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Pritchard's Approximation in Array Modeling

by

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Pritchard's Approximation in Array Modeling

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Abstract

An investigation into the applicability and accuracy of Pritchard's approximation for closely packed transducer arrays is undertaken. A new, "modal" Pritchard approximation is developed, based upon normal modes of the acoustic medium, and is tested for arrays of acoustically hard spheres to ascertain its accuracy in determining the mutual acoustic radiation impedance between array elements. For $ka \approx 1$, it is found that the modal Pritchard approximation works quite well in approximating the mutual radiation impedance of a two element array, even for relatively close spacing, but for arrays of three or more scatterers in close proximity, the approximation may have relatively large errors. The effect of neglecting inter-element scattering is analyzed for the monopole to monopole scattering of various configurations of a three element array and a sixteen element double line array.

1 Theoretical Background

The so-called Pritchard [9] and "generalized Pritchard" methods [2] are popular techniques used to approximate the acoustic interactions between elements in an underwater acoustic array. Origins of the method can be traced back to an asymptotic analysis for small spherical scatterers in an acoustic medium performed by Karnovsky [7], who in turn references (as does Pritchard in his paper) earlier work done (including baffled pistons) by Rayleigh in 1903 [10].

In Pritchard's paper, the approximation is based upon an analytical treatment of acoustic interactions in an array of baffled circular pistons loaded by a semi-infinite fluid medium. Extensions to more general arrays are possible however if the following three criteria are met:

- The mutual impedance between any two elements of the array can be considered without regard to other array elements in the construction of a global impedance matrix for the entire array.
- The dimensions of an array element (in our case, the sphere radius a) is small relative to the acoustic wavelength ($ka \ll 1$, where $k = \omega/c$ is the wavenumber of the time harmonic problem and c is the acoustic wave speed).
- The ratio of array element dimension to inter-element distances (d) is small ($\frac{a}{d} \ll 1$).

While it is true that if the second and third conditions are met, so must the first, the method is often applied when the latter two conditions are not strictly satisfied, with very good results. This is most likely due to the first condition being met. In Pritchard's paper the first of these approximations is exactly satisfied, since his array elements are coplanar with the baffle. When the assigned velocity of all but one of the pistons vanishes, the remainder of the array becomes essentially part of the baffle, and the exact mutual radiation impedance between it and any of the remaining array elements can be found analytically without regard to the other array elements, because there is no scattering.

The Pritchard approximation for the mutual radiation impedance between two array elements (for example pistons 1 and 2) can be formulated mathematically as follows. A series representation of the mutual acoustic radiation impedance between the two pistons to leading order for small ka and small a/d is:

$$Z_{12} \approx R_{22}\left[\frac{\sin kd}{kd} + i\frac{\cos kd}{kd}\right] = R_{22}h_0^{(2)}(kd)$$

in which Z_{12} is the mutual radiation impedance on piston 1 due to a radiated pressure from piston 2, $h_0^{(2)}$ is the zeroth order spherical Hankel function of the second kind (**an $e^{i\omega t}$ time dependence is used throughout**), and R_{22} is the real (resistive) part of the self radiation impedance of piston 2. Pritchard demonstrated that the above approximation is quite accurate in comparisons with exact calculations over a range of center to center distances between pistons in the baffled array.

Benthien [2] reported on extensions of Pritchard's "method" to other types of arrays. In particular, Benthien found excellent agreement in comparing a Pritchard approximation to experimental as well as numerically determined results based upon the boundary element technique CHIEF[13], for a three element array of flextensional transducers. In Benthien's methodology, the mutual radiation impedances between array elements are found by numerically determining the free field self radiation impedance of a single array element (Z_{self}). This uses the code CHIEF. The resulting matrix is then used to obtain the mutual radiation impedance of that array element on a second (identical) array element by multiplying the self radiation resistance matrix (R_{self}) by $h_0^{(2)}(kd)$ where d is the distance between the two array elements.

A more complicated and perhaps more accurate calculation would apply a Pritchard type approximation on a *finite element* by *finite element* basis rather than from *array element* to *array element*, but is not done for two reasons. One reason is that the calculations involved in finding the mutual radiation impedance between array elements for a large array would be greatly increased. A second reason is based upon the underlying assumption of the Pritchard approximation itself, namely that $a \ll d$ where a is a characteristic length of the array element. This implies that finite element to finite element spacing between two array elements whose center to center spacing is d should be very nearly d as $d \pm a \approx d$.

In the Benthien formulation, the mutual radiation impedance matrix so constructed is applied to the velocity degrees of freedom (one for each surface finite element facet) for each element of the array to produce a pressure forcing function on each of the array element fluid loaded surfaces. One can then solve for the unknown values of the velocities by matrix inversion techniques. Following Benthien, one could also normalize the matrix equations utilizing the *in vacuo* eigenvectors of the mass/stiffness finite element matrices of the array element prior to solving the system.

The work of Blottman et al [3] should be mentioned. These authors determined the mutual radiation impedance between a two element array of transducers numerically wherein the *in vacuo* normal modes of the transducers were used as degrees of freedom of the system. In this regard, Blottman's work resembles that of Benthien's "normalization". Unlike the work of Benthien, it is assumed that inter-mode (cross-) coupling of the self radiation impedance for a single transducer is negligible (i.e. *in vacuo* normal modes don't couple when heavy fluid loading is applied). *Modal* mutual impedances between array elements are found by employing a boundary element technique (EQI/ATILA)[4] which (unlike a Pritchard approximation) numerically accounts for scattering effects to all orders. While the work by Blottman et al would appear to have little to do with the Pritchard approximation, the idea of applying a "modal" mutual radiation impedance is of interest. In particular, this report considers the possibility of a modal Pritchard approximation which might improve upon the Pritchard approximations currently in use.

2 The mutual radiation impedance matrix and the Pritchard approximation

Given a fluid loaded array of M elements/transducers, and that the total pressure on array element n is represented by P_n, its value can be formally written in terms of the mutual and self radiation impedances, and the prescribed normal surface velocities on each of the array elements (V_m for array element m), by the sum

$$P_n = \sum_{m=1}^{M} Z_{nm} V_m$$

These pressures can in turn be used in conjunction with surface velocities to calculate pressures at arbitrary field points in the acoustic medium.

Crucial to Pritchard's method is an accurate representation of the self radiation impedance of an array element since it is the basis for determination of all mutual radiation impedances imposed by that element on other elements in the array (it is used to find the Z_{nm} along that row of the global impedance matrix). If the self radiation impedance is approximated by the *free field* radiation impedance for the array element, the effect of inter-element array backscattering is necessarily neglected. Backscatter might however, be taken into account at some level. At one extreme, one could numerically find the self radiation impedance of a single element which accounts for all scattering between array elements, but that would be tantamount to solving the full array problem numerically, precluding the need for a Pritchard approximation altogether. (Incidentally, this is exactly what Pritchard was able to do for his baffled array.) A second alternative, neglects all backscattering except what is between those two array elements for which the mutual radiation impedance is being found. For a two body array, this and the former methods of finding the self radiation impedance are equivalent. A difficulty with this latter technique is that the self radiation impedance expression used in a Pritchard approximation for each array element would depend upon the geometry (distance and orientation) between it and each member of the array. Calculations would then have to be done on every combination of two element arrays possible in the full array, leading to a considerable amount of work for "large" arrays.

If the free field self radiation impedance *is* used in the Pritchard approximation, and the velocity on the surface of each array element is known, the field pressure at any point in the acoustic medium can be found by a simple superposition of the radiated pressures resulting from each of the array elements. The surface pressure on the nth array element can be expressed as the sum of its self radiated pressure ($Z_{nn}V_n$) plus contributions of radiated pressures from all other array elements, through the mutual radiation impedance matrix. Pritchard's approximation essentially substitutes the contribution to the radiation impedance from the mth array element on the nth element by a point source with amplitude equal to the resistive part of the mth array element's self-radiation impedance. This can be written as

$$P_n = \sum_{m=1}^{M} Z_{nm} V_m \approx Z_{nn_{free}} V_n + \sum_{m=1, m\neq n}^{M} R_{mm} h_0^{(2)}(kd_{nm}) V_m$$

where M is the number of elements in the array, and d_{nm} is the distance between array elements n and m.

To gauge the accuracy of the Pritchard approximation, an exact form of the mutual radiation impedance between elements of an array is needed which is more general than the "flat" array considered by Pritchard in his original work. An analytical form for the mutual radiation impedance between two acoustically hard spheres is given by New and Eisler [5] which employs a Green's function analysis coupled with spherical addition formulas [8]. Self and mutual acoustic radiation impedances between the spheres are found and represented by infinite series involving spherical Bessel functions and spherical harmonics. Results are then given over a range of center to center distances for cases in which one or both of the spheres are oscillating in a monopole fashion.

The surface radiation impedance used by New and Eisler is based upon the definition [14]

$$Z_{r_j} = \frac{1}{V_j V_j^*} \iint p(\overline{r_j}) v^*(\overline{r_j})\, ds_j$$

where $p(\overline{r})$ and $v(\overline{r})$ are, respectively, the pressure and normal velocity on the surface (s_j), and where $v(\overline{r})$ is further defined as $v(\overline{r}) = V\beta(\overline{r})$ in which V is an amplitude applied to the spatially varying function $\beta(\overline{r})$. (Asterisks refer to complex conjugates.)

It is a simple matter to find values for the self and mutual radiation impedance for the two element array analyzed in New and Eisler's work since the two spheres are identical. For this array, the total radiation impedance seen by sphere 1 is:

$$Z_{r_1} = Z_{11} + \frac{V_2}{V_1} Z_{12}$$

The scaled radiation impedance on sphere one when both spheres have identical velocities, is given by the sum $Z_{11} + Z_{12}$, while Z_{11} is found by calculating the surface radiation impedance of sphere one while assigning a zero velocity on sphere two. To find the mutual radiation impedance of sphere two on sphere one (Z_{12}), the difference between these two radiation impedances is taken $(Z_{r_1}|_{V_1=V_2=1} - Z_{r_1}|_{V_1=1,V_2=0})$. (Equivalently, Z_{12} could be found by determining the pressure on sphere 1 when $V_1 = 0$ and $V_2 = 1$.)

In reporting their results, New and Eisler make the following observation regarding the two sphere array and the radiation impedance on sphere 1 (Z_{r_1}):

> ... the contribution of the mutual term Z_{12} is certainly significant. However, the primary contribution to the deviation of Z_{r_1} from the free field value is *not* scattering, but merely a consequence of the added pressure field from sphere number 2. One can conclude, therefore, that, in the transition region ($ka_1 \approx ka_2 \approx 1$), the effects of scattering on the radiation impedance of one spherical

transducer in the presence of a second spherical transducer are barely significant, even for close spacings.

This would appear to validate a "Pritchard" type approximation for an array of spheres, up to and including $ka \approx 1$, and small inter-array element spacings.

In our analysis, spherical harmonic functions and the spherical addition formula are used to find an appropriate "modal" Pritchard approximation based upon **normal modes of the acoustic medium**. Our work explores the nature of this approximation with comparisons to exact solutions found by a T-matrix formalism [11], in which scattering effects between array elements are fully taken into account.

3 Derivation of the Modal Pritchard Approximation

The time harmonic radiated or scattered pressure field is represented in terms of outgoing spherical Hankel functions applied to the set of spherical harmonics. For a single radiator/scatterer with a local coordinate system written with the index "j", the functional form of the pressure field (p_j) is written

$$p_j(r_j, \theta_j, \phi_j) = \sum_{n=0}^{N} \sum_{m=-n}^{n} A^j_{mn} h_n(kr_j) \Omega^m_n(\theta_j, \phi_j)$$

where

$$\Omega^m_n(\theta, \phi) = P^m_n(\cos\theta) e^{im\phi}$$

are spherical harmonic functions.

Representation of a single outgoing spherical wave using the spherical addition formula, is given by[8]

$$h_n(kr_2)\Omega^m_n(\theta_2, \phi_2) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{\substack{p=|n-\nu| \\ p\geq|m-\mu|}}^{n+\nu}{}' \\ a(\nu, p, n, \mu, m) j_\nu(kr_{1<}) h_p(kr_{1>}) \Omega^{\mu+m}_p(\theta_{12}, \phi_{12}) \Omega^\mu_\nu(\theta_1, \phi_1)$$

(the prime on the summation over p indicates jumps of 2 in the sum)

(r_1, θ_1, ϕ_1) = spherical coordinates relative to system 1

(r_2, θ_2, ϕ_2) = spherical coordinates relative to system 2

$(r_{12}, \theta_{12}, \phi_{12})$ = origin of system 2 relative to 1

$r_{1_>} = \max\{r_2, r_{12}\} \qquad r_{1_<} = \min\{r_2, r_{12}\}$

and where the coefficients $a(\ldots)$ are related to the Wigner 3-j symbols used in quantum mechanics.

Derivation of the modal Pritchard approximation begins with a description of a two element array. Spheres have common radius a, and velocities specified on spheres 1 and 2 are given by individual normal modes:

$$V_1 = \frac{i}{\omega\rho}\frac{\partial p}{\partial r}\bigg|_{r_1=a} = V_{n_1}^{m_1}\Omega_{n_1}^{m_1}(\theta_1,\phi_1)$$

$$V_2 = \frac{i}{\omega\rho}\frac{\partial p}{\partial r}\bigg|_{r_2=a} = V_{n_2}^{m_2}\Omega_{n_2}^{m_2}(\theta_2,\phi_2)$$

Letting

$$\bar{\Omega}_n^m = \Omega_n^m(\theta_{12},\phi_{12})$$

and noting that

$$\Omega_n^m(\pi-\theta_{12},\pi-\phi_{12}) = (-1)^{n+m}\bar{\Omega}_n^m$$

the pressure from sphere 2 translated to sphere 1 coordinates (with $r_1 < r_{12}$) is given by:

$$p_2^1 = \sum_{t_2=0}^{\infty}\sum_{s_2=-t_2}^{t_2} A_{s_2t_2}^2 \left\{ \sum_{\nu_1=0}^{\infty}\sum_{\mu_1=-\nu_1}^{\nu_1} j_{\nu_1}(kr_1) \sum_{\substack{\sigma_1=|t_2-\nu_1| \\ \sigma_1\geq|s_2-\mu_1|}}^{t_2+\nu_1}{}' a(\nu_1,\sigma_1,t_2,\mu_1,s_2) h_{\sigma_1}(kd)\Omega_{\nu_1}^{\mu_1}(\theta_1,\phi_1)\bar{\Omega}_{\sigma_1}^{s_2-\mu_1} \right\}$$

Similarly, the pressure from sphere 1 translated to sphere 2 coordinates is

$$\begin{aligned} p_1^2 \;=\; & \sum_{t_1=0}^{\infty}\sum_{s_1=-t_1}^{t_1} A_{s_1t_1}^1 \left\{ \sum_{\nu_2=0}^{\infty}\sum_{\mu_2=-\nu_2}^{\nu_2} j_{\nu_2}(kr_2) \right. \\ & \left. \sum_{\substack{\sigma_2=|t_1-\nu_2| \\ \sigma_2\geq|s_1-\mu_2|}}^{t_1+\nu_2}{}' a(\nu_2,\sigma_2,t_1,\mu_2,s_1) h_{\sigma_2}(kd)\Omega_{\nu_2}^{\mu_2}(\theta_2,\phi_2)\bar{\Omega}_{\sigma_2}^{s_1-\mu_2}(-1)^{\sigma_2+s_1-\mu_2} \right\} \end{aligned}$$

Upon application of the boundary conditions:

$$\begin{aligned} \frac{i}{\omega\rho}\frac{\partial}{\partial r}\left[p_1+p_2^1\right]\Big|_{r_1=a} &= V_{n_1}^{m_1}\Omega_{n_1}^{m_1}(\theta_1,\phi_1) \\ \frac{i}{\omega\rho}\frac{\partial}{\partial r}\left[p_2+p_1^2\right]\Big|_{r_2=a} &= V_{n_2}^{m_2}\Omega_{n_2}^{m_2}(\theta_2,\phi_2) \end{aligned}$$

and formally applying normal modes with division by derivatives of Hankel functions applied at ka leads to the system of equations

$$\frac{-i\rho c V_{n_1}^{m_1}}{h'_{n_1}(ka)}\delta_{mm_1}\delta_{nn_1} = A^1_{mn}\delta_{mm_1}\delta_{nn_1}$$

$$+ \frac{j'_{n_1}(ka)}{h'_{n_1}(ka)}\sum_{t_2=0}^{\infty}\sum_{s_2=-t_2}^{t_2} A^2_{s_2t_2}\left[\sum_{\substack{\sigma_1=|t_2-n_1|\\ \sigma_1\geq|s_2-m_1|}}^{t_2+n_1}{}' a(n_1,\sigma_1,t_2,m_1,s_2)h_{\sigma_1}(kd)\bar{\Omega}_{\sigma_1}^{s_2-m_1}\right]$$

$$\frac{-i\rho c V_{n_2}^{m_2}}{h'_{n_2}(ka)}\delta_{mm_2}\delta_{nn_2} = A^2_{mn}\delta_{mm_2}\delta_{nn_2}$$

$$+ \frac{j'_{n_2}(ka)}{h'_{n_2}(ka)}\sum_{t_1=0}^{\infty}\sum_{s_1=-t_1}^{t_1} A^1_{s_1t_1}\left[\sum_{\substack{\sigma_2=|t_1-n_2|\\ \sigma_2\geq|s_1-m_2|}}^{t_1+n_2}{}' a(n_2,\sigma_2,t_1,m_2,s_1)h_{\sigma_2}(kd)\bar{\Omega}_{\sigma_2}^{s_1-m_2}(-1)^{\sigma_2+s_1-m_2}\right]$$

(The δ_{ij} terms above are Kronecker delta functions which are one when $i = j$ and zero otherwise.)

Truncating the series, letting

$$B^1_{mn} = \frac{-i\rho c V_{n_1}^{m_1}}{h'_{n_1}(ka)}\delta_{mm_1}\delta_{nn_1} \qquad B^2_{mn} = \frac{-i\rho c V_{n_2}^{m_2}}{h'_{n_2}(ka)}\delta_{mm_2}\delta_{nn_2}$$

and rewriting these equations in matrix form (e.g. $\vec{\mathbf{A}}^1 = [A^1_{00}, A^1_{-11}, A^1_{10}, A^1_{11}, A^1_{-22}, \cdots]^T$)

$$\begin{bmatrix} I & K_{12} \\ K_{21} & I \end{bmatrix}\begin{bmatrix} \vec{\mathbf{A}}^1 \\ \vec{\mathbf{A}}^2 \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{B}}^1 \\ \vec{\mathbf{B}}^2 \end{bmatrix}$$

where the $[(s_1,t_1),(s_2,t_2)]$ entry of the K_{12} matrix is

$$K_{12}((s_1,t_1),(s_2,t_2)) = \frac{j'_{s_1}(ka)}{h'_{s_1}(ka)}\sum_{\substack{\sigma_1=|t_2-t_1|\\ \sigma_1\geq|s_2-s_1|}}^{t_2+t_1}{}' a(t_1,\sigma_1,t_2,s_1,s_2)h_{\sigma_1}(kd)\bar{\Omega}_{\sigma_1}^{s_2-s_1}$$

and the $[(s_2,t_2),(s_1,t_1)]$ entry of the K_{21} is

$$K_{21}((s_2,t_2),(s_1,t_1)) = \frac{j'_{s_2}(ka)}{h'_{s_2}(ka)}\sum_{\substack{\sigma_2=|t_1-t_2|\\ \sigma_2\geq|s_1-s_2|}}^{t_1+t_2}{}' a(t_2,\sigma_2,t_1,s_2,s_1)h_{\sigma_2}(kd)\bar{\Omega}_{\sigma_2}^{s_1-s_2}(-1)^{\sigma_2+s_1-s_2}$$

Solution of the above matrix equation for the unknown amplitudes provides an "exact" answer to the array problem, and is equivalent to the solution found by employing the T-matrix formalism of Scandrett and Baker [12].

For $ka \ll 1$ asymptotic forms for the ratio of the spherical Bessel functions are [1]

$$\frac{j_n'(ka)}{h_n'(ka)} \sim \begin{cases} O[(ka)^3] & \text{for } n = 0 \text{ or } 1 \\ O[(ka)^{2n+1}] & \text{for } n \geq 2 \end{cases} \qquad \text{as} \qquad ka \to 0$$

Therefore the magnitude of the matrix entries K_{12} and K_{21} are of order $(ka)^3$, and an approximate inverse to the matrix can be found which is accurate to order $(ka)^6$

$$\begin{bmatrix} I & K_{12} \\ K_{21} & I \end{bmatrix}^{-1} = \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & K_{12} \\ K_{21} & 0 \end{bmatrix} \right\}^{-1} = \begin{bmatrix} I & -K_{12} \\ -K_{21} & I \end{bmatrix} + O[(ka)^6]$$

and an explicit calculation for the unknown scattering amplitudes is possible:

$$\vec{\mathbf{A}}^1 \approx \vec{\mathbf{B}}^1 - K_{12}\vec{\mathbf{B}}^2 \qquad \text{and} \qquad \vec{\mathbf{A}}^2 \approx \vec{\mathbf{B}}^2 - K_{21}\vec{\mathbf{B}}^1$$

To obtain the radiation impedance component on the (n_1, m_1) mode of sphere 1 given that sphere 2 is radiating in the (n_2, m_2) mode, the pressure amplitudes are substituted into the radiation impedance integral formula, and all terms of order $(ka)^6$ or higher are neglected. One is left with the expression:

$$\begin{aligned} Z_{r_1} = \frac{-i\rho c}{V_{n_1}^{m_1}} \iint_{s_1} \|\Omega_{n_1}^{m_1}(\theta_1, \phi_1)\|^2 \, ds_1 \Bigg\{ & \frac{h_{n_1}(ka)}{h_{n_1}'(ka)} V_{n_1}^{m_1} \\ -\delta_{n_1 0} \frac{j_{n_1}(ka)}{h_{n_1}'(ka)} V_{n_1}^{m_1} & \left[\frac{j_0'(ka)}{h_0'(ka)} [h_0(kd)]^2 + 3\cos(2\theta_{12}) \frac{j_1'(ka)}{h_1'(ka)} [h_1(kd)]^2 \right] \\ \frac{-iH(4 - n_1 - n_2)}{(ka)^2 h_{n_1}'(ka) h_{n_2}'(ka)} V_{n_2}^{m_2} & \sum_{\substack{\sigma = |n_2 - n_1| \\ \sigma \geq |m_2 - m_1|}}^{n_2 + n_1}{}' \, a(n_1, \sigma, n_2, m_1, m_2) h_\sigma(kd) \bar{\Omega}_\sigma^{m_2 - m_1} \Bigg\} + O[(ka)^6] \end{aligned}$$

where $H(4 - n_1 - n_2)$ is a Heaviside function which is zero for $n_1 + n_2 \geq 4$ and equals one otherwise.

The middle term in the above expression is the sole contribution to scattering from one sphere to the other, and is present only when one of the spheres is radiating with a nonzero amplitude in the monopole mode (at least to $O[(ka)^6]$). The two parts of this term result from monopole and dipole backscatter from sphere two which affect the amplitude of the *scattered* monpole pressure amplitude on sphere one. The backscatter terms are to leading order $(ka)^5$, and if neglected, one obtains an approximation for the radiation impedance to $O[(ka)^5]$ which completely neglects all scattering effects, and is consistent with using the free field radiation impedance for the self radiation impedance, as outlined in the previous section.

$$Z_{r_1} \approx \frac{-i\rho c}{V_{n_1}^{m_1}} \iint_{s_1} \|\Omega_{n_1}^{m_1}(\theta_1,\phi_1)\|^2 \, ds_1 \left\{ \frac{h_{n_1}(ka)}{h'_{n_1}(ka)} V_{n_1}^{m_1} \right.$$
$$\left. \frac{-iH(3-n_1-n_2)}{(ka)^2 h'_{n_1}(ka) h'_{n_2}(ka)} V_{n_2}^{m_2} \sum_{\substack{\sigma=|n_2-n_1| \\ \sigma\geq|m_2-m_1|}}^{n_2+n_1}{}' \, a(n_1,\sigma,n_2,m_1,m_2) h_\sigma(kd) \bar{\Omega}_\sigma^{m_2-m_1} \right\}$$

The first term which is multiplied by $V_{n_1}^{m_1}$ is the self radiation impedance

$$Z_{11} \approx -i\rho c \iint_{s_1} \|\Omega_{n_1}^{m_1}(\theta_1,\phi_1)\|^2 \, ds_1 \frac{h_{n_1}(ka)}{h'_{n_1}(ka)}$$

and equals the free field radiation impedance of sphere one. The summation term multiplying $V_{n_2}^{m_2}$ is the mutual radiation impedance of sphere 2 on sphere 1, and can be thought of as an approximation to the *translation* of the *radiated pressure* from sphere 2 onto sphere 1:

$$\begin{aligned} Z_{12(n_2,m_2)\to(n_1,m_1)} \approx & -i\rho c \iint_{s_1} \|\Omega_{n_1}^{m_1}(\theta_1,\phi_1)\|^2 \, ds_1 \\ & \frac{-iH(3-n_1-n_2)}{(ka)^2 h'_{n_1}(ka) h'_{n_2}(ka)} \sum_{\substack{\sigma=|n_2-n_1| \\ \sigma\geq|m_2-m_1|}}^{n_2+n_1}{}' \, a(n_1,\sigma,n_2,m_1,m_2) h_\sigma(kd) \bar{\Omega}_\sigma^{m_2-m_1} \end{aligned}$$

The matrix form for the non-neglected terms in the modal Pritchard approximation to the mutual radiation impedance is:

$$\begin{bmatrix}
Z_{(00)(00)} & Z_{(00)(1-1)} & Z_{(00)(10)} & Z_{(00)(11)} & Z_{(00)(2-2)} & Z_{(00)(2-1)} & Z_{(00)(20)} & Z_{(00)(21)} & Z_{(00)(22)} & 0 & \cdots \\
Z_{(1-1)(00)} & Z_{(1-1)(1-1)} & Z_{(1-1)(10)} & Z_{(1-1)(11)} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Z_{(10)(00)} & Z_{(10)(1-1)} & Z_{(10)(10)} & Z_{(10)(11)} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Z_{(11)(00)} & Z_{(11)(1-1)} & Z_{(11)(10)} & Z_{(11)(11)} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Z_{(2-2)(00)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Z_{(2-1)(00)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Z_{(20)(00)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Z_{(21)(00)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Z_{(22)(00)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

It is interesting to compare the real part of the self impedance to the amplitude of the mutual impedance in the special case that both spheres are radiating identically ($\vec{\mathbf{B}}^1 = \vec{\mathbf{B}}^2$ or $n_1 = n_2 = n$ and $m_1 = m_2 = m$). In this case, the real part of $Z_{11} = R_{11}$ becomes

$$\begin{aligned} R_{11} &\approx \rho c \iint_{s_1} \|\Omega_{n_1}^{m_1}(\theta_1,\phi_1)\|^2 \, ds_1 \mathcal{I}m \left\{ \frac{h_{n_1}(ka)}{h'_{n_1}(ka)} \right\} \\ &= -\rho c \iint_{s_1} \|\Omega_{n_1}^{m_1}(\theta_1,\phi_1)\|^2 \, ds_1 \frac{1}{(ka)^2 \|h'_{n_1}(ka)\|^2} \\ &\approx \frac{(ka)^{2n+2}}{[(n+1)(1)(3)\cdots(2n-1)]^2} \rho c \iint_{s_1} \|\Omega_{n_1}^{m_1}(\theta_1,\phi_1)\|^2 \, ds_1 \end{aligned}$$

while Z_{12} has the asymptotic form

$$\begin{aligned} Z_{12} &\approx \frac{(ka)^{2n+2}}{[(n+1)(1)(3)\cdots(2n-1)]^2}\rho c \iint_{s_1} \|\Omega_{n_1}^{m_1}(\theta_1,\phi_1)\|^2\, ds_1 \\ &\times H(3-2n)\sum_{\sigma=0}^{2n}{}' \; a(n,\sigma,n,m,m)h_\sigma(kd)\bar{\Omega}_\sigma^0 \end{aligned}$$

When only the first term in the series for Z_{12} (the $\sigma = 0$ case which results in an $h_0(kd)$ term) is kept, the "standard" Pritchard approximation results. In this instance note that $a(n,0,n,m,m) = 1$ for all $n \geq 0$ and $-n \leq m \leq n$. The remaining terms in the series have an angular dependence which affects the amplitude of Z_{12} as the relative angles between array elements vary. Such an angular dependence is *not* built into the standard Pritchard approximation - only the relative *distance* between elements as expressed by the factor $h_0(kd)$ is present.

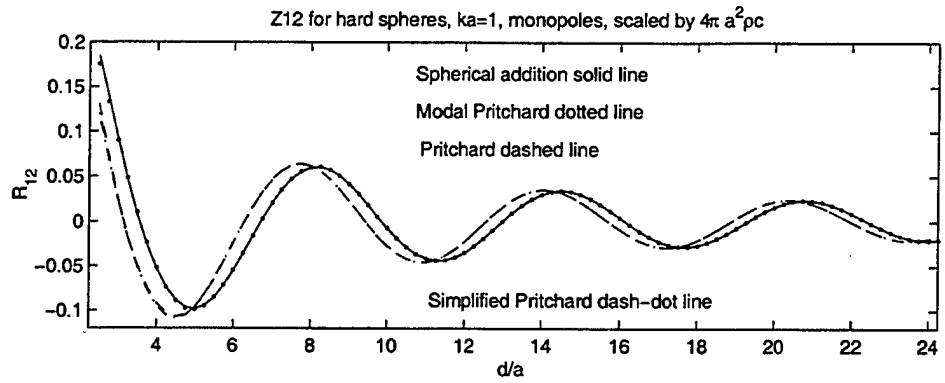
4 Results

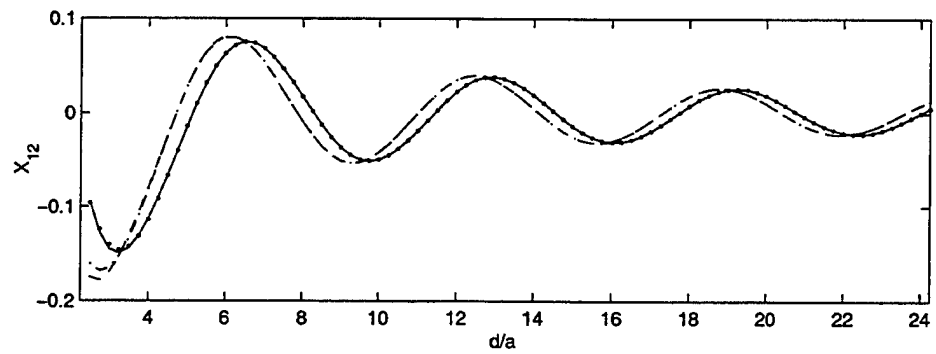
4.1 Two body interactions

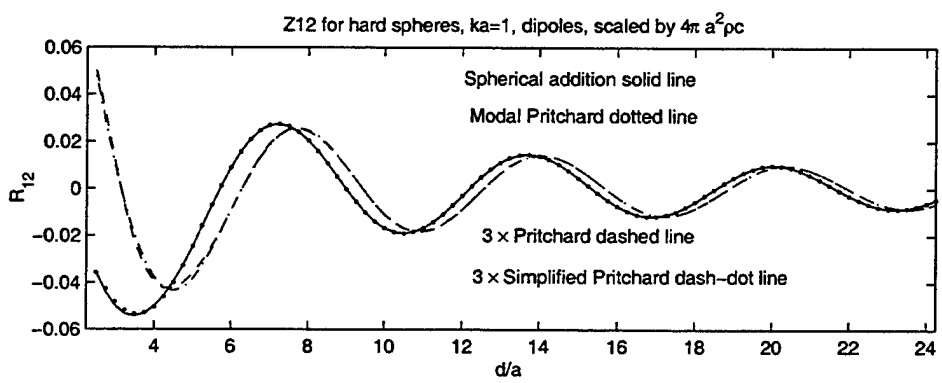
In this section several graphs will be displayed in an effort to demonstrate the modal Pritchard approximation given above. The first set of numerical experiments involve the New and Eisler problem of an array of two, identical, acoustically hard spheres separated by a distance d. Each sphere has radius $a = 1/2$, and is assigned a modal velocity in such a way that the mutual radiation impedance can be found.

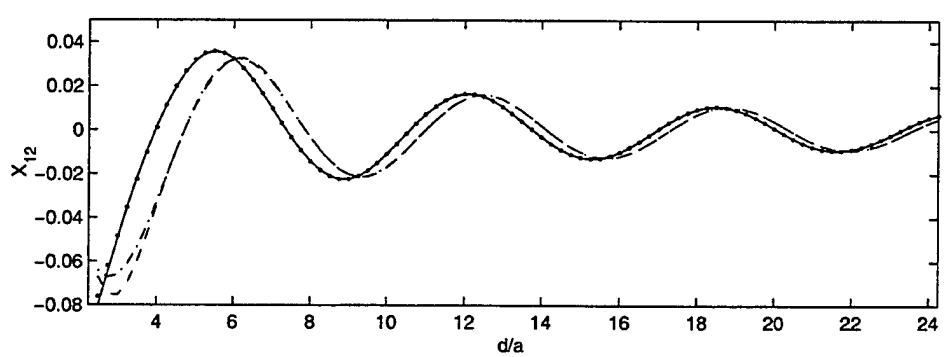
Besides the modal Pritchard method, there are two other approximations displayed. The results labelled "Pritchard's method" and "Simplified Pritchard" are found as follows. For the "Pritchard method", an "exact" value for R_{11} which includes all backscattering is used in the approximation, while the "simplified Pritchard" employs only the *free-field* values of the self radiation resistance term R_{11}, and therefore neglects backscatter.

The first series of graphs display the monopole to monopole, axially symmetric dipole to dipole, and axially symmetric quadrupole to quadrupole mutual radiation impedances for the two spheres which are aligned with the z axis. The final graph of quadrupole to quadrupole is included only to show the accuracy of the modal Pritchard approximation even through the quadrupole term, in spite of the fact that this term is of order $(ka)^6$ in the expansion for small ka of the mutual radiation impedance and therefore would be neglected in an application of the derived modal Pritchard approximation.


Z12 for hard spheres, ka=1, monopoles, scaled by 4π a²ρc
Spherical addition solid line
Modal Pritchard dotted line
Pritchard dashed line
Simplified Pritchard dash-dot line
R12
d/a


X12
d/a


Z12 for hard spheres, ka=1, dipoles, scaled by 4π a²ρc
Spherical addition solid line
Modal Pritchard dotted line
3 × Pritchard dashed line
3 × Simplified Pritchard dash-dot line
R12
d/a


X12
d/a

It would appear from the above graph that backscattering effects in the R_{11} determination are negligible by comparing results of the "simplified Pritchard" and "Pritchard" methods. Additionally, the "modal Pritchard" seems to very closely match the "exact" spherical addition formula results which includes all scattering effects. The results of the monopole to monopole mutual radiation impedance highlight what appears to be a phase shift between the "modal" and "simplified" Pritchard methods. Because of their analytical form, a comparison between Pritchard and the exact solution can be analytically performed by comparing the simplified Pritchard with the modal Pritchard. The formulas for the modal and simplified Pritchards (scaled by $4\pi a^2 \rho c$) as a function of the distance between spheres (d) is

$$\begin{aligned}
\text{modal Pritchard monopole to monopole} &= h_0(kd)\frac{-1}{[kah_0'(ka)]^2} \\
\text{simplified Pritchard monopole to monopole} &= h_0(kd)\mathcal{R}e\left[\frac{-ih_0(ka)}{h_0'(ka)}\right]
\end{aligned}$$

Taking the ratio of these two expressions, one has

$$\frac{h_0(kd)\dfrac{-1}{[kah_0'(ka)]^2}}{h_0(kd)\mathcal{R}e\left[\dfrac{-ih_0(ka)}{h_0'(ka)}\right]} = -\frac{j_0'(ka)+iy_0'(ka)}{j_0'(ka)-iy_0'(ka)} \approx 1+\frac{2i}{3}(ka)^3 \quad \text{as} \quad ka \to 0$$

One can also see from the above expression, that the amplitude of the ratio is one with a constant phase shift. When $ka = 1$, this phase shift is about 24.6 degrees, which is what is seen in the monopole to monopole figure above.

For the dipole to dipole case, the relative position of the spheres introduces an angular dependence in the mutual radiation impedance predicted by the modal Pritchard approximation which is not present in the simplified Pritchard approximation. As in the monopole to monopole case, one finds that the simplified Pritchard approximation is essentially equivalent to the Pritchard method which accounts for backscattering, but to compare with the exact and/or modal Pritchard approximation, the results must be magnified by a factor of three. To see where the factor of three comes from, one can again compare the (scaled) modal and simplified Pritchard approximations

$$\begin{aligned}
\text{modal Pritchard dipole to dipole} &= \frac{-1}{3\,[kah_1'(ka)]^2}\left\{h_0(kd) - 2h_2(kd)\Omega_2^0(\theta_{12},\phi_{12})\right\} \\
\text{simplified Pritchard dipole to dipole} &= \frac{h_0(kd)}{3}\mathcal{R}e\left[\frac{-ih_1(ka)}{h_1'(ka)}\right]
\end{aligned}$$

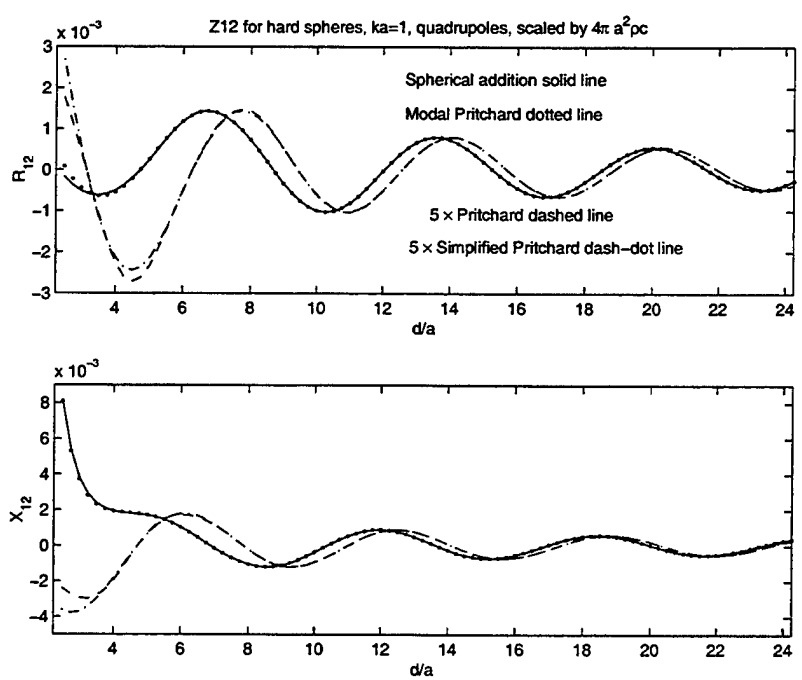
If one assumes $kd \gg 1$, the approximation $h_2(kd) \approx -h_0(kd)$ can be made, and with the fact that $\Omega_2^0(\theta_{12}) = [1+3\cos(2\theta_{12})]/4$, the ratio of the two expressions can be written:

$$\frac{h_0(kd) - \dfrac{3[1+\cos(2\theta_{12})]/2}{[kah_1'(ka)]^2}}{h_0(kd)\mathcal{R}e\left[\dfrac{-ih_1(ka)}{h_1'(ka)}\right]} = -\frac{3[1+\cos(2\theta_{12})]}{2}\left\{\frac{j_1'(ka)+iy_1'(ka)}{j_1'(ka)-iy_1'(ka)}\right\}$$

$$\approx \frac{3[1+\cos(2\theta_{12})]}{2} + O[(ka)^3] \quad \text{as} \quad ka \to 0$$

In addition to the phase shift embodied by the ratio of the spherical Hankel functions, one sees an amplitude variation with respect to the angle θ_{12} between the spheres (in this instance, the axis of the dipole aligns with the z axis from which the angle θ_{12} is measured). For the geometry considered in the graph, the angle $\theta_{12} = 0$ leading to a factor of 3 needed to bring the Pritchard approximation in line with the analytic solution. The amplitude and phase shift are unfortunately not the only differences between the solutions. For small distances d the approximation used for $h_2(kd)$ is no longer valid, and this can be readily seen in the graph of the dipole to dipole mutual radiation for small values of d/a.

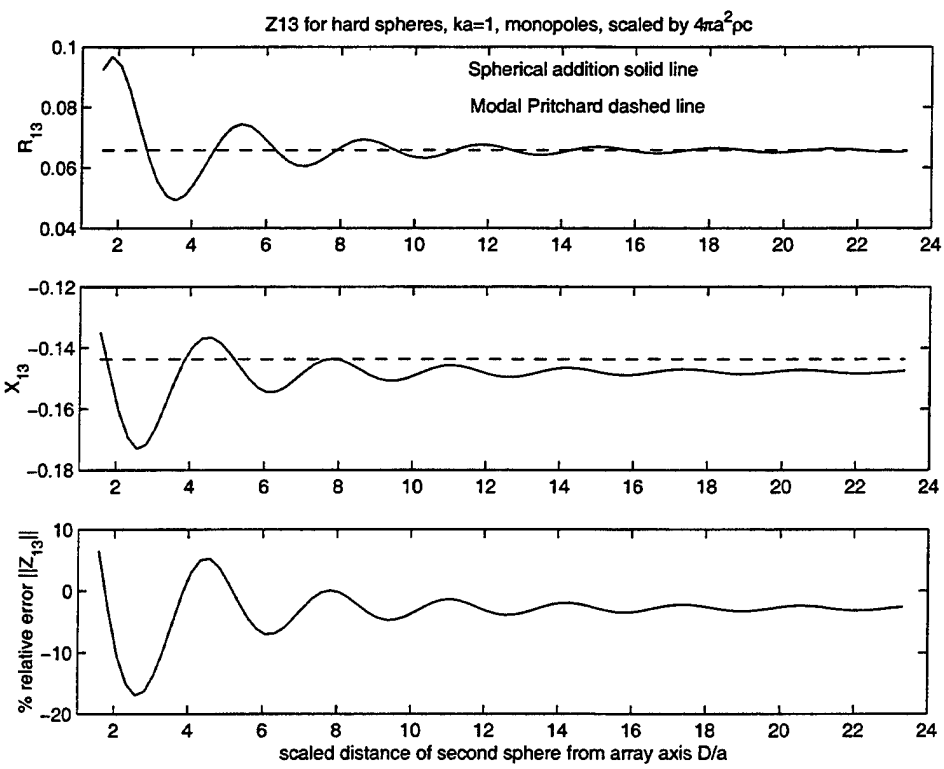
Below is the axially symmetric quadrupole to quadrupole mutual radiation impedances of the various Pritchard approximations along with the spherical addition result. The modal Pritchard result would actually be zero if we neglected all terms of order ka^5 since quadrupole to quadrupole radiation is of higher order. This graph is only included to show the continued success of the approximation embodied by the formula derived in the previous section for Z_{12}. Note also, that the magnitude of these quadrupole impedances are an order of magnitude less significant than the dipole terms.



4.2 Three body interactions

One might hope that the modal Pritchard method works well even for situations in which there are more than two elements in a given array. This would indeed be true if "third party" scattering were negligible. Third party is meant to imply single or multiple scattering of an incident pressure from one array element to another (thereby affecting the mutual radiation impedance between the two), by way of scattering from a third obstacle or array element. This is an important effect when considering closely packed arrays of transducers. If one considers only two element arrays in the production of mutual radiation impedances between array elements, such scattering effects are necessarily neglected.

The most elementary, and perhaps most enlightening case to consider, are three element arrays, in which one of the elements acts solely as an acoustically hard obstacle, and is moved around relative to the remaining two, which are fixed. For such an array, one can anticipate the negligible effects of the third array element when it is far from the two active elements, and one can quantify the effect it has on the mutual radiation impedance when it is moved to the nearfield. Because the two active array elements are fixed in space, the modal Pritchard approximation, (and for that matter all of the Pritchard approximations), produce constant mutual radiation impedances between the two elements, regardless of the positioning of the third. Because of its accuracy in the two element array problem, only the modal Pritchard approximation will be compared with the spherical addition results, to see how important third party scattering is in the determination of mutual radiation impedances.



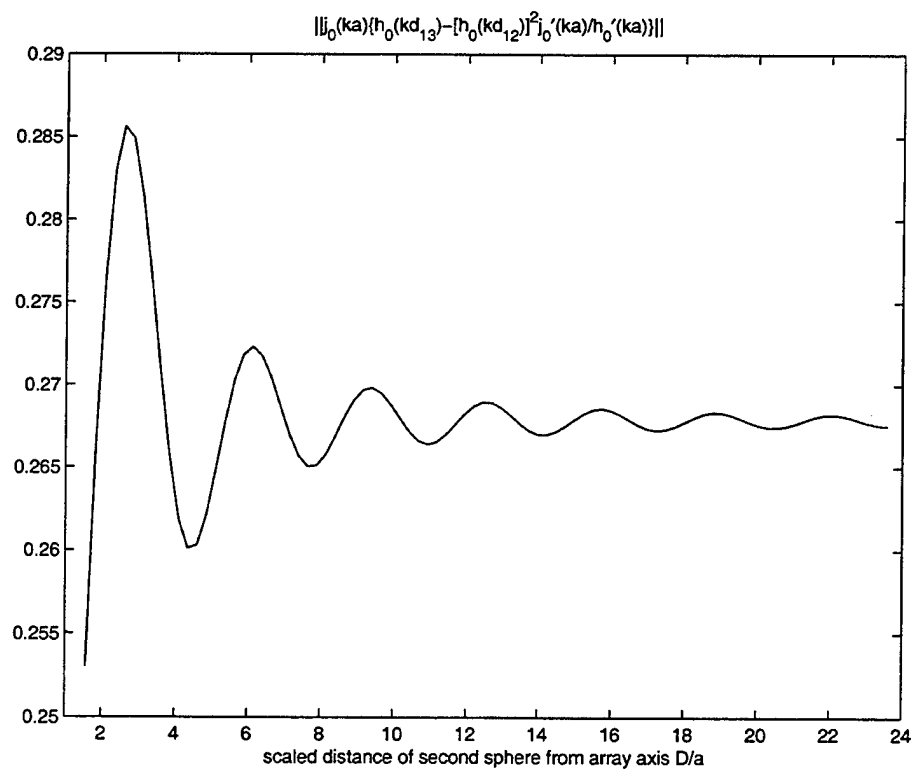
In the first test case, the two active array elements (numbered elements 1 and 3) are aligned to the z axis, and are at a constant center to center distance of $\lambda/2$, which for $ka = 1$ implies a separation of $\pi/2$. The third element (element number 2) of the array starts at a distance of $\lambda/4$ away from the axis of the array, halfway between the two active array elements. It then moves axially away from the two active array elements. Above, is the comparison of the monopole to monopole mutual radiation impedance of sphere 3 on sphere 1.

Notice that there is a sizeable difference in the resistance and reactance terms of the mutual radiation impedance when sphere number 2 is in close proximity to the radiating spheres. In fact, in the third graph of the above series, the relative error of the *magnitude* of the mutual radiation impedance

$$\text{Percent error equals} \qquad 100 \times \frac{\|Z_{13}\|_{\text{modal Pritchard}} - \|Z_{13}\|_{\text{spherical addition}}}{|Z_{13}\|_{\text{spherical addition}}}$$

shows relative errors as great as 15%.

The deviations of the mutual radiation impedance amplitudes can be explained by a relatively simple argument. Depending upon the total distance from sphere 3 to 2 to 1, the scattered pressure is either in phase (when Z_{13} is at a maximum), or out of phase, with the direct pressure from sphere 3 to sphere 1.

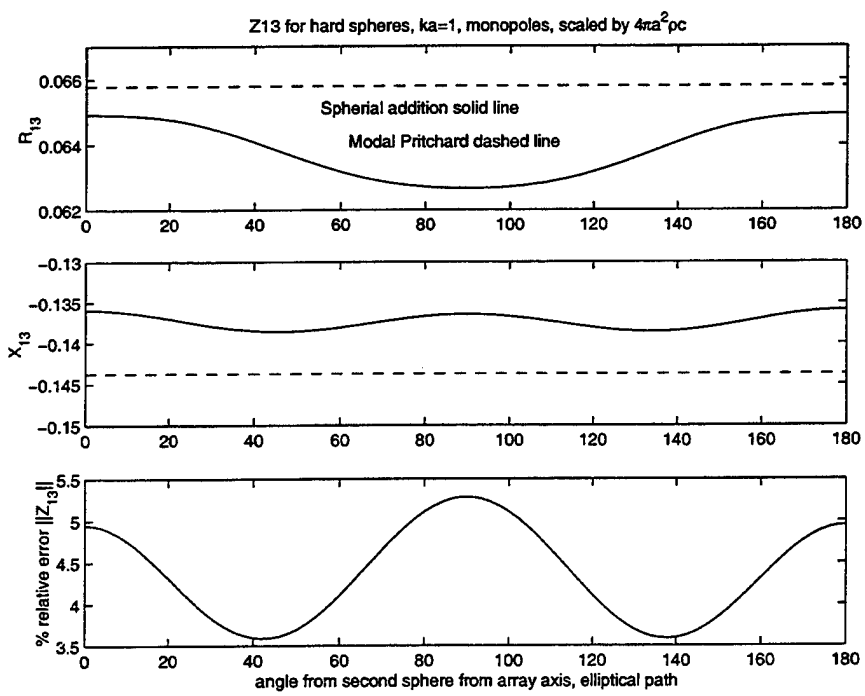


This occurs when the total scattered distance from sphere 3 to 2 to 1 is an integral multiple of the acoustic wavelength. This may seem counter-intuitive since the scattering sphere is acoustically hard, but can be explained by a simple argument involving monopole radiators and the spherical addition formula. Considering only the monopole terms, the sum of the directly incident and once scattered monopole pressures is given by the formula

$$Aj_0(ka)\left\{h_0(kd_{13}) - \frac{j_0{}'(ka)}{h_0{}'(ka)}\left[h_0(kd_{12})\right]^2\right\}$$

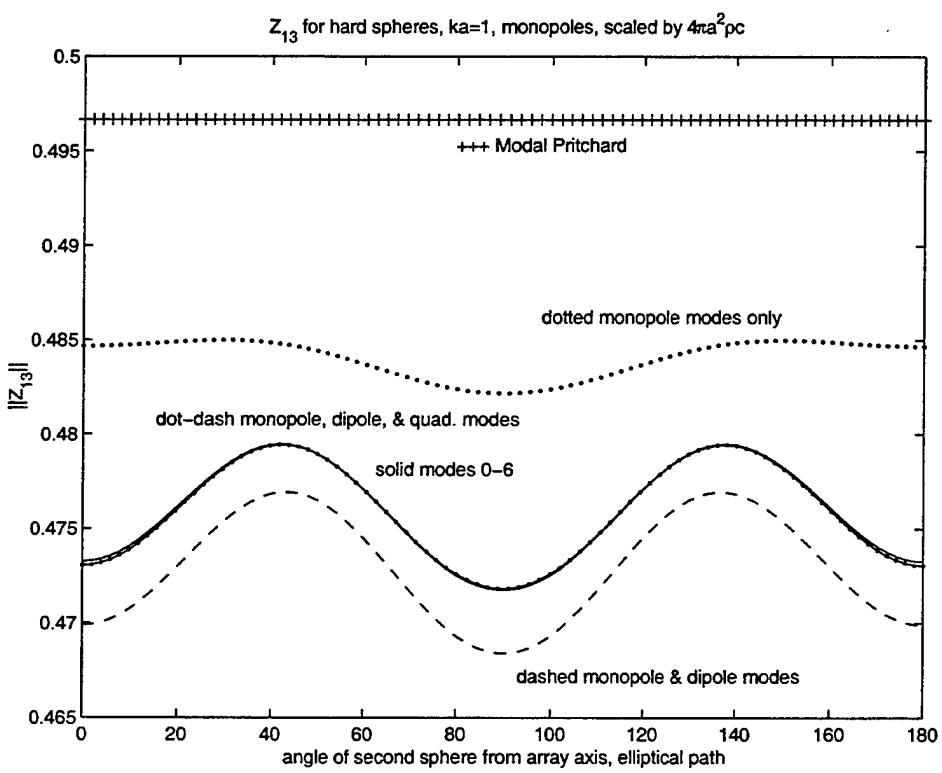
where d_{13} is the distance between spheres 1 and 3, d_{12} is the distance between spheres 1 and 2 (and also the distance between spheres 2 and 3), and A is the amplitude of the monopole pressure radiated from sphere 3. The value of d_{12} can be represented in terms of sphere 2's distance from the z axis (D) by $d_{12} = \sqrt{(\lambda/4)^2 + D^2}$. In the figure above, the amplitude of the above factor for $A = 1$ is produced, and to a large degree explains the variation seen in the amplitude of the mutual radiation impedance term (Z_{13}) found in the previous graph.

A second three element array is also considered, in which the active elements are the same as before, but in which the scattering element moves in an elliptical path around the two active elements in such a way that the two active elements are foci of the ellipse. In this case, the scattering distance from sphere 3 to 2 to 1 has the constant value $3\lambda/2$. In terms of the preceeding three body graph, the point which would intersect the elliptical path occurs when $D/a \approx 4.44$, or when the relative error in Z_{13} is at its maximum value (corresponding to the Pritchard approximation being greater than the spherical addition result).



Graphs of the resistamce and reactance as well as the relative error in the amplitude of Z_{13} are given for angles 0 to 180 degrees. At 0 degrees, the center values for spheres 1, 2, and 3 are respectively 0, λ, and $\lambda/2$, while at 180 degrees, the z values of these spheres is 0, $-\lambda/2$, and $\lambda/2$.

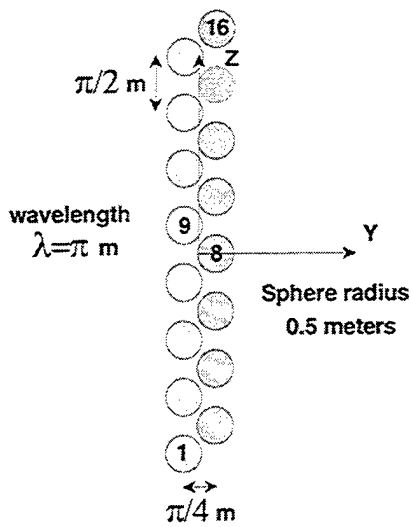
In this instance the variation due to scattering is nearly uniform, with maximum deviations occuring at the poles and perpendicular to the "active" array axis. Unlike the previous three body case, monopole scattering alone does not explain the variation of the amplitude of Z_{13}. The graph below shows the result of limiting the number of terms retained in the spherical addition formulation to monopole only, monopole and dipole only, and monopole, dipole and quadrupole terms. The graph illustrates that the monopole to dipole back to monopole scattering is the primary reason for the deviations seen in the amplitude of the Z_{13} term (the monopole-monopole-monopole and monopole-quadrupole-monopole terms are nearly constant throughout the range of angles). Note also that the result of keeping only through the quadrupole terms very nearly matches the result of keeping harmonics up through order 6.



4.3 Double line array of hard spheres

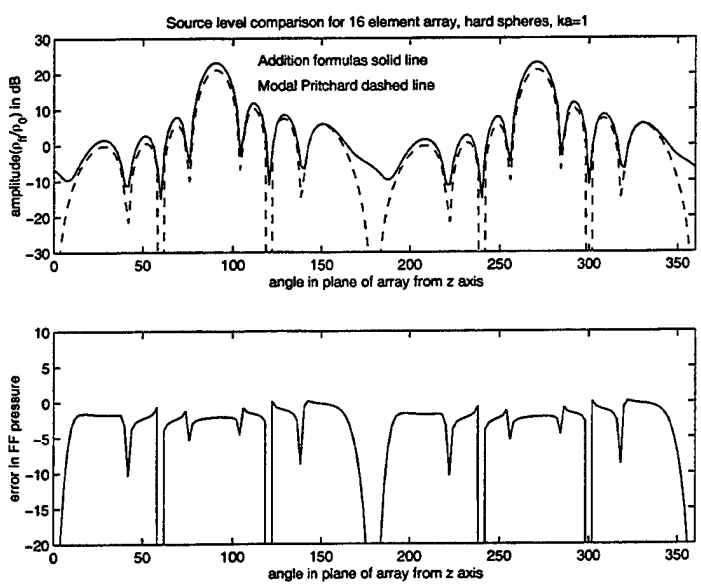
As a final example, consider a double line array of hard spheres which are spaced $\lambda/2$ apart along the lines, and which has the spacing $\lambda/4$ between the two lines. The double

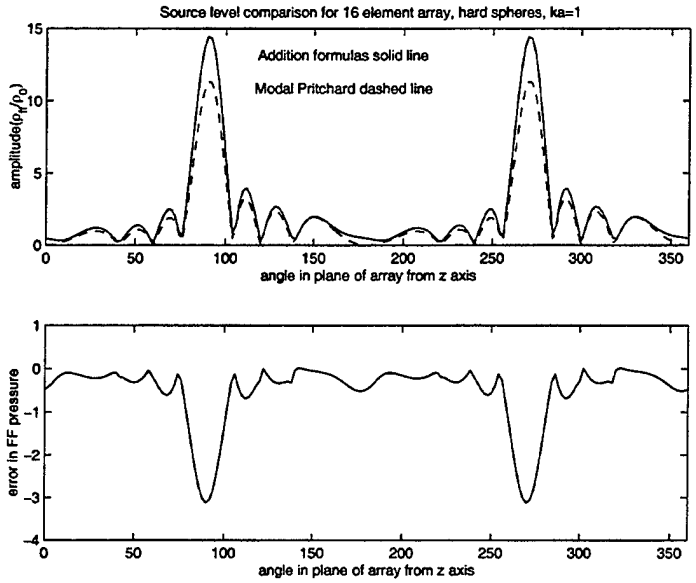
line of spheres is parallel with the z axis. A graphic of the array is shown below, with the numbering of the spheres starting at the bottom left, and ending with the 16th element at the top right.



Orientation of the spheres in the 16 element array

In the first graph below, the source level (in dB) of the array given that each sphere is oscillating in a breathing mode with surface velocity one is shown. The spherical addition result is compared to the modal Pritchard calculation, and as can be seen, the modal Pritchard approximation does quite well, with a maximum error of about 2 dB at broadside. The far field results are scaled by the far-field amplitude of a single oscillating sphere. A second graph shows the magnitude of the far field pressure, scaled by a single radiating sphere.

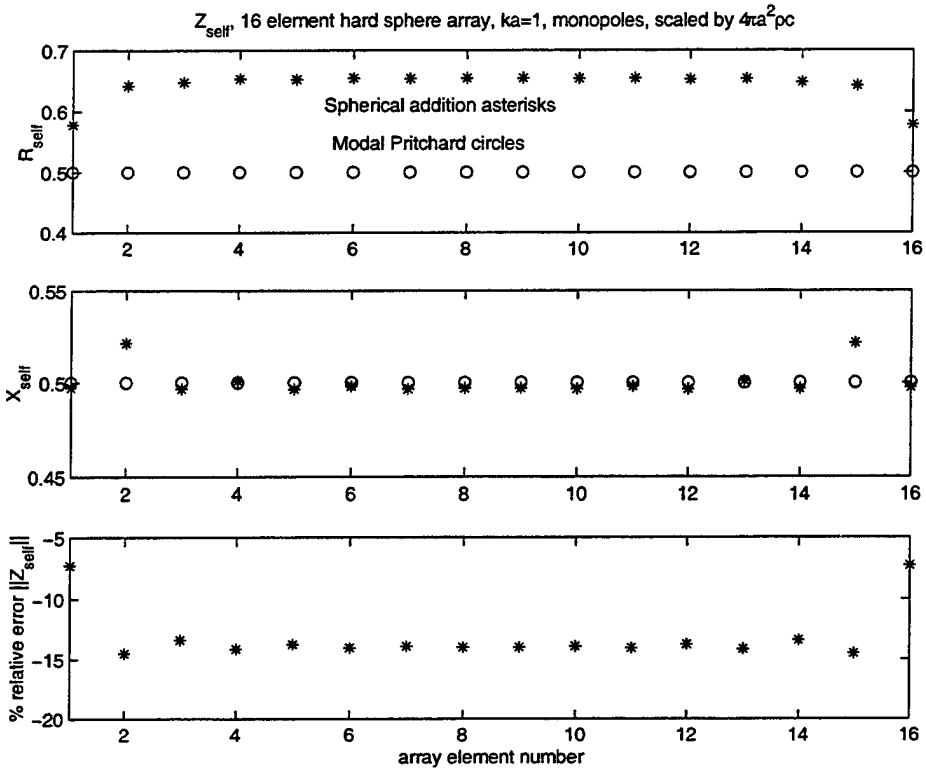


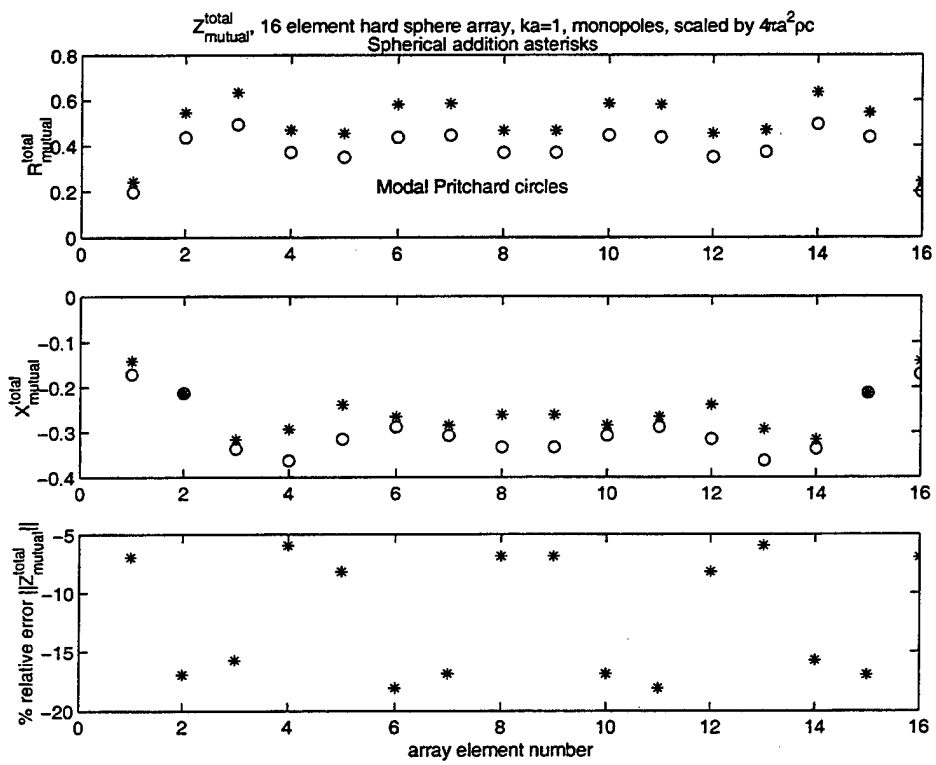


It is possible for this array to compare the self and mutual radiation impedances for the monopole terms without too much difficulty. In the next two pictures, the self-radiation impedance of each of the array elements is given, using both the modal Pritchard approximation and the full spherical addition results. Recall that the modal Pritchard result neglects third party scattering *and* backscattering. In the first graph of the self radiation impedance one can see that there is nearly a uniform 15% relative error in all but the spheres found at the extremities of the array. The reason for the variation from the modal Pritchard is because of backscatter not being taken into account, and that at array elements 1 and 16, the backscatter is less than for any of the remaning array elements. The second graph shows on an element by element basis, the total mutual radiation on each sphere due to radiation of pressure from all other array elements. Essentially, it is the sum of the off diagonal elements in the monopole impedance matrix of the array in any given row, e.g. for array element i, this value is essentially

$$p_{rad}^{mutual} = \sum_{j=1, j\neq i}^{16} Z_{ij}$$

since all array elements have identical (monopole excited) velocities of one.


Z_{self}, 16 element hard sphere array, ka=1, monopoles, scaled by $4\pi a^2\rho c$
Spherical addition asterisks
Modal Pritchard circles
R_{self}
X_{self}
% relative error $||Z_{self}||$
array element number


Z^{total}_{mutual}, 16 element hard sphere array, ka=1, monopoles, scaled by $4\pi a^2\rho c$
Spherical addition asterisks
Modal Pritchard circles
R^{total}_{mutual}
X^{total}_{mutual}
% relative error $||Z^{total}_{mutual}||$
array element number

5 Conclusions and recommendations

The work reported would indicate that improvements can be made to existing methodolgy in array modeling. In particular, for low frequency active array calculations with close array spacing, it would appear that the spherical addition formula should be used to incorporate inter-element scattering. It would appear however, that the number of harmonics needed is only through the quadrupole term for accuracy up to a level of $(ka)^6$, and so if a T-matrix algorithm is adopted, a total of 81 matrix elements should properly characterize the array problem.

Great accuracy has been achieved by using a Pritchard approximation to model the mutual radiation impedance between array elements for several of the reasons delineated in the above report. Improvement would seem to be possible, however, by adopting a modal type Pritchard approximation. There are still limitations to the methodology, however, which should be addressed in an actual application, such as the need to model third party scattering. The manner in which such a modal approximation might be incorporated into a finite element code is as follows.

The finite element code must be used to determine the self radiation impedance of a given array element. Almost any boundary element code would do, for example EQI or CHIEF, but it must couple to the finite element code of the array element. Given nodal values of the normal surface velocity $(\vec{V}_j)$(or $i\omega\times$ the normal surface displacement), and the free field self impedance radiation matrix, the *nodal* pressure vector $(\vec{P}_j)$ on the jth array element is given by

$$(\vec{P}_j)_{nodal} = Z_{jj}\vec{V}_j$$

The nodal pressure values must be converted to modal pressure amplitudes which can be used by the spherical addition formula for translations to other array elements. First introduce an intermediate matrix (A) of spherical Hankel/spherical harmonic functions evaluated at the surface nodes of the jth array element (r_s, θ_s, and ϕ_s)

$$A_{s,(n,m)} = h_n^{(2)}(kr_s)\Omega_n^m(\theta_s, \phi_s)$$

The size of this matrix is $M \times K$ where M is the number of surface nodes, and K is the number of harmonics (9 if use only harmonics through the quadrupole term). The equation to be solved is

$$(\vec{P})_{nodes} = A(\vec{P})_{modes}$$

The least squares solution to this problem is found by solving the normal equations, resulting in

$$(\vec{P})_{modes} = \underbrace{(A^H A)^{-1} A^H}_{C}(\vec{P})_{nodes}$$

where A^H is the Hermitian conjugate of A, and C is the sought after translation matrix from nodal pressures to modal pressures.

Now with the modal radiated pressures, the addition formula can be used to determine the nodal pressures resulting from the modal radiated pressures through an application of a truncated spherical addition formula. The translation from array element j to array element i's nodal coordinates r_t, θ_t, and ϕ_t is given by

$$\begin{aligned}(\vec{P}_i)_t^{nodal} &= \left\{ \sum_{l_1,m_1}^{K} (\vec{P}_j)_{l_1,m_1}^{modal} \sum_{l_2,m_2}^{K} H(3-l_1-l_2) j_{l_2}(kr_t)\Omega_{l_2}^{m_2}(\theta_t,\phi_t) \right. \\ &\times \left. \sum_{\substack{\sigma=|l_2-l_1| \\ \sigma\geq|m_2-m_1|}}^{l_2+l_1}{}' \; a(l_1,\sigma,l_2,m_1,m_2) h_\sigma(kd)\bar{\Omega}_\sigma^{m_2-m_1} \right\}\end{aligned}$$

where d is the distance between the ith and j^{th} array element, and $H(\ )$ is the Heaviside function. Introducing the matrix G which is $M \times K$ (assuming the ith array element also has M surface nodes, it is given by

$$G_{t,(l_1,m_1)} = \sum_{l_2,m_2}^{K} H(3-l_1-l_2) j_{l_2}(kr_t)\Omega_{l_2}^{m_2}(\theta_t,\phi_t) \sum_{\substack{\sigma=|l_2-l_1| \\ \sigma\geq|m_2-m_1|}}^{l_2+l_1}{}' \; a(l_1,\sigma,l_2,m_1,m_2) h_\sigma(kd)\bar{\Omega}_\sigma^{m_2-m_1}$$

Combining the three matrices into a modal approximation for the mutual radiation impedance leads to

$$Z_{ij} \approx G_{ik} C_{kj} Z_{jj}$$

which corresponds to the modal Pritchard approximation used in this report, and which should be compared to the standard Pritchard approximation given by

$$Z_{ij} \approx \mathcal{R}e\,(Z_{jj})\, h_0(kd_{ij})$$

6 References:

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