THE MAXIMUM PRINCIPLE IN THE THEORY
OF OPTIMAL PROCESSES

By V. G. Boltyanskiy

-USSR-

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FOREWORD

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(USSR)

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This paper borders on our joint work [1] on the theory of
optimal processes. The investigations of the secondary variations
by R. V. Gamkrelidze led to Pontryagin's hypothesis of the maximum
principle. Here, complete proof of the maximum principle as a necesa-
sary condition of optimum is presented. To obtain this result the
whole work is repeated with different excitations (variations).

Here is considered a point of motion \( x = (x_1, \ldots, x^n) \) in an
n-dimensional phase space \( X \) described by a law:

\[
\dot{x} = f^i(x^1, \ldots, x^n, u) = f^i(x, u), \quad i = 1, \ldots, n, \tag{1}
\]

where \( u \) is a control parameter which varies within the boundaries of a
certain domain \( U \). The domain \( U \) may be any topological space, for
instance, any arbitrary bounded set of the \( r \)-dimensional space of the
variable \( u = (u_1, \ldots, u^r) \); the functions \( f^i \) are assumed to be continu-
ously dependent on two variables \( x, u \) and continuously differentiable
with respect to \( x^1, \ldots, x^n \). If the law of the control is known, i.e.,
if it is given the variable \( u(t) \in U \), then the system (1) uniquely
describes the law of motion of this variable point; the control \( u(t) \)
must be piece-wise continuous. Similarly with earlier work [1] the
problem is stated:

"Find a control \( u(t) \) which secures the minimal time for the
process of transfer of a point \( x \) from a given initial coordinate
\( x_0 \in X \)." The control defined above and the trajectory corresponding
to it are called optimal control and optimal trajectory. Let \( u(t) \), 
\( t_0 \leq t \leq t_1 \), be a certain control within the domain \( U \). We will choose
the instants of time \( \tau_1, \ldots, \tau_k \), where \( t_0 < \tau_1 < \cdots < \tau_k < t_1 \) and let
\( \delta \tau_{i,j}, j = 1, \ldots, s_1, i = 1, \ldots, k \), be arbitrary non-negative numbers
and \( v_{i,j} \) be arbitrary points of the domain \( U \). We will denote by \( I_{i,j} \)
a segment
\[ \tau_i + \varepsilon \delta \tau_{i,j} + \cdots + \tau_{i+1} \leq t \leq \tau_i + \varepsilon \delta \tau_{i,j} + \cdots + \delta \tau_{i,j} \]
and assume
\[
\bar{u}(t) = \begin{cases} v_{i,j} & \text{if } t \in I_{i,j}, \\ u(t) & \text{if } t \text{ does not belong to any segment } I_{i,j}. \end{cases}
\]
Here \( \varepsilon \) is a positive quantity which affects the control \( \bar{u}(t) \).
Subsequently we will consider \( \varepsilon \) as a variable of the first order of
infinitesimal; the quantities of higher order of infinitesimality
will be denoted by a row of dots and ignored. We will note that the
control \( \bar{u}(t) \) is derived by variations of control \( u(t) \) in the neighborhood of the instants of time \( \tau_1, \ldots, \tau_k \). We will denote by \( \bar{x}(t) \) a
trajectory originated at the same coordinate \( x_0 \) and corresponding to
varied control \( \bar{u}(t) \) and we will consider a vector
\[
\bar{x}(t_1 - \varepsilon \delta t) - x(t_1),
\]  
where \( \delta t \) is a non-negative number.
The linear with respect to \( \varepsilon \) part of the vector (2) we will denote by
\[
\varepsilon \Delta x:
\]
\[
\bar{x}(t_1 - \varepsilon \delta t) = x(t_1) + \varepsilon \Delta x + \ldots
\]  
If we consider now all possible control \( \bar{u}(t) \) resulting from
variation of the control \( u(t) \) and independently from this we vary \( \delta t \)
giving it all possible non-negative values, then the vectors \( \Delta x \),
originated in the point \( x_1 = x(t_1) \), form a certain cone with its vertex
in the point \( x_1 \). By means of fairly simple logic the following statement could be proven: Lemma. "The cone of attainability is a convex cone". Subsequently the theorem presented is an intensification of the theorem in [1], which states that dimension of a linear manifold \( P' \) constructed there is not greater than \( n - 1 \).
Theorem 1. "If a cone of attainability \( K \) fulfills the whole
space \( X \) of the variables \( x^1, \ldots, x^n \), then the trajectory \( x(t) \) and
the corresponding control \( u(t), t_0 \leq t \leq t_1 \) are not optimal."
Thus, if \( x(t) \) and \( u(t) \) are optimal than a convex cone \( K \) cannot
occupy the whole space \( X \) and therefore the whole cone \( K \) lies in one
half-space describable by a certain hyperplane passing through a
point \( x_1 \). Let \( a_{x}(x^2 - x_1^2) = 0 \) be the equation of this hyperplane.
We assume that the whole cone \( K \) lies in a negative half-space
\( a_{x}(x^n - x_1^n) = 0; \) we may change the algebraic signs of all coefficients
\( a_{x} \), if necessary.
If we assume, just as in [1],
\[
a_{x}b_1(t) = b_0, \quad b_0\gamma_e(t) = \phi_e(t),
\]  

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then we receive \( \psi(t) = a \psi \) and the equation of a negative half-space will have a mode

\[
\psi(t) (x^a - x^0) \leq 0.
\]

Consequently we will get the following necessary condition for optimum:

\[
\psi(t) \Delta x^0 \leq 0
\]

where \( \Delta x \) is determined by equation (3) for any varied control \( \bar{u}(t) \) and any \( \Delta t \geq 0 \).

This condition we may regard as a generalization of the conditions (6) and (7) of [1]. In the work presented in [1] the variations of \( \psi u(t) \) also produced a certain cone of attainability, which however may be smaller than the cone \( K \) discussed here. Consequently it is possible to have a case when quantities \( a \psi \) and functions \( \psi(t) \) would satisfy all the conditions imposed in the [1] and would not conform with the condition (4). In other words, the condition (4) is more strict than the conditions (6) and (7) of [1].

From (4) it is easy to deduce the following inequality

\[
\psi(t) / f(x(t), u(t)) \geq 0,
\]

or, we may say the function \( H(x, \psi, u) = \psi f(x, u) \) will have a nonnegative value at the point \((x(t), \psi(t), u(t))\). L. S. Pontryagin pointed out that the function \( H(x, \psi, u) \) plays an important role in the theory of optimal processes. In particular, the relations (1) and (5) of [1] may be written in the form

\[
\dot{x}^i = \partial H / \partial \psi^i, \quad \dot{\psi} = - \partial H / \partial x^i, \quad i = 1, \ldots, n.
\]

Theorem 2. (L. S. Pontryagin's Maximum Principle). For optimal trajectory \( x(t) \) and control \( u(t) \), \( t_0 \leq t \leq t_1 \), it is necessary to have such variable vector \( \psi(t) \), that the relations (5) will be satisfied and besides with any \( t, \ t_0 \leq t \leq t_1 \), the following condition will be also satisfied

\[
H(x(t), \psi(t), u(t)) = \max_{u \in U} H(x(t), \psi(t), u) \geq 0.
\]

A sketchy proof of this theorem follows. We will choose a vector \( \psi(t) \) as outlined above. We assume that for a certain \( \psi(t), \ t_0 \leq t \leq t_1 \), and a certain \( \psi U \) we will have

\[
H(x(t), \psi(t), u(t)) \leq H(x(t), \psi(t), u).
\]

We will vary the control \( u(t) \) on the single interval \( I_{1,1} \) of the duration \( \varepsilon \) which starts at \( \psi(t) \), and we will assume \( u_{1,1} \equiv 0 \). Then we will have: \( x(t) = x(t) \) with \( t_0 \leq t \leq t_1 \) and

\[
\bar{x}(t) - x(t) = [f(x(t), u(t)) - f(x(t), u(t))] (t - \tau) + \ldots
\]

we will have: \( \bar{x}(t) = x(t) \) with \( t_0 \leq t \leq t_1 \) and

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\[
\bar{x}(t) - x(t) = [f(x(t), u(t)) - f(x(t), u(t))] (t - \tau) + \ldots
\]
Particularly,
\[ x(\tau + \varepsilon \delta t_{1,1}) - x(\tau + \varepsilon \delta t_{1,1}) = [f(x(\tau), u) - f(x(\tau), u(t))] \varepsilon \delta t_{1,1} + \ldots \]
from which it follows, in accordance with definition of \( R \), that
\[ \psi_x(\tau + \varepsilon \delta t_{1,1}) [x^x(\tau + \varepsilon \delta t_{1,1}) - x^x(\tau + \varepsilon \delta t_{1,1})] = A \epsilon + \ldots \]  
where \( \Delta R \).

Since on the interval \( \tau^* + \varepsilon \delta t_{1,1} \leq t \leq t_{1,1} \) the control \( \tilde{u}(t) \) coincides with \( u(t) \), then on this interval we will have:
\[ \dot{x}^x(t) = x^x(t) + \varepsilon \dot{x}^x(t); \quad \ldots \]
\[ \tilde{u}^i(t) = \frac{\partial \tilde{y}^i(t, u(t))}{\partial x^j(t)} \dot{x}^x(t), \quad i = 1, \ldots, n. \]

Thus, with \( \tau^* + \varepsilon \delta t_{1,1} \leq t \leq t_{1,1} \) in accordance with the second part of (5) we will have:
\[ \frac{d}{dt} (\psi_x(t) \dot{x}^x(t)) = \frac{\partial \psi_x(t)}{\partial t} \dot{x}^x(t) + \psi_x(t) \ddot{x}^x(t) = \]
\[ = - \frac{\partial \psi_x(t)}{\partial t} \ddot{x}^x(t) + \psi_x(t) \frac{\partial}{\partial x} \dot{x}^x(t) = 0. \]

Therefore on the whole interval \( \tau^* + \varepsilon \delta t_{1,1} \leq t \leq t_{1,1} \) it holds the relation
\[ \psi_x(t) \dot{x}^x(t) = \psi_x(\tau + \varepsilon \delta t_{1,1}) \dot{x}^x(\tau + \varepsilon \delta t_{1,1}) = A \epsilon + \ldots \]
see (6), and consequently
\[ \psi_x(t_1) [x^x(t_1) - x^x(t_1)] = \psi_x(t_1) [x^x(t_1) + \ldots] = A \epsilon + \ldots \]

Hence, the vector \( \Delta x = \dot{x}^x(t_1) \), derived according to formulas (2) and (3) with \( \delta t = 0 \), what is permissible, satisfies the condition
\[ \psi_x(t_1) \Delta x^x = A \epsilon ; \] but this contradicts the condition (4).

Closely related to the Maximum Principle is the following important property of the function \( H \), which states that along the optimal trajectory \( H \geq 0 \), i.e. the maximum, the existence of which was proven, is non-negative.

Theorem 3. If \( x(t) \) is an optimal trajectory, \( u(t) ; t_0 \leq t \leq t_1 \) is an optimal control, and \( \psi(t) \) is a variable vector, the existence of which is confirmed in theorem 2, then the function \( H \) maintains along \( x(t), \psi(t), u(t) \) a constant non-negative value:
\[ H(x(t), \psi(t), u(t)) = \text{const}. \]

This relation is proven first for every segment of piece-wise continuous control \( u(t) \). Subsequently it is proven that at every point of discontinuity of the control \( u(t) \) the function \( H \) is not changing its value.

These results are obtained in a seminar on the theory of oscillations and automatic control personally conducted by L. S. Pontryagin. L. S. Pontryagin pointed out to me one possible simplification in my derivation of the Maximum Principle proof. Owing to that suggestion my proof, as it is now, holds for any arbitrary topological space \( U \); when my original variant had a redundant construction not used for any useful purpose yet limiting the proof to the case of \( U \) being bounded domain of the vector space with piece-wise-smooth frontier and convex interior angles at the joints.
V. A. Steklov's Institute of Mathematics of the Academy of Sciences USSR.

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