times N$ covariance matrix is given by

$$R = \sigma_J^2 \mathbf{J}\mathbf{J}^{\mathrm{H}} + I_N \tag{62}$$

where σ_J^2 is the power of the jammer referenced to the internal noise power of each antenna element receiver, $\mathbf{J} = (1/\sqrt{N})\ (1\ e^{j\psi_J}\ e^{j2\psi_J}\ \ldots\ e^{j(N-1)\psi_J})^{\mathrm{T}}$, ψ_J is the array phasing of the jammer, and I_N is the $N \times N$ identity matrix which represents the covariance matrix of the internal noise contribution of each antenna element receiver.

For the simulation, we choose $\mathbf{s} = (1/\sqrt{N})\ (1\ 1\ \ldots\ 1)^{\mathrm{T}}$, the number of samples per receive channel, $K = 15$, the number of sensor inputs, $N = 8$, and the probability of false alarm $= 10^{-4}$. In Fig. 1, the probability of detection (P_D) performance of a single point scatterer Kelly detector (not the M out K Kelly detector) and the MGLRT of a distributed target(s) are plotted versus the input signal-to-internal noise power ratio for the case when there is only one scatterer present in the $N \times K$ data block and the jamming power is zero. It is seen that the MGLRT detector of distributed targets slightly out-performs the single point scatterer Kelly detector. In Fig. 2, the setup is the same as Fig. 1 except now we have two distinct (different range cells) equi-powered scatterers. The sum of the SNR's of each scatterer equals the input SNR. We see clearly that the single point scatterer Kelly detector is significantly effected by the presence of a desired signal (one of the scatterers) in the secondary data. In fact, as the power of the second scatterer increases, the SNR losses become much greater (this was also shown in [7]). We show the single point scatterer Kelly performance in order to demonstrate that the M out of K Kelly detector could perform poorly in a multiple scatterer scenario because of the poor detection performance in the first level of detection. We now present results for the M out of K Kelly detector versus the GLRT of distributed target(s) which show this.

In Fig. 3, we compare the P_D performance of a 1 out of K Kelly detector with the MGLRT detector of a distributed target(s) of a single scatterer when the interference scenario consists of a single 30 dB jammer (with angle $.6\pi$) and internal noise. We see that the new detector slightly out performs the 1 out of K Kelly detector. The setup is the same for Fig. 4 as for Fig. 3 except two distinct equi-powered scatterers are present. We compare the 1 out of K and 2 out of K Kelly detectors with the GLRT of distributed target(s). The 2 out of K Kelly detector is superior to the 1 out of K Kelly detector and the new detector is clearly superior to the 2 out of K Kelly detector.

We also point out that the new detector's detection performance is about the same for when there are one or two scatterers present. Also the detection performance is about the same for the example when jamming is present or not. For the jamming scenario, the N'th diagonal element of the Cholesky factor of R is approximately equal to one. Hence based on the results of Section 3A, the desired thresholds for a given P_F for the two noise scenarios are approximately equal.

What is lost by using the MGLRT of a distributed target or target(s) instead of a single point scatterer Kelly detector? Clearly, the range resolution is degraded. With the new detector one only knows that there is a detection among the K range cells. If we are looking for multiple targets, we obviously desire to know how many are present within the K range cells. This information might be obtainable if after a detector indication by the new detector, the single point scatterer Kelly is applied to each range cell in order to see if individual range cells are detected thus restoring to some degree the original range resolution capability of the receiver.

Finally, the new detection scheme could be used to reduce the processing load of a range cell-by range cell single point scatterer adaptive detector; the new detector would be used as a first level detector indicating whether a target or targets are present in a set of range cells. A second level of detection using the single point scatterer adaptive detector would be invoked if a detection recurred in the first level of detection.

VII. A COMMENT ON THE MGLRT FOR A SINGLE POINT TARGET

Almost all of the adaptive detection literature has been concerned with detection of a single point target; i.e., the target is modeled as a single point scatterer that is completely contained within a single range cell. The data that can contain the desired target returns is called the primary data vector. Returns from other range cells are used to form secondary data vectors. The effects of contaminated (by desired signals) secondary data on performance of the Kelly detector and the AMF detector were examined in [7] for the case where the target is modeled as a point scatterer. It was shown that significant degradation in detection performance can occur when multiple targets of equal scattering strength are contained within the same block of primary and secondary data. The two detectors, the Kelly and AMF, do not take into account the possibility of contamination of the secondary data by other target returns. The search for an effective adaptive detector against secondary data that has been contaminated has been a long standing unsolved problem. In this section we derive the MGLRT for this problem. We assume that the steering vectors associated with the contamination are identical. In practice, this is often the case, since in a given range interval which is used to construct the primary and secondary data, multiple targets are usually transiting at the same speed and direction (such as a formation of air targets).

Interestingly, it will be found that the derived MGLRT does not use the secondary data at all. Hence the derived GLRT is not useful as an adaptive detector since one normally uses the secondary data to estimate the unknown covariance matrix of the received interference. As a result, the MGLRT formalism for solving this problem is not the solution and the search for an effective adaptive detector against secondary data is still an open problem.

We now briefly outline the derivation of the MGLRT for single point target and show that the test statistic is independent of the secondary data set. We collect data from N sensors and form a primary data vector associated with a given range. We also collect data from the N sensors at $K - 1$ other range cells and form $K - 1$ secondary data vectors where other targets returns may be present. The hypotheses are

$$
\begin{aligned}
H_0: \quad & \mathbf{z}_0 = \mathbf{x}_0 \\
& \mathbf{z}_k = \mathbf{x}_k + a_k^* \mathbf{s} \quad k = 1, 2, \ldots, K-1 \\
H_1: \quad & \mathbf{z}_0 = \mathbf{x}_0 + a_0^* \mathbf{s} \\
& \mathbf{z}_k = \mathbf{x}_k + a_k^* \mathbf{s} \quad k = 1, 2, \ldots, K-1,
\end{aligned}
\tag{63}
$$

where $\mathbf{z}_0$ is the primary data, $\mathbf{z}_k (k = 1, 2, \ldots, K)$ are the secondary data vectors, $\mathbf{x}_0$ and $\mathbf{x}_k (k = 1, 2, \ldots, K)$ are the interferences and $a_k^* (k = 0, 1, \ldots, K-1)$ are the complex signal amplitude associated with the k'th range cell. Similarly to the development of Section II, (with the same assumptions on the interference), we can derive the MGLRT statistic as

$$
\beta = \frac{\min\limits_{\alpha_0} \left\| (Z_s - \mathbf{s}\alpha_0^{\mathrm{H}})(Z_s - \mathbf{s}\alpha_0^{\mathrm{H}})^{\mathrm{H}} + \mathbf{z}_0\mathbf{z}_0^{\mathrm{H}} \right\|_p}{\min\limits_{\alpha_1} \left\| (Z - \mathbf{s}\alpha_1^{\mathrm{H}})(Z - \mathbf{s}\alpha_1^{\mathrm{H}})^{\mathrm{H}} \right\|_p}
\tag{64}
$$

where $Z_s = (\mathbf{z}_1 \mathbf{z}_2 \ldots \mathbf{z}_{K-1})$ is the $N \times (K-1)$ matrix of secondary data, $Z = (\mathbf{z}_0 \vdots Z_s)$ is the $N \times K$ total data matrix, $\boldsymbol{\alpha}_0 = (a_1, a_2, \ldots, a_{K-1})^{\mathrm{T}}$, and $\boldsymbol{\alpha}_1 = (a_0, a_1, \ldots, a_{K-1})^{\mathrm{T}}$. Let Q_{z_0} be a unitary matrix such that $Q_{z_0}\mathbf{z}_0 = |\mathbf{z}_0|\mathbf{1}_0$ where $|\mathbf{z}_0| = (\mathbf{z}_0^{\mathrm{H}}\mathbf{z}_0)^{1/2}$. Define $\tilde{Z}_s = Q_{z_0}Z_s$, $\tilde{Z} = Q_{z_0}Z = (|\mathbf{z}_0|\mathbf{1}_0 \vdots \tilde{Z}_s)$, and $\mathbf{s}_0 = Q_{z_0}\mathbf{s}$. Thus (64) can be rewritten as

$$
\beta = \frac{\min\limits_{\alpha_0} \left\| (\tilde{Z}_s - \mathbf{s}_0\alpha_0^{\mathrm{H}})(\tilde{Z}_s - \mathbf{s}_0\alpha_0^{\mathrm{H}})^{\mathrm{H}} + |\mathbf{z}_0|^2\mathbf{1}_0\mathbf{1}_0^{\mathrm{T}} \right\|_p}{\min\limits_{\alpha_1} \left\| (\tilde{Z} - \mathbf{s}_0\alpha_1^{\mathrm{H}})(\tilde{Z} - \mathbf{s}_0\alpha_1^{\mathrm{H}}) \right\|_p}.
\tag{65}
$$

The solution for α_1 was derived in Section II:

$$
\alpha_1^{\mathrm{H}} = \mathbf{s}_0^{\mathrm{H}}\tilde{Z}.
\tag{66}
$$

The solution for α_0 in the numerator of (65) is now considered. The numerator can be rewritten as

$$
\mathrm{NUM} = \min_{\alpha} \left\| (\tilde{Z} - \mathbf{s}_0\alpha^{\mathrm{H}})(\tilde{Z} - \mathbf{s}_0\alpha^{\mathrm{H}})^{\mathrm{H}} \right\|_p,
\tag{67}
$$

where $\alpha = (0 \vdots \alpha_0)^{\mathrm{T}}$. It is straightforward to show that there exists an α of this form such that $\tilde{Z} - \mathbf{s}_0\alpha^{\mathrm{H}}$ has rank $N-1$. Hence, we look for solutions for α on the space where $\tilde{Z} - \mathbf{s}_0\alpha^{\mathrm{H}}$ has rank $N-1$.

We will need the following lemma in order to proceed.

Lemma 2: If $\tilde{Z} - \mathbf{s}_0\alpha^{\mathrm{H}}$ has rank $N-1$ and $\tilde{Z}$ has rank N, then there exists an N-length vector $\tilde{\mathbf{s}}_0$ such that

$$
(I_N - \mathbf{s}_0\tilde{\mathbf{s}}_0^{\mathrm{H}})\tilde{Z} = \tilde{Z} - \mathbf{s}_0\alpha^{\mathrm{H}},
\tag{68}
$$

with $\tilde{s}_0^H s_0 = 1$.

proof: See Appendix B □

Using Lemma 2, we see that finding α is equivalent to finding

$$\tilde{s}_0 = \arg \min_{\tilde{s}_0} \| (I_N - s_0 \tilde{s}_0^H) \tilde{Z} \|_{psv}. \tag{69}$$

Set $\tilde{s}_0 = (\tilde{s}_{01}\ \tilde{s}_{02}\ \tilde{s}_{03} \ldots \tilde{s}_{0N})^T$. Because $\alpha^H = \tilde{s}_0^H \tilde{Z} = \tilde{s}_0^H (|\mathbf{z}_0| \mathbf{1}_0 \ \vdots\ \tilde{Z}_s)$ and the first element of α is zero, it follows that $\tilde{s}_{01}^* \ |\mathbf{z}_0| = 0$. Since $|\mathbf{z}_0| \neq 0$ with probability one, the first element of $\tilde{s}_0$ must be zero or $\tilde{s}_{01} = 0$ which is a constraint on the minimization problem posed by (69) along with the constraint $\tilde{s}_0^H s_0 = 1$. Using a methodology similar to that of finding the solution for (23), we can show that (69) is equivalent to solving the following problem

$$\min_{\tilde{s}} \tilde{s}_0^H \tilde{s}_0 \quad , \text{ constraints } \tilde{s}_{01} = 0 \quad , \tilde{s}^H s_0 = 1. \tag{70}$$

It is straightforward to show that the optimal solution is given by

$$\tilde{s}_{0n} = \frac{s_{0n}}{\sum_{n=2}^{N} |s_{0n}|^2} \quad , n = 2, 3, \ldots, N. \tag{71}$$

This solution is substituted into the following equivalent GLRT test statistic

$$\beta = \frac{\| (I_N - s_0 \tilde{s}_0^H) \tilde{Z} \tilde{Z}^H (I - s_0 \tilde{s}_0^H) \|_p}{\| (I_N - s_0 s_0^H) \tilde{Z} \tilde{Z}^H (I - s_0 s_0^H) \|_p}. \tag{72}$$

It is straightforward to show that β reduces to the following simple form:

$$\beta = \tilde{s}_0^H \tilde{s}_0. \tag{73}$$

The elements of $\tilde{s}_0$ are derived from the elements of s_0 which equals $Q_{z_0} s$. Thus $\tilde{s}_0$ is only a function of the primary data $\mathbf{z}_0$ and *not* a function of any of the secondary data, Z_s. Hence this detector is not useful since it essentially throws away the data that would be used to estimate R.

Recall for the adaptive detector of a range distributed target that under H_1 (signal plus noise) that the MGLRT projected the data matrix, Z, onto the null space of the desired signal vector. Hence, a hueristic adaptive detection scheme for a point target with contaminated secondary data would be to project the secondary data matrix onto the null space of s under the H_1 hypothesis. The resultant heuristic detector has the form

$$\beta = \frac{\| (I_N - \mathbf{s}\mathbf{s}^H) Z_s Z_s^H (I_N - \mathbf{s}\mathbf{s}^H) + \mathbf{z}_0 \mathbf{z}_0^H \|}{\| (I_N - \mathbf{s}\mathbf{s}^H) Z Z^H (I_N - \mathbf{s}\mathbf{s}^H) \|_p} . \tag{74}$$

This also turns out not to be an effective detector. In fact, the detection statistic is approximately equal to $|\mathbf{s}^H \mathbf{z}_0|^2$ where again the secondary data is not used.

VIII. CONCLUSIONS

A Modified Generalized Likelihood Ratio Test (MGLRT) for the adaptive detection of a target or targets that are distributed in range was derived. The unknown parameters associated with the hypothesis test are the complex amplitudes in range of desired target and the unknown covariance matrix of the additive interference which is assumed to be characterized as complex zero-mean correlated Gaussian random variables. The target's or targets' complex amplitudes are assumed to be distributed across the entire input data block (sensor $\times$ range). Results on probabilities of false alarm and detection were derived. The MGLRT statistic was found to be dependent on the external noise environment and thus does not have the desirable quality of being a CFAR detector.

Methods were presented which would make it almost CFAR (upper-bounded P_F) if only one input data block were used and CFAR if multiple input data blocks were used. Some simulation results were presented. It was shown that the derived MGLRT of range distributed targets is much more effective in detecting targets distributed in range than a M out of K detector cascaded with a single point target Kelly detector. In fact a solution to abate the deleterious effect of the secondary data being contaminated by desired signal is to detect all the target scatterers within the data block using the MGLRT for range distributed targets (such as a formation of targets).

Finally, the MGLRT associated with detecting a single point target with signal contaminated secondary data was also derived. It was shown that surprisingly, this detector is ineffective in solving this problem.

Appendix A
PROOF OF LEMMA A1

Lemma A1: Let A be a $K \times N$ matrix, and U_1, U_2 be two $N \times M$ matrices such that

1. $Sp(U_1) = Sp(U_2)$ where Sp denotes Span of,
2. $M \leq N$,
3. $U_i^{\mathrm{H}} U_i = I_M$, $i = 1, 2$,

then

$$\|AU_1\|_{psv} = \|AU_2\|_{psv}$$

proof: Since $Sp(U_1) = Sp(U_2)$, we can find an $M \times M$ matrix B such that $U_1 = U_2 B$. Now

$$U_1^{\mathrm{H}} U_1 = B^{\mathrm{H}} U_2^{\mathrm{H}} U_2 B = B^{\mathrm{H}} B = I_M. \tag{A1}$$

Thus B is unitary and

$$\|AU_1\|_{psv} = \|AU_2 B\|_{psv} = \|AU_2\|_{psv}. \tag{A2}$$

□

Appendix B
PROOF OF LEMMA 2

Before proofing Lemma 2, we prove the following.

Lemma B1: Let A be a $N \times N$ nonsingular matrix, $\mathbf{s}$ an $N \times 1$ vector, and $\mathbf{b}$ a $K \times 1$ vector with $K \geq N$. Define $B \equiv [A : 0_{N,K-N}] - \mathbf{s}\mathbf{b}^{\mathrm{H}}$. If B has rank $N - 1$, then the $N + 1$st through K'th elements of $\mathbf{b}$ are zero.

proof: The matrix BB^{H} also has rank $N - 1$. We can write

$$BB^{\mathrm{H}} = (A - \mathbf{s}\mathbf{b}_0^{\mathrm{H}})(A - \mathbf{s}\mathbf{b}_0^{\mathrm{H}}) + \alpha\mathbf{s}\mathbf{s}^{\mathrm{H}}, \tag{B1}$$

where $\mathbf{b}_0$ is a N-length vector equal to the first N elements of $\mathbf{b}$ and α equals the sum of the magnitude squares of the last $K - N$ elements of $\mathbf{b}$. Now if B has rank $N - 1$ there exists an N-length vector, $\mathbf{u}$, such that $\mathbf{u}^{\mathrm{H}}B = 0$. Thus

$$\mathbf{u}^{\mathrm{H}}BB^{\mathrm{H}}\mathbf{u} = \mathbf{u}^{\mathrm{H}}(A - \mathbf{s}\mathbf{b}_0^{\mathrm{H}})(A - \mathbf{s}\mathbf{b}_0^{\mathrm{H}})^{\mathrm{H}}\mathbf{u} + \alpha\,|\mathbf{u}^{\mathrm{H}}\mathbf{s}|^2 = 0. \tag{B2}$$

Since $(A - \mathbf{s}\mathbf{b}_0^{\mathrm{H}})(A - \mathbf{s}\mathbf{b}_0^{\mathrm{H}})^{\mathrm{H}}$ is a positive semi-definite matrix, (B2) implies if $\alpha \neq 0$ that $\mathbf{u}^{\mathrm{H}}\mathbf{s} = 0$. In this case it follows that $\mathbf{u}^{\mathrm{H}}AA^{\mathrm{H}}\mathbf{u} = 0$ which contradicts the assumption that A is nonsingular. Therefore $\alpha = 0$ and Lemma B1 follows □

We now prove Lemma 2. Let $Z = [L : 0_{N,K-N}]Q$ be the Q - R decomposition of Z where L is an $N \times N$ LTM and Q is a $K \times K$ unitary matrix. Now if $Z - \mathbf{s}\mathbf{a}^{\mathrm{H}}$ has rank $N - 1$ then $[L : 0_{N,K-N}] - \mathbf{s}\mathbf{a}_0^{\mathrm{H}}$ has rank $N - 1$ where $\mathbf{a}_0^{\mathrm{H}} = \mathbf{a}^{\mathrm{H}}Q^{\mathrm{H}}$. From Lemma B1, the last $K - N$ elements of $\mathbf{a}_0$ must be zero. Thus there exists a $\tilde{\mathbf{s}}$ such that

$$\mathbf{a}_{0r}^{\mathrm{H}} = \tilde{\mathbf{s}}^{\mathrm{H}}L, \tag{B3}$$

where $\mathbf{a}_{0r}$ are the first N elements of $\mathbf{a}_0$ and L is nonsingular. Hence

$$\mathbf{a}^{\mathrm{H}} = \tilde{\mathbf{s}}^{\mathrm{H}}[L : 0]Q = \tilde{\mathbf{s}}^{\mathrm{H}}Z, \tag{B4}$$

and the lemma follows. □

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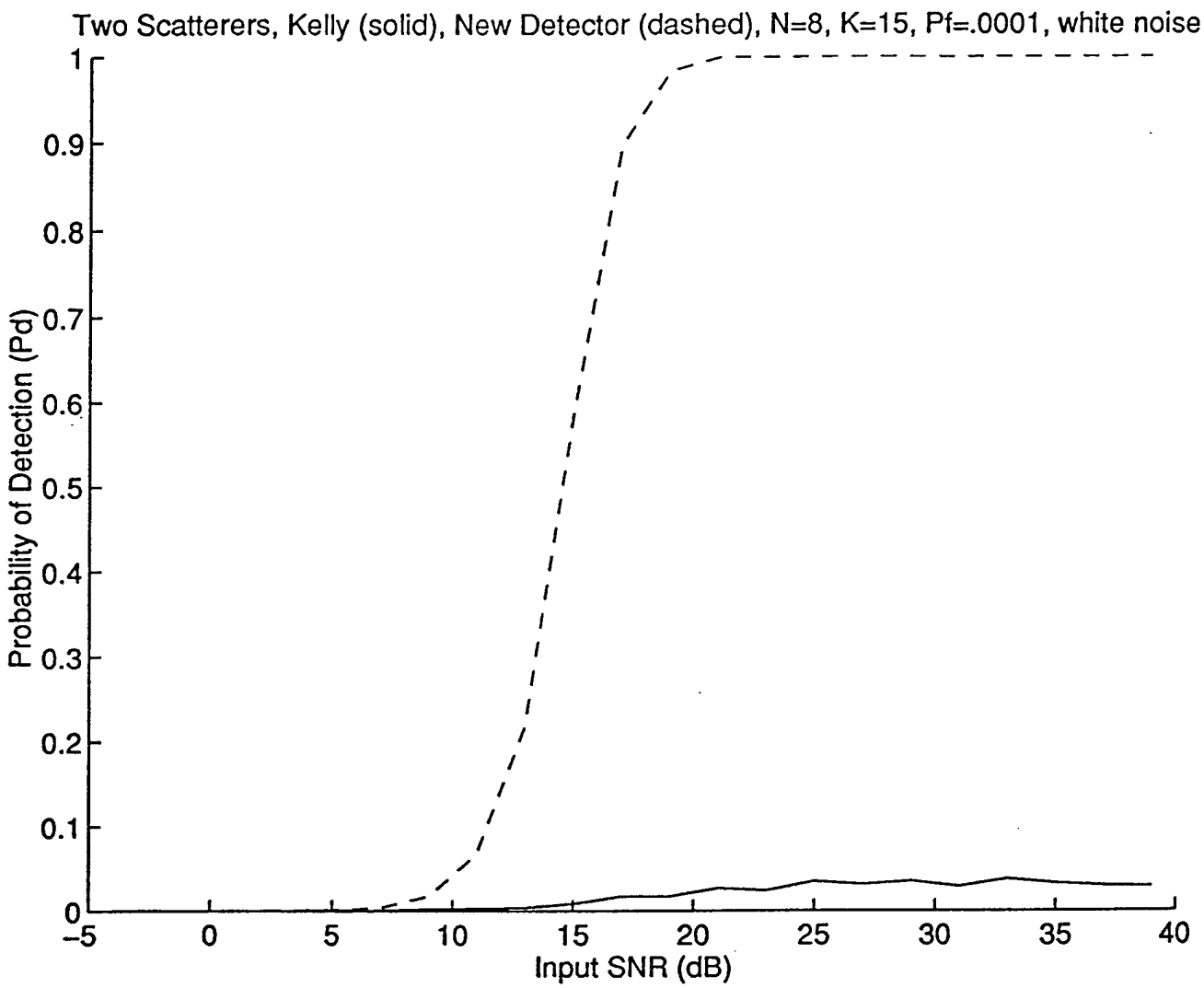

Fig. 1 — Performance comparison of a single point scatterer Kelly detector with the GLRT detector of a distributed target(s): one scatterer, $N = 8$, $K = 15$, $P_F = 10^{-2}$, white noise interference.

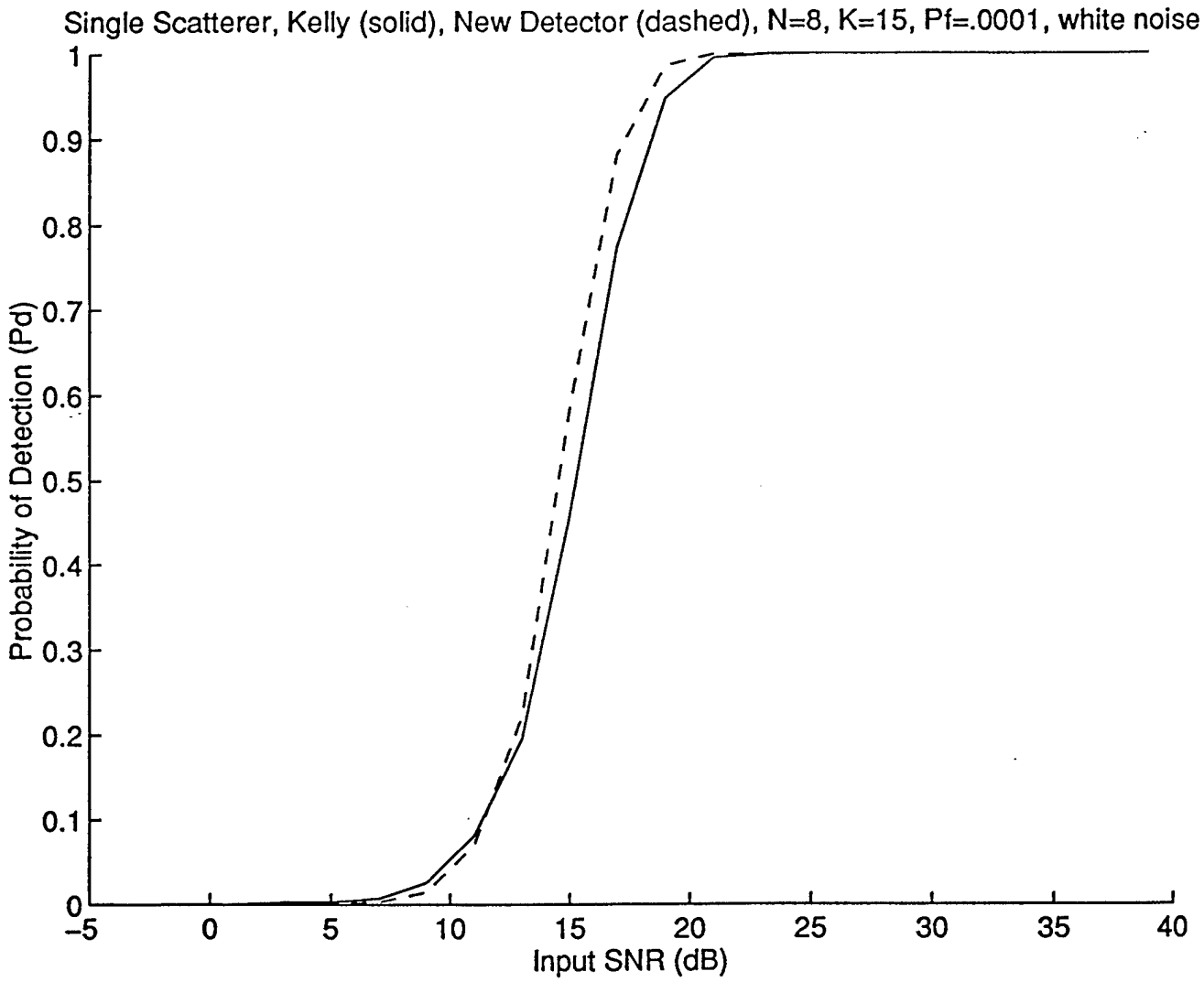

Fig. 2 — Performance comparison of a single point scatterer Kelly detector with the GLRT detector of a distributed target(s): two scatterer, $N = 8$, $K = 15$, $P_F = 10^{-2}$, white noise interference.

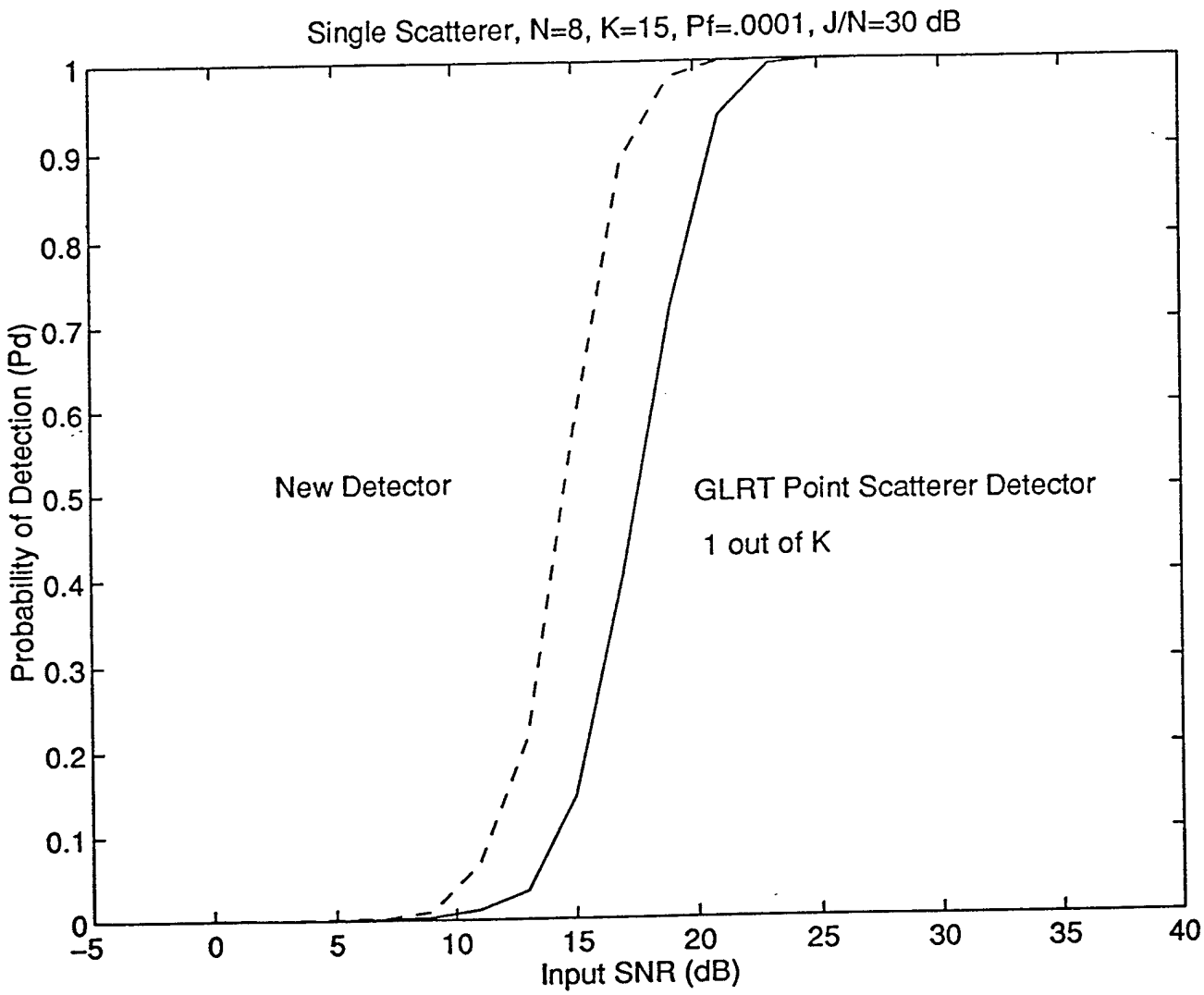

Fig. 3 — Performance comparison of a 1 out of K Kelly detector with the GLRT detector of a distributed target(s): one scatterer, $N = 8$, $K = 15$, $P_F = 10^{-2}$, interference is a 30 dB jammer plus white noise.

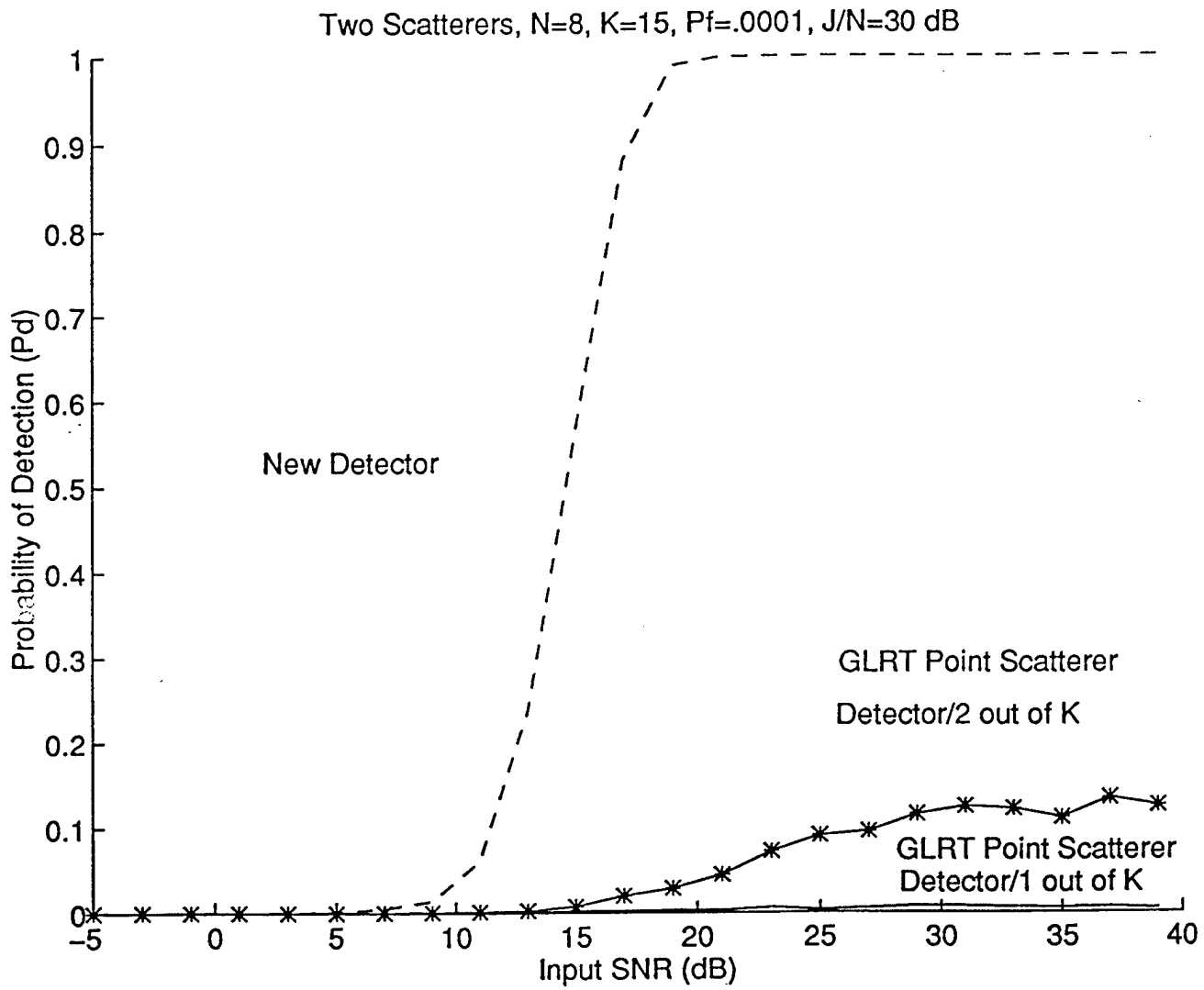

Fig. 4 — Performance comparison of a 1 out of K Kelly detector, 2 out of K Kelly detector with the GLRT detector of a distributed target(s): two scatterers, $N = 8$, $K = 15$, $P_F = 10^{-2}$, interference is a 30 dB jammer plus white noise.