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by George H. Goedecke, Michael DeAntonio, and Harry J. Auvermann

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First-Order Acoustic Wave Equations and Scattering by Atmospheric Turbules

George H. Goedecke and Michael DeAntonio
New Mexico State University

Harry J. Auvermann
Information Systems Technology Directorate, ARL

Abstract

A new turbulence model is used to describe the acoustical scattering from atmospheric turbulence. A complete set of fluid equations, including the heat flow equation with zero conductivity, is presented for an ideal gas atmosphere. From this set, a complete set of coupled linear differential equations is derived for the acoustic pressure, temperature, mass density, and velocity in the presence of stationary turbulence. From these acoustic wave equations, expressions for acoustic scattering cross sections are derived for individual localized stationary scalable turbules of arbitrary morphology and orientation. Averages over random turbule orientations are also derived. Criteria for comparability of orientationally averaged turbules with different envelope functions are presented and applied, and cross sections for Gaussian and exponential envelopes are compared. The azimuthal dependence of the velocity scattering cross section for a spherically symmetric nonuniformly rotating turbule is illustrated. It is shown that, for incoherent scattering, a collection of randomly oriented turbules of arbitrary morphology may be replaced by an "equivalent" collection of spherically symmetric, nonuniformly rotating turbules with randomly directed rotation axes.

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1. Introduction

The motivation for the work described in this report was to investigate the utility of a new model of atmospheric turbulence when used to predict scattering of acoustical signals by the turbulent atmosphere. Much of the information in this report we provide in a journal article [1], in which we emphasize the aspects of the work strictly related to acoustical scattering. In this report, however, we also include aspects of the work not emphasized in the article. For example, as part of the work, we examined in detail the fluid flow equations to establish the relative order of magnitude of the various terms retained when the acoustic wave equation is derived; for the most part, we did not include this analysis in the journal article. We also derived the properties of structure functions predicted by the new turbulence model; we included in the article only a minor portion of these findings. Instead, the structure function findings are recounted in a companion technical report [2]. In short, all our findings are included in this report and its companion [2], but only those parts closely connected to acoustical scattering are included in the journal article [1].

Several different approximate wave equations for acoustic pressure or acoustic potential have been derived to describe acoustics in a turbulent atmosphere [3–11]. Several of these equations are valid only to first order in stationary turbulent temperature variation and/or velocity. These include Monin's equation [3] and its short-wavelength limit [4], and Abdullayev's and Ostashev's [8] and Ostashev's [10] generalizations of Monin's equation to include water vapor and other additives and nonstationarity of turbulence. These first-order wave equations usually provide an adequate basis for describing linear atmospheric acoustics because of two general features of turbulent atmosphere: first, the ratio of turbulent velocity to acoustic wavespeed and that of turbulent temperature variation to ambient temperature are generally very small for the atmosphere, and second, turbulent time scales are generally much longer than acoustic time scales. Section 2 presents the basic theory of acoustic wave equations and the scattering properties of turbulence. Section 3 develops the scattering properties of model velocity and temperature distributions.

Section 2 of this report has four purposes. First, we provide a complete set of fluid equations in terms of pressure, density, temperature, and fluid velocity, and a complete set of coupled linear differential equations that describe linear acoustics in the presence of turbulence in an ideal gas atmosphere. Second, we show that Monin's first-order acoustic wave equation results from this set. Third, we provide a coherent development of general properties of the acoustic scattering predicted by these wave equations, properties that are independent of the detailed morphology of the turbulent velocity, density, pressure, and temperature fields, but dependent on the wavelength and the length scales of the turbulence and the scattering volume. Fourth, we establish the basic formulas to be used below and in succeeding reports that will treat acoustic scattering by individual localized turbules (or *vortons*) and by ensembles of such turbules.

Section 2.1 contains a derivation of a complete set of basic fluid equations for an ideal gas. In section 2.2, we assume a stationary turbulence model, derive equations relating the turbulent fields, and then derive the desired linear acoustic differential equations. We then show that Monin's equation results in the first-order limit. In section 2.3, we develop the general expressions for the Born approximation scattering amplitudes and cross sections predicted by the equations considered. We show that the length scale a_s of the scattering volume is of central importance in predicting the general behavior of the acoustic scattering at wavelength λ by turbulence of scale length a. For example, we show that the short-wavelength limit of Monin's equation should be used only if both $a \gg \lambda$ and $a \geq a_s$, and that the standard predictions from Monin's equation are valid for any ratio a/λ only if $a \ll a_s$. We also obtain the interesting general result that the first Born forward and backward scattering due to solenoidal turbulent flow velocity is essentially zero if $a \ll a_s$, but nonzero if $a \geq a_s$.

In contrast with the turbulence model presented here, standard treatments of the scattering of sound waves by atmospheric turbulence have relied on describing the turbulence entirely in terms of the autocorrelation or structure functions of the turbulent temperature and velocity fields [4,7,10]. This approach has been particularly successful for isotropic, homogeneous, fully developed turbulence. It has also been used to describe scattering by inhomogeneous and/or anisotropic turbulence [12–14]. Turbulence has also been described as collections of individual localized eddies or vortices, often called *turbules* [15,16]. Section 3 describes research aiming at the description of scattering and propagation of sound waves in anisotropic inhomogeneous atmospheric turbulence.

Section 3 considers acoustic scattering by individual turbules. Specifically, we describe a general scalable localized static turbule model, valid for solenoidal turbule velocities and arbitrary morphology. In section 3.1, we use the results of section 2 to obtain general expressions for the acoustic scattering amplitudes and cross sections due to scattering by the velocity and temperature variation fields of such a turbule, in first Born approximation. Then in section 3.2, we describe general expressions for the isotropic ensemble averages of these cross sections. Criteria of comparability for ensembles having different envelope functions are specified: namely, they should have the same size and the same kinetic and thermal energy content. We present two examples, with Gaussian and exponential envelopes, respectively, and contrast their differential and total scattering cross sections. In section 3.3, we also derive the azimuthally dependent velocity scattering cross section for a simple rotating turbule with a spherically symmetric Gaussian envelope. In addition, we prove a general theorem: for many applications, an ensemble of randomly oriented turbules of arbitrary morphology may be replaced by an "equivalent" ensemble of simple turbules, each of which has spherical symmetry but rotates nonuniformly about a fixed but randomly directed axis. In section 4, we briefly summarize and discuss our results.

2. Basic Theory

2.1 Fluid Equations

In descriptions of acoustic propagation, the atmosphere is most often treated as an ideal gas. That is, viscosity and thermal conductivity are neglected; the fluid is assumed to be in local thermal equilibrium. The fundamental fluid equations for such a gas are the mass conservation equation, the Euler equation, the ideal gas equation of state, and the thermal energy balance or heat flow equation without the heat conduction term:

$$D_t \rho + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{1}$$

$$D_t \mathbf{v} + \rho^{-1} \boldsymbol{\nabla} p - \mathbf{g} = 0, \tag{2}$$

$$p = \rho k_{\mathrm{B}} T / M, \tag{3}$$

$$D_t U + (p + U) \boldsymbol{\nabla} \cdot \mathbf{v} = 0. \tag{4}$$

Here, $\rho(\mathbf{r}, t)$ is the mass density, $p(\mathbf{r}, t)$ is the pressure, $T(\mathbf{r}, t)$ is the temperature, $U(\mathbf{r}, t)$ is the internal energy density, $\mathbf{v}(\mathbf{r}, t)$ is the fluid velocity, k_{B} is Boltzmann's constant, $\mathbf{g}$ is the vector of acceleration of gravity, and M is the average molecular mass, which is constant for the assumed homogeneously mixed atmosphere. The operator D_t is the convective derivative,

$$D_t = \partial_t + \mathbf{v} \cdot \boldsymbol{\nabla}. \tag{5}$$

The heat flow equation does not seem to have been used in previous treatments of acoustic propagation.

For a fluid in local thermal equilibrium,

$$U = \frac{\rho}{M} F \frac{k_{\mathrm{B}} T}{2}, \tag{6}$$

where F is the number of degrees of freedom per molecule excited at temperature T. We have

$$c_v = (\partial U / \partial T)_\rho \;=\; \rho k_{\mathrm{B}} F / 2M, \quad c_p = c_v + \rho k_{\mathrm{B}} / M, \quad \gamma = c_p / c_v = 1 + (2/F). \tag{7}$$

Combining equations (3), (4), (6), and (7) yields

$$D_t p + \gamma p \boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{8}$$

$$D_t T + (\gamma - 1) T \boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{9}$$

as alternative forms of the heat flow equation. Furthermore, combining equations (1) and (8) yields

$$D_t \ln(p/\rho^\gamma) = 0, \tag{10}$$

which is clearly the equation for isentropic flow, as given by Tatarskii [7]. Thus, we could reverse our derivation and start from equations (1–3) and (10), which would then yield equation (8). The set of equations (1–3) and (8) or (1–3) and (10) is complete, since it includes six equations for the six fields (ρ, p, T, v_i).

2.2 Acoustic Equations

2.2.1 Linear Acoustics

For linear acoustics, the standard approach is to write each field as a sum of a turbulent part and an acoustic part, and retain terms in the resulting equations only to first order in the presumably small acoustic part. Because the time scales of the turbulent flow are much longer than those of the acoustic fields, the time dependence of the former is often neglected; we do that here. We define

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r})\big(1 + \epsilon(\mathbf{r}, t)\big), \tag{11}$$

$$p(\mathbf{r}, t) = p_0(\mathbf{r})\big(1 + \eta(\mathbf{r}, t)\big), \tag{12}$$

$$T(\mathbf{r}, t) = T_0(\mathbf{r})\big(1 + \delta(\mathbf{r}, t)\big), \tag{13}$$

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_0(\mathbf{r}) + \mathbf{u}(\mathbf{r}, t), \tag{14}$$

where the fields with subscripts $_0$ represent the turbulent flow, and unsubscripted fields represent the acoustic flow. We also take the turbulent flow to be solenoidal,

$$\boldsymbol{\nabla} \cdot \mathbf{v}_0 = 0, \tag{15}$$

as is usually done [7]. Then substitution of equations (11–15) into equations (1–3) and (8) yields both zero-order equations relating the turbulent fields, and acoustic equations linear in $(\epsilon, \eta, \delta, \mathbf{u})$. The zero-order equations are

$$\mathbf{v}_0 \cdot \boldsymbol{\nabla} \rho_0 = 0, \quad \mathbf{v}_0 \cdot \boldsymbol{\nabla} p_0 = 0, \quad p_0 = \rho_0 k_{\mathrm{B}} T_0 / M, \tag{16}$$

$$\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \rho_0^{-1} \nabla p_0 - \mathbf{g} = 0. \tag{17}$$

The acoustic equations are

$$\begin{aligned} \partial_t \epsilon + \mathbf{v}_0 \cdot \nabla \epsilon + \mathbf{u} \cdot \nabla \ln \rho_0 + \nabla \cdot \mathbf{u} &= 0, & (18) \\ \partial_t \eta + \mathbf{v}_0 \cdot \nabla \eta + \mathbf{u} \cdot \nabla \ln p_0 + \gamma \nabla \cdot \mathbf{u} &= 0, & (19) \end{aligned}$$

$$\partial_t \mathbf{u} + \mathbf{v}_0 \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{v}_0 + \gamma^{-1} c_0^2 \nabla \eta + \delta \nabla p_0 / \rho_0 = 0, \tag{20}$$

$$\delta = \eta - \epsilon, \tag{21}$$

where we have defined the adiabatic sound speed $c_0(\mathbf{r})$ by

$$c_0^2 \equiv \gamma p_0 / \rho_0 = \gamma k_{\mathrm{B}} T_0 / M. \tag{22}$$

Combining equations (18) and (19) yields an alternative equation relating η, ϵ, and $\mathbf{u}$:

$$(\partial_t + \mathbf{v}_0 \cdot \nabla)(\eta - \gamma \epsilon) + \mathbf{u} \cdot \nabla \ln(p_0 / \rho_0^\gamma) = 0. \tag{23}$$

This shows clearly that $\eta \neq \gamma \epsilon$ unless p_0/ρ_0^γ is uniform, which is not the case in general.

The acoustic equations (18–21) are a complete set for the acoustic fields $(\epsilon, \eta, \delta, \mathbf{u})$, provided $\big(T_0(\mathbf{r}),\ \mathbf{v}_0(\mathbf{r})\big)$ are given and equations (16) and (17) and boundary conditions are used to obtain $\big(p_0(\mathbf{r}),\ \rho_0(\mathbf{r})\big)$.

In what follows, we describe situations in which the turbulence is localized in some bounded volume V_T. Outside V_T, the pressure, temperature, and mass density take on constant uniform values $(p_\infty, T_\infty, \rho_\infty)$, the flow velocity $\mathbf{v}_0 = 0$, and the adiabatic sound speed c_∞ is defined by

$$c_\infty^2 = \gamma p_\infty / \rho_\infty = \gamma k_{\mathrm{B}} T_\infty / M. \tag{24}$$

2.2.2 First-Order Acoustic Equations

In the atmosphere, v_0/c_∞ and $|T_0 - T_\infty|/T_\infty$ are usually of the same order, both much less than unity, so it is a good approximation to discard terms from the acoustic equations that are second order or higher in these ratios. The principal effect of gravity in the atmosphere is to produce internal gravity waves, which we neglect. Thus equation (17) shows that ∇p_0 is a second-order quantity. Dropping terms containing ∇p_0 from the acoustic equations (18–20) leaves only two relevant coupled equations, namely, the first-order equations (19) and (20). Putting in time dependence $\exp(-i\omega t)$ yields

$$\begin{aligned} i\omega\eta &= \gamma \nabla \cdot \mathbf{u} + \mathbf{v}_0 \cdot \nabla \eta, & (25) \\ i\omega\mathbf{u} &= \gamma^{-1} c_0^2 \nabla \eta + \mathbf{v}_0 \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{v}_0. & (26) \end{aligned}$$

It is simple to eliminate $\mathbf{u}$ from these equations and get a wave equation for η valid to first order in v_0/c_∞. First, replace $\mathbf{u}$ by $(i\omega\gamma)^{-1}c_\infty^2\boldsymbol{\nabla}\eta$ in the last two terms in equation (26); this is valid to first order. Then evaluate $\gamma\boldsymbol{\nabla}\cdot\mathbf{u}$ and substitute it in equation (25). This procedure yields

$$\begin{aligned}(\boldsymbol{\nabla}^2+k^2)\eta &= \boldsymbol{\nabla}\cdot[(\Delta T_0/T_\infty)\boldsymbol{\nabla}\eta] \\ &\quad + (ik/c_\infty)\{-\mathbf{v}_0\cdot\boldsymbol{\nabla}\eta + k^{-2}\boldsymbol{\nabla}\cdot[\mathbf{v}_0\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\eta+\boldsymbol{\nabla}\eta\cdot\boldsymbol{\nabla}\mathbf{v}_0]\},\end{aligned} \tag{27}$$

where

$$k \equiv \omega/c_\infty, \quad \Delta T_0/T_\infty = 1 - T_0/T_\infty = 1 - c_0^2/c_\infty^2. \tag{28}$$

Rearranging terms on the right hand side of equation (27), replacing $\boldsymbol{\nabla}^2\eta$ by $-k^2\eta$ in those terms, and using equation (15) yields Monin's equation for η, written here in summation notation with $\partial_i \equiv \partial/\partial x_i$:

$$(\boldsymbol{\nabla}^2+k^2)\eta = \partial_i[(\Delta T_0/T_\infty)\partial_i\eta] + 2i\omega^{-1}\partial_i(v_{0j}\partial_j\partial_i\eta). \tag{29}$$

This is the accepted wave equation for the acoustic pressure variation that is valid to first order in the small stationary quantities $(v_0/c_\infty,\ \Delta T_0/T_\infty)$. The same equation is valid to this order for $p' \equiv p_0\eta$, since $\boldsymbol{\nabla}p_0$ is second order. This first-order equation is valid for all turbulent length scales a. If these length scales are all very large compared to the acoustic wavelengths ($ka \gg 1$), then derivatives of ΔT_0 and $\mathbf{v}_0$ may be neglected; then equation (29) reduces to

$$(\boldsymbol{\nabla}^2+k^2)\eta \approx -k^2(\Delta T_0/T_\infty)\eta - (2ik/c_\infty)v_{0j}\partial_j\eta, \tag{30}$$

which is the same as the first-order wave equation given by Tatarskii [4].

Note that our formulation uses a constant reference temperature T_∞ as the temperature outside V_T. Conventional treatments use the average temperature $\overline{T}$ in V_T as the reference, which might or might not be equal to T_∞.

2.3 Scattering

2.3.1 General Formulation

Equations (29) and (30) are in the generic form

$$(\boldsymbol{\nabla}^2+k^2)\eta(\mathbf{r}) = -4\pi S(\mathbf{r})\eta(\mathbf{r}) \tag{31}$$

where $S(\mathbf{r})$ is an operator that is nonzero only inside a bounded volume V_T, or goes to zero rapidly outside this volume. The operator S may contain derivatives that operate on $\eta(\mathbf{r})$. This standard form allows a causal Green's function implicit solution for $\eta(\mathbf{r})$, given an incident plane wave

$$\eta_{in}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{32}$$

or any other incident wave that satisfies the Helmholtz equation. For an incident plane wave, the implicit solution is

$$\eta(\mathbf{r}_1) = \exp(i\mathbf{k}\cdot\mathbf{r}_1) + \int d^3r_2 r_{12}^{-1} \exp(ikr_{12}) S(\mathbf{r}_2)\eta(\mathbf{r}_2), \tag{33}$$

where the scattered wave $\eta_s(\mathbf{r}_1)$ is given by the integral. Here, and in what follows, $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$, and three-dimensional spatial integrals with no limits indicated involve integration over all space. Actually, the integration need extend only over the support V_T of $S(\mathbf{r}_2)$.

In the far field, the scattered wave has the form

$$\eta_s(\mathbf{r}) = r^{-1}e^{ikr} f(\hat{\mathbf{r}}), \tag{34}$$

where the scattering amplitude $f(\hat{\mathbf{r}})$ for the scattering volume V_s' is given by

$$f(\hat{\mathbf{r}}) = \int_{V_s'} d^3r_1 \exp(-ik\hat{\mathbf{r}}\cdot\mathbf{r}_1) S(\mathbf{r}_1)\eta(\mathbf{r}_1), \tag{35}$$

where $\hat{\mathbf{r}} \equiv \mathbf{r}/r$ is the radially outward unit vector.

The scattering volume V_s' is that portion of the volume V_T that is observed by the detector. In this report, we assume that the detector is in the far field of V_s'. The criteria for this are

$$r \gg (a_s')^2/\lambda, \quad r \gg \lambda, \tag{36}$$

whichever is greater, where a_s' is the length scale of V_s' and $\lambda = 2\pi/k$ is the acoustic wavelength. In actual experimental situations, these criteria are seldom fulfilled.

The differential and total scattering cross sections ($\sigma(\hat{\mathbf{r}})$, σ_s) of the turbulence are given respectively by

$$\sigma(\hat{\mathbf{r}}) = |f(\hat{\mathbf{r}})|^2, \qquad \sigma_s = \int d\Omega \sigma(\hat{\mathbf{r}}) \tag{37}$$

where the integration extends over 4π sr (solid angle).

2.3.2 Born Approximation

In general, the correct acoustic wave equation would involve an operator

$$S = \Sigma_n S_n, \tag{38}$$

where S_n is n^{th} order in the small quantities $(v_0/c_\infty,\ \Delta T_0/T_\infty)$. However, the wave equations (29) and (30) include only S_1, so it makes no sense to go beyond the first Born approximation in solving for the scattering amplitude $f(\hat{\mathbf{r}})$.

Scattering amplitudes. We get the first Born scattering amplitude by inserting the incident (plane) wave (eq (32)) in the integrand of equation (35):

$$f(\hat{\mathbf{r}}) = \int_{V_s} d^3 r_1 \exp(-ik\hat{\mathbf{r}} \cdot \mathbf{r}_1) S_1(\mathbf{r}_1) \exp(i\mathbf{k} \cdot \mathbf{r}_1). \tag{39}$$

Here, the volume V_s is now determined by the intersection of the portion V_s' of V_T observed by the detector, and the portion V_I of V_T directly illuminated by the source. If $V_I < V_T$, then $V_s < V_s'$. The volume V_s might be regarded as the intersection volume of two cones, the detector cone of view and the source cone, as discussed by Tatarskii [7]. For convenience in what follows, we ascribe a single characteristic length scale a_s to the volume V_s.

For wave equation (29), the operator $S_1(\mathbf{r})$ may be written

$$S_1(\mathbf{r}) = S_{1T}(\mathbf{r}) + S_{1v}(\mathbf{r}), \tag{40}$$

where

$$S_{1T}(\mathbf{r}) \ = \ -\frac{1}{4\pi}\partial_i\Big((\Delta T_0/T_\infty)\partial_i\Big) = -(4\pi T_\infty)^{-1}[(\partial_i \Delta T_0)\partial_i + \Delta T_0 \partial_i \partial_i], \tag{41}$$

$$S_{1v}(\mathbf{r}) \ \equiv \ -(i/2\pi\omega)\partial_i(v_{0j}\partial_j\partial_i) = -(i/2\pi\omega)[(\partial_i v_{0j})\partial_j\partial_i + v_{0j}\partial_j\partial_i\partial_i] \tag{42}$$

are the operators associated with the turbulent temperature variation and velocity, respectively.

It is clear from equation (39) that, for each derivative operator ∂_q that appears on the right of the functions ΔT_0 or v_{0j} in S_1, we get a factor ik_q in $f(\hat{\mathbf{r}})$. However, it is more complicated to describe the general effect of a derivative operator that appears on the left of these functions. It must be correct to substitute equations (41) and (42) directly into equation (39), to get

$$f(\hat{\mathbf{r}}) = f_T(\hat{\mathbf{r}}) + f_v(\hat{\mathbf{r}}), \tag{43}$$

$$f_T(\hat{\mathbf{r}}) = (-k^2/4\pi T_\infty)\int_{V_s} d^3r_1 \exp(-i\mathbf{K}\cdot\mathbf{r}_1)[ik^{-1}\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}_1\Delta T_0(\mathbf{r}_1) - \Delta T_0(\mathbf{r}_1)], \tag{44}$$

$$f_v(\hat{\mathbf{r}}) = (-k^2/2\pi c_\infty)\int_{V_s} d^3r_1 \exp(-i\mathbf{K}\cdot\mathbf{r}_1)[-ik^{-1}\hat{k}_i\hat{k}_j\partial_{1i}v_{0j}(\mathbf{r}_1) + \hat{k}_j v_{0j}(\mathbf{r}_1)], \tag{45}$$

where

$$\mathbf{K} \equiv k\hat{\mathbf{r}} - \mathbf{k}. \tag{46}$$

But then we cannot draw conclusions about the general behavior of $f(\hat{\mathbf{r}})$ that are independent of the detailed morphology of the turbulence. On the other hand, it must also be correct to integrate by parts in equation (39) for $f(\hat{\mathbf{r}})$. For the S_{1T} of equation (41), this yields

$$\begin{aligned} f_T(\hat{\mathbf{r}}) &= (k^2/4\pi T_\infty)\hat{k}_i\hat{r}_i\int_{V_s} d^3r_1\Delta T_0(\mathbf{r}_1)\exp(-i\mathbf{K}\cdot\mathbf{r}_1) \\ &\quad -(i/4\pi T_\infty)k_i\oint_{\Sigma_s} d\Sigma_i[\Delta T_0(\mathbf{r}_1)\exp(-i\mathbf{K}\cdot\mathbf{r}_1)]_{\Sigma_s}, \end{aligned} \tag{47}$$

where the surface integral extends over the surface Σ_s bounding the volume V_s.

Consider three cases, as follows:

Case i. All the length scales a of the localized turbulent eddies that are totally or partially included in V_s are much less than a_s. This is usually the case for inhomogeneities in the inertial range.

Imagine that a turbulent eddy or turbule is localized in a volume a^3 "centered" about a random location $\mathbf{b}$. Then, for $a \ll a_s$, the fields $(\Delta T_0,\ v_0)$ of that turbule are zero on the surface Σ_s for almost all $\mathbf{b}$. Thus, the surface integral in equation (47) is negligibly small compared to the volume integral, and the scattering amplitude is given by just the first term of equation (47) to good approximation:

$$f_T(\hat{\mathbf{r}}) = (k^2/4\pi T_\infty)\hat{k}_i\hat{r}_i\int_{V_s} d^3r_1\Delta T_0(\mathbf{r}_1)\exp(-i\mathbf{K}\cdot\mathbf{r}_1). \tag{48}$$

This is proportional to $\hat{k}_i\hat{r}_i = \cos\theta$, where θ is the scattering angle. This result is valid for all ka, for $a \ll a_s$. The corresponding velocity scattering amplitude for this case is

$$f_v(\hat{\mathbf{r}}) = -(k^2/2\pi c_\infty)(\hat{k}_i\hat{r}_i)\hat{k}_j\int_{V_s} d^3r_1 v_{0j}(\mathbf{r}_1)\exp(-i\mathbf{K}\cdot\mathbf{r}_1), \tag{49}$$

which also displays proportionality to $\cos\theta$. This neglect of the surface integrals is usual in standard treatments, such as that by Tatarskii [7]. Scattering amplitudes and cross-section results based on equations (48) and (49) have previously been given [17] for a particular temperature distribution and a particular velocity distribution.

Case ii. All the length scales a of the turbulent eddies in V_T are greater than a_s, and $a_s \gg \lambda$, so $ka \gg 1$. This is usually the case for large-scale inhomogeneities.

In this case, the surface integral in equation (47) is not negligible; but we can neglect $(\partial_i \Delta T_0,\ \partial_i v_{0j})$ in equations (41) and (42). This is equivalent to using wave equation (30) rather than equation (29). Dropping the first term in the integrand of equation (44), we get

$$f_T(\hat{\mathbf{r}}) \approx (k^2/4\pi T_\infty) \int_{V_s} d^3 r_1 \Delta T_0(\mathbf{r}_1) \exp(-i\mathbf{K}\cdot\mathbf{r}_1) \tag{50}$$

to good approximation. This is the same as equation (48), except for a factor of $\cos\theta$.

The corresponding velocity scattering amplitude for this case is to good approximation

$$f_v(\hat{\mathbf{r}}) \approx -(k^2/2\pi c_\infty)\hat{k}_j \int_{V_s} d^3 r_1 v_{0j}(\mathbf{r}_1) \exp(-i\mathbf{K}\cdot\mathbf{r}_1), \tag{51}$$

which is the same as equation (49) except for a factor of $\cos\theta$.

Case iii. $\lambda \geq a \geq a_s$, so $ka \leq 1$, $ka_s \leq 1$. This case occurs when the detector has a limited field of view and turbule sizes are small because of interference of the ground.

In this case, the surface integral in equation (47) is not negligible compared to the volume integral, and neither is the first term in equation (44) compared to the second. One would thus expect a dip in the scattering near $\theta = 90°$, but not a zero; the scattering amplitude $f_T(\hat{\mathbf{r}})$ of equation (44) directly involves the derivative of ΔT_0 (which it does not for cases (i) and (ii) above) and similarly for $f_v(\hat{\mathbf{r}})$. The full expressions (44) and (45) should be used.

It seems worthwhile to note that, if $a \gg a_s$, then both $\Delta T_0(\mathbf{r})$ and $\mathbf{v}_0(\mathbf{r})$ do not change appreciably within V_s; a turbulent eddy with such a large scale length appears to the observer to be a uniform wind field and a uniform temperature shift over the volume V_s. For such cases, the fields $\Delta T_0(\mathbf{r}_1)$ and

V_{0j} may be taken outside the integrals in equations (44), (45), (50), and (51), and evaluated at any point inside V_s, say at some center point $\mathbf{r}_s$. For example, for very large-scale inhomogeneities, with $a \gg a_s$, $ka \gg 1$, equation (50) reduces to

$$f_T(\hat{\mathbf{r}}) = (k^2/4\pi T_\infty)\Delta T_0(\mathbf{r}_s) \int_{V_s} d^3r_1 \exp(-i\mathbf{K}\cdot\mathbf{r}_1). \tag{52}$$

In summary, the scale length a_s of the scattering volume is at least as important as the scale lengths a of the turbulent eddies and the acoustic wavelength $\lambda = 2\pi/k$ in determining the general behavior of the first Born acoustic scattering amplitudes. If $a \ll a_s$, then for any ka values, equations (48) and (49) result. If $a > a_s$ and $ka \gg 1$, then equations (50) and (51) result, which are the same as equations (48) and (49), except for the absence of the $\cos\theta$ factor, and the same as would be achieved directly if we started from the approximate wave equation (30) instead of from equation (29). If $a \geq a_s$, but the ka are not large, then the complete equations (44) and (45) must be used.

Cross sections. From equations (37) and (43), the first Born differential cross sections are given by

$$\sigma(\hat{\mathbf{r}}) = \sigma_T(\hat{\mathbf{r}}) + \sigma_v(\hat{\mathbf{r}}) + \sigma_{Tv}(\hat{\mathbf{r}}) \tag{53}$$

where

$$\sigma_T(\hat{\mathbf{r}}) = |f_T(\hat{\mathbf{r}})|^2, \quad \sigma_v(\hat{\mathbf{r}}) = |f_v(\hat{\mathbf{r}})|^2, \quad \sigma_{Tv}(\hat{\mathbf{r}}) = f_T(\hat{\mathbf{r}})f_v^*(\hat{\mathbf{r}}) + \text{c. c.} \tag{54}$$

If we are interested in the ensemble average over assumed stochastic homogeneous turbulent fields $\big(T_0(\mathbf{r}),\ \mathbf{v}_0(\mathbf{r})\big)$, and we assume that fluctuations in $T_0(\mathbf{r})$ are uncorrelated to fluctuations in $\mathbf{v}_0(\mathbf{r})$, and $\langle\mathbf{v}_0(\mathbf{r})\rangle = 0$, then $\langle\sigma_{Tv}(\hat{\mathbf{r}})\rangle = 0$.

Consider, for example, the case for $\langle\sigma_T(\hat{\mathbf{r}})\rangle$ for $a \ll a_s$. From equations (54) and (48),

$$\langle\sigma_T(\hat{\mathbf{r}})\rangle = (k^4/16\pi^2T_\infty^2)\cos^2\theta\langle|\Delta\widetilde{T}_0(\mathbf{K})|^2\rangle \tag{55}$$

where

$$\Delta\widetilde{T}_0(\mathbf{K}) \equiv \int_{V_s} d^3r_1 \Delta T_0(\mathbf{r}_1)\exp(-i\mathbf{K}\cdot\mathbf{r}_1) \tag{56}$$

is the Fourier transform of $\Delta T_0(\mathbf{r}_1)$ with respect to the volume V_s. Then for homogeneous turbulence,

$$\langle|\Delta\widetilde{T}_0(\mathbf{K})|^2\rangle = \int_{V_s} d^3r_1 \int_{V_s} d^3r_2 B_T(\mathbf{r}_{12})\exp(-i\mathbf{K}\cdot\mathbf{r}_{12}), \tag{57}$$

where

$$B_T(\mathbf{r}_{12}) \equiv \langle \Delta T_0(\mathbf{r}_1)\Delta T_0(\mathbf{r}_2)\rangle \tag{58}$$

is the autocorrelation function of the turbulent temperature variations. Since $a \ll a_s$, the scale length of $B_T(\mathbf{r}_{12})$ is also much less than a_s. That is, $B_T(\mathbf{r}_{12})$ goes to zero for $r_{12} >$ some a', where $a' \ll a_s$. Therefore, the integration variables $(\mathbf{r}_1, \mathbf{r}_2)$ in equation (57) may be changed to $\mathbf{r} \equiv \mathbf{r}_{12}$, $\mathbf{R} \equiv (\mathbf{r}_1+\mathbf{r}_2)/2$, and the limits on the integral over $\mathbf{r}$ may be changed to all space, while the integral over $\mathbf{R}$ remains over V_s, with negligible error. Thus we get

$$\langle \sigma_T(\hat{\mathbf{r}})\rangle = (k^4/16\pi^2T_\infty^2)\cos^2\theta V_s\tilde{B}_T(\mathbf{K}) \tag{59}$$

where

$$\tilde{B}_T(\mathbf{K}) \equiv \int d^3rB_T(\mathbf{r})\exp(-i\mathbf{K}\cdot\mathbf{r}) \tag{60}$$

is the full Fourier transform of the autocorrelation $B_T(\mathbf{r})$ integrated over all space, to good approximation. This formula is generally used in conventional treatments of homogeneous turbulence [7].

However, consider the opposite limits, $a \gg a_s$, $ka \gg 1$. Then from equations (54) and (52),

$$\langle \sigma_T(\hat{\mathbf{r}})\rangle \approx (k^4/16\pi^2T_\infty^2)B_T(\mathbf{0})|\int_{V_s} d^3r_1\exp(-i\mathbf{K}\cdot\mathbf{r}_1)|^2. \tag{61}$$

This is not only missing the $\cos^2\theta$ factor, but it also has a quite different dependence on $\mathbf{K}$ from that in the expression in equation (59).

In the companion report [2], we will assume that V_s is large enough so that $a \ll a_s$ for all turbulent scale lengths of interest, so that equations (48) and (49) apply. This will allow us to compare results using ensembles of localized turbulent eddies or turbules of different scale lengths with the standard structure function results.

Forward and backward velocity scattering. Consider equation (49) for the Born scattering amplitude due to turbulent velocity in case (i), $a \ll a_s$. In the forward direction $\hat{\mathbf{r}} = \hat{\mathbf{k}}$ or $\mathbf{K} = 0$, we have

$$f_v(\hat{\mathbf{k}}) = (k^2/2\pi c_\infty)(\cos\theta)\hat{k}_j\int_{V_s} d^3rv_{0j}(\mathbf{r}). \tag{62}$$

Consider the following integral:

$$I_j \equiv \int_{V_s} d^3r\partial_i(v_{0i}r_j) = \int_{V_s} d^3rv_{0j}(\mathbf{r}). \tag{63}$$

Since the velocity is solenoidal, $\partial_i v_{0i} = 0$. But by the divergence theorem,

$$I_j = \int_{\Sigma_s} d\Sigma_i (v_{0i} r_j)_{\Sigma_s}. \tag{64}$$

By the same reasoning used earlier for $a \ll a_s$, this surface integral vanishes (is negligibly small) because for almost all positions of the center of any localized eddy, v_{0i} is zero on the bounding surface Σ_s. Therefore, if all $a \ll a_s$, $f_v(\hat{\mathbf{k}}) \approx 0$.

Consider the Fourier transform of $v_{0i}(\mathbf{r})$ in the volume V_s:

$$\tilde{v}_{0i}(\boldsymbol{\kappa}) \equiv \int_{V_s} d^3 r e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}} v_{0i}(\mathbf{r}). \tag{65}$$

Again, because $\partial_i v_{0i} = 0$, we have

$$\kappa_i \tilde{v}_{0i}(\boldsymbol{\kappa}) = i \int_{V_s} d^3 r \partial_i [v_{0i}(\mathbf{r}) e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}}] = i \int_{\Sigma_s} d\Sigma_i [v_{0i}(\mathbf{r}) e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}}]_{\Sigma_s}. \tag{66}$$

For the same reason as before, the surface integral is vanishingly small if $a \ll a_s$. From equation (49), the Born scattering amplitude in the backward direction $\hat{\mathbf{r}} = -\hat{\mathbf{k}}$ is given by

$$f_v(-\hat{\mathbf{k}}) = -(k^2/2\pi c_\infty)(\cos\theta)\hat{k}_i \tilde{v}_{0i}(-2\mathbf{k}), \tag{67}$$

and is therefore zero (negligibly small). This zero result for backward velocity scattering was noted by Tatarskii [7].

On the other hand, for cases (ii) and (iii), $a \geq a_s$, the surface integrals (64) and (66) are not negligibly small, and the results $f_v(\pm\hat{\mathbf{k}}) \approx 0$ do not hold. In fact, for $a \gg a_s$, we have $\hat{k}_j I_j \approx V_s \hat{k}_j v_{0j}(\mathbf{r}_s)$, and $\kappa_i \tilde{v}_{0i}(\boldsymbol{\kappa}) = i v_{0i}(\mathbf{r}_s) \int_{\Sigma'_s} d\Sigma_i [e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}}]_{\Sigma'_s}$. Both are nonzero in general. This is what would be expected if we were observing a volume V_s in which there is a uniform wind field.

3. Scattering from Model Turbules

In section 2, we describe some general properties of acoustical scattering by atmospheric inhomogenieties, which are true regardless of the internal configuration or morphology of the inhomogeneity. In this section, we describe scattering properties of inhomogeneities with prescribed morphology, as well as properties of isotropic ensembles of inhomogeneities with similiar morphology.

We define a turbule as a localized eddy or vortex, characterized by a location $\mathbf{b}$, a scale length a, and flow velocity and temperature fields $(\mathbf{v}_0, \Delta T_0)$. Here, "localized" means that these fields go to zero rapidly for $|\mathbf{r} - \mathbf{b}| > a$.

In actuality, the fields, the location, and the scale length depend on time. As is often done in discussions of interactions of acoustic waves with turbulence, we neglect this time dependence, and instead treat the location and the morphological parameters in $(\mathbf{v}_0, \Delta T_0)$ as stationary but stochastic variables.* The fields $(\mathbf{v}_0, \Delta T_0)$ are assumed independent, and may have arbitrary morphology.

We assume, as usual, that the turbule flow is solenoidal:

$$\nabla \cdot \mathbf{v}_0 = 0 \qquad \longrightarrow \qquad \mathbf{v}_0 = \nabla \times \mathbf{A}_0. \tag{68}$$

Here, $\mathbf{A}_0$ is a vector field.

Since we are interested ultimately in ensembles of self-similar turbules of different scale lengths, we assume that the functional forms of $(\mathbf{A}_0, \ \Delta T_0)$ satisfy

$$\mathbf{A}_0(\mathbf{r}) = a\mathbf{A}(\boldsymbol{\xi}), \qquad \Delta T_0(\mathbf{r}) \equiv T_\infty - T_0(\mathbf{r}) = T(\boldsymbol{\xi}), \text{where } \ \boldsymbol{\xi} \equiv (\mathbf{r} - \mathbf{b})/a. \tag{69}$$

Here, $T_0(\mathbf{r})$ is the temperature field, and T_∞ is the uniform constant reference temperature outside the turbule. These are general scalable forms, chosen for convenience so that the functions $(\mathbf{A}(\boldsymbol{\xi}), T(\boldsymbol{\xi}))$ have the dimensions (velocity, temperature), respectively. Then equation (68) yields

$$\mathbf{v}_0(\mathbf{r}) = \nabla_\xi \times \mathbf{A}(\boldsymbol{\xi}). \tag{70}$$

*This means that Doppler shifts cannot be described.

3.1 General Expressions

We use the expressions for the first Born scattering amplitudes derived in section 2, equations (48) and (49), for $a \ll a_s$, where a_s is the scale length of the scattering volume:

$$\begin{aligned} f_T(\hat{\mathbf{r}}) &= -(k^2/4\pi T_\infty)\cos\theta\Delta\widetilde{T}_0(\mathbf{K}), && (71)\\ f_v(\hat{\mathbf{r}}) &= -(k^2/2\pi c_\infty)\cos\theta\,\hat{\mathbf{k}}\cdot\tilde{\mathbf{v}}_0(\mathbf{K}). && (72)\end{aligned}$$

Here, $(\Delta\widetilde{T}_0(\mathbf{K}),\ \tilde{\mathbf{v}}_0(\mathbf{K}))$ are the Fourier transforms of $(\Delta T_0(\mathbf{r}),\ \mathbf{v}_0(\mathbf{r}))$, respectively, and for acoustic wave time dependence $\exp(-i\omega t)$,

$$k = \omega/c_\infty,\ \mathbf{K} = k(\hat{\mathbf{r}} - \hat{\mathbf{k}}),\ K = 2k\sin(\theta/2), \tag{73}$$

where θ is the (polar) scattering angle, that is, the angle between the incident plane-wave propagation direction $\hat{\mathbf{k}}$ and the observation direction $\hat{\mathbf{r}}$. Also,

$$c_\infty = (\gamma k_{\mathrm{B}} T_\infty/M)^{\frac{1}{2}} \tag{74}$$

is the adiabatic sound speed outside the turbule; here, γ is the specific heat ratio, M is the mean molecular mass, and k_{B} is Boltzmann's constant. The Fourier transforms are defined by

$$\Delta\widetilde{T}_0(\mathbf{K}) \equiv \int d^3r_1 \Delta T_0(\mathbf{r}_1)e^{-i\mathbf{K}\cdot\mathbf{r}_1},\ \ \tilde{\mathbf{v}}_0(\mathbf{K}) \equiv \int d^3r \mathbf{v}_0(\mathbf{r}_1)e^{-i\mathbf{K}\cdot\mathbf{r}_1}, \tag{75}$$

where the integrals here have been extended over all space, since $a \ll a_s$.

Combining equations (68–75) yields the following general expressions for the scattering amplitudes for a turbule localized around $\mathbf{r} = \mathbf{b}$:

$$f_{T,v}(\hat{\mathbf{r}}, \mathbf{b}) = e^{-i\mathbf{K}\cdot\mathbf{b}} f_{T,v}(\hat{\mathbf{r}}), \tag{76}$$

where

$$\begin{aligned} f_T(\hat{\mathbf{r}}) &= -(k^2a^3/4\pi T_\infty)\cos\theta\widetilde{T}(\mathbf{K}a), && (77)\\ f_v(\hat{\mathbf{r}}) &= -i(k^3a^4/2\pi c_\infty)\sin\theta\cos\theta\big[\hat{\boldsymbol{\varphi}}\cdot\widetilde{\mathbf{A}}(\mathbf{K}a)\big] && (78)\end{aligned}$$

are the scattering amplitudes for a turbule localized around the origin.

The scaled Fourier transforms $(\widetilde{T}, \widetilde{\mathbf{A}})$ are given by

$$\widetilde{T}(\mathbf{y}) \equiv \int d^3\xi T(\boldsymbol{\xi})e^{-i\boldsymbol{\xi}\cdot\mathbf{y}},\ \ \widetilde{\mathbf{A}}(\mathbf{y}) \equiv \int d^3\xi \mathbf{A}(\boldsymbol{\xi})e^{-i\boldsymbol{\xi}\cdot\mathbf{y}}, \tag{79}$$

and

$$\hat{\boldsymbol{\varphi}} \equiv -\mathbf{e}_1 \sin\varphi + \mathbf{e}_2 \cos\varphi \tag{80}$$

is a unit vector in the direction of increasing azimuthal scattering angle φ. The $\mathbf{e}_i$ ($i = 1, 2, 3$) are Cartesian basis vectors.

Note the $\sin\theta\cos\theta$ dependence in f_v; this occurs for arbitrary turbule morphology, and implies that the first Born scattering is zero in the forward and backward directions, as well as at $\theta = \pi/2$, since $\widetilde{\mathbf{A}}(\mathbf{y})$ is bounded everywhere for a localized bounded $\mathbf{A}(\boldsymbol{\xi})$. That $f_v(\hat{\mathbf{k}}) = f_v(-\hat{\mathbf{k}}) = 0$ was shown in section 2 to be a general consequence for localized solenoidal turbulent velocities for $a \ll a_s$; equations (68–78) constitute another general proof of that result.

Another useful expression for f_v results from using equation (73) in equation (78):

$$f_v(\hat{\mathbf{r}}) = -i(k^2a^3/2\pi c_\infty)\cos\theta\cos(\theta/2)(Ka)(\hat{\boldsymbol{\varphi}}\cdot\widetilde{\mathbf{A}}(\mathbf{K}a)). \tag{81}$$

The differential scattering cross sections are given, as usual, by

$$\sigma(\hat{\mathbf{r}}) = |f(\hat{\mathbf{r}},\mathbf{b})|^2, \quad f(\hat{\mathbf{r}},\mathbf{b}) = f_v(\hat{\mathbf{r}},\mathbf{b}) + f_T(\hat{\mathbf{r}},\mathbf{b}). \tag{82}$$

Thus

$$\sigma(\hat{\mathbf{r}}) = \sigma_T(\hat{\mathbf{r}}) + \sigma_v(\hat{\mathbf{r}}) + \sigma_{Tv}(\hat{\mathbf{r}}) \tag{83}$$

where

$$\sigma_T(\hat{\mathbf{r}}) = |f_T(\hat{\mathbf{r}})|^2 = (k^4a^6/16\pi^2T_\infty^2)\cos^2\theta|\widetilde{T}(\mathbf{K}a)|^2 \tag{84}$$

$$\sigma_v(\hat{\mathbf{r}}) = |f_v(\hat{\mathbf{r}})|^2 = (k^4a^6/4\pi^2c_\infty^2)\cos^2\theta\cos^2(\theta/2)(Ka)^2|\hat{\boldsymbol{\varphi}}\cdot\widetilde{\mathbf{A}}(\mathbf{K}a)|^2, \tag{85}$$

$$\sigma_{Tv}(\hat{\mathbf{r}}) = f_T(\hat{\mathbf{r}})f_v^*(\hat{\mathbf{r}}) + \text{c. c.} \tag{86}$$

Note that these cross sections are independent of turbule location $\mathbf{b}$. Also note the implicit dependence of σ_v on the azimuthal scattering angle φ.

The total scattering cross sections are the integrals of these over 4π sr (solid angle).

We cannot go further unless we choose specific functional forms for $(T(\boldsymbol{\xi}), \mathbf{A}(\boldsymbol{\xi}))$, or unless we average general forms over orientations and morphologies. In section 3.2, we consider isotropic ensembles of turbules of scale length a. In section 3.3, we choose instead one simple specific form, in order to illustrate the azimuthal dependence of the velocity scattering that must occur (but which disappears for isotropic ensembles). We also use this simple form to show that general isotropic ensembles may often be replaced by ensembles of spherically symmetric, nonuniformly rotating turbules.

3.2 Isotropic Ensembles

In a follow-up report [2] we will compare (1) the scattering produced by isotropic ensembles of turbules of different scale lengths and locations with (2) the scattering predicted by the conventional structure functions of isotropic homogeneous fully developed turbulence. We will need ensemble averages over the orientation parameters of the individual turbules; thus we derive these here.

3.2.1 Orientation Averages

Imagine a static turbule of scale length a, such that in a "primed" frame of reference (its "rest frame"), it has arbitrary morphology. A subensemble is formed by the application of different static rotations $R(\psi)$ to this generic form, with all rotations equally probable for an isotropic case. Here $\psi \equiv (\psi_1, \psi_2, \psi_3)$ are the three Euler angles that characterize a rotation. Consider first the temperature field. In the primed frame, we may write in general

$$|\widetilde{T}'(\mathbf{y}')|^2 = \Sigma_{\ell m} a_{\ell m}(y) Y_{\ell m}(\theta', \varphi') = |\widetilde{T}(\mathbf{y})|^2, \tag{87}$$

where the last equality follows because of scalarity, and the sum is a generic expansion in spherical harmonics in the primed frame, with given but arbitrary $a_{\ell m}(y)$. But we know that the spherical harmonics in the two frames are related by

$$Y_{\ell m}(\theta', \varphi') = \Sigma_{m'} D^{\ell}_{mm'}(\psi) Y_{\ell m'}(\theta, \varphi), \tag{88}$$

where the $D^{\ell}_{mm'}(\psi)$ are the irreducible representations of $R(\psi)$ for integer ℓ. For all rotations equally probable, we then want

$$\left\langle D^{\ell}_{mm'} \right\rangle_{\psi} = (8\pi^2)^{-1} \int_0^{2\pi} d\psi_1 \int_0^{2\pi} d\psi_3 \int_0^{\pi} d\psi_2 \sin\psi_2 D^{\ell}_{mm'}(\psi) = \delta_{\ell,0}\delta_{m,0}\delta_{m',0}. \tag{89}$$

Thus,

$$\left\langle |\widetilde{T}(\mathbf{y})|^2 \right\rangle_{\psi} = a_{00}(y) \equiv \pi^3 (\delta T)^2 \widetilde{B}^2_T(y), \tag{90}$$

which is a function of $y = |\mathbf{y}|$ only. Here, δT is a temperature variation amplitude parameter. We do not specify the dimensionless real-valued envelope function $\widetilde{B}_T(y)$ at this time; we present two examples (Gaussian and exponential) in section 3.2.3.

Consider now the velocity field. For an isotropic ensemble, each component of an otherwise unrestricted vector field must be statistically independent and have zero mean, and each component must obey the same statistics. Thus we may put

$$\begin{aligned}\langle \tilde{A}_i(\mathbf{y})\rangle_\psi &= 0,\\ \langle \tilde{A}_i(\mathbf{y})\tilde{A}_j^*(\mathbf{y})\rangle_\psi &= \tfrac{1}{3}\pi^3\delta_{ij}v^2\widetilde{B}_v^2(y),\end{aligned} \tag{91}$$

where v is a velocity amplitude parameter. The envelope function $\widetilde{B}_v(y)$ may be different from $\widetilde{B}_T(y)$. It is important to note that equation (91) yields the usual form for $\langle \tilde{v}_{0i}(\mathbf{K})\tilde{v} *_{0i} (\mathbf{K})\rangle \equiv \Phi_{ij}(\mathbf{K})$, namely,

$$\Phi_{ij}(\mathbf{K}) = (\delta_{ij} - \hat{K}_i\hat{K}_j) \text{ (function of } |\mathbf{K}|). \tag{92}$$

This follows because of equation (70).

From equations (84) to (86), we have, for an isotropic ensemble of scale length a,

$$\overline{\sigma}_{Tv}(\hat{\mathbf{r}}) = 0, \tag{93}$$

$$\overline{\sigma}_T(\hat{\mathbf{r}}) = (\pi a^2)(\delta T/4T_\infty)^2(ka)^4\cos^2\theta\widetilde{B}_T^2(Ka), \tag{94}$$

$$\overline{\sigma}_v(\hat{\mathbf{r}}) = \tfrac{1}{3}(\pi a^2)(v/2c_\infty)^2(ka)^4\cos^2\theta\cos^2(\theta/2)(Ka)^2\widetilde{B}_v^2(Ka). \tag{95}$$

The implicit φ-dependence of $\sigma_v(\hat{\mathbf{r}})$ in equation (85) for an individual turbule is eliminated by the isotropic averaging process.

3.2.2 Comparable Turbules

It is important to be able to compare the scattering produced by ensembles of the same scale length, but different morphologies (i.e., ensembles characterized by different envelope functions $\widetilde{B}(y)$). In this report, we choose comparable turbules to be those having the same rms radius and the same average energy content.

One way to define an rms radius is to write

$$a_{rms}^2 = a^2\int d^3\xi\xi^2 B(\xi)/\int d^3\xi B(\xi), \tag{96}$$

where $B(\xi)$ is the inverse Fourier transform of $\widetilde{B}(y)$,

$$B(\xi) = (2\pi)^{-3}\int d^3y e^{i\boldsymbol{\xi}\cdot\mathbf{y}}\widetilde{B}(y). \tag{97}$$

This is equivalent to

$$(a_{rms}/a) = \Big[-\nabla_y^2\widetilde{B}(y)/\widetilde{B}(y)\Big]_{y=0}^{\frac{1}{2}}. \tag{98}$$

We adopt this definition.

The kinetic energy content of the ensemble average turbule of scale length a, to lowest order in $\mathbf{v}_0$ (second order), is proportional to

$$\int d^3r\langle v_0^2(\mathbf{r})\rangle. \tag{99}$$

Substitution of equations (69), (79), and (91) shows, after a little algebra, that this integral is proportional to

$$\mathcal{E}_v \equiv \tfrac{1}{3}v^2(2\pi)^{-3/2}\int d^3y y^2 \widetilde{B}_v^2(y). \tag{100}$$

We take the excess internal (thermal) energy content of the ensemble average temperature turbule to be proportional to

$$\mathcal{E}_{\mathrm{T}} \equiv \left[(8/(2\pi)^{3/2})\int d^3\xi\langle T^2(\boldsymbol{\xi})\rangle\right]^{\frac{1}{2}} = \left[(8/(2\pi)^{9/2})\int d^3y\langle|\widetilde{T}(y)|^2\rangle\right]^{\frac{1}{2}}. \tag{101}$$

3.2.3 Examples

(a) *Gaussian envelope.* For convenience, we let

$$\widetilde{B}_T(y) = \widetilde{B}_v(y) \equiv \widetilde{B}_g(y) = e^{-y^2/4}. \tag{102}$$

Such forms have been used in the literature [15]. Equations (98), (101), and (102) then yield

$$a_{rms}/a = \sqrt{3/2}, \quad \mathcal{E}_{Tg} = \delta T_g, \quad \mathcal{E}_{vg} = v_g^2. \tag{103}$$

The ensemble average differential scattering efficiencies follow from equations (94) and (95):

$$\overline{\sigma}_{Tg}(\hat{\mathbf{r}})/\pi a^2 = (\delta T_g/4T_\infty)^2(ka)^4\cos^2\theta\exp\left[-(ka)^2(1-\cos\theta)\right], \tag{104}$$

$$\overline{\sigma}_{vg}(\hat{\mathbf{r}})/\pi a^2 = \tfrac{1}{3}(v_g/2c_\infty)^2(ka)^6\sin^2\theta\cos^2\theta\exp\left[-(ka)^2(1-\cos\theta)\right], \tag{105}$$

where we define a scattering efficiency as the ratio of a cross section to πa^2. These differential scattering efficiencies are plotted in figures 1 and 2 as functions of scattering angle θ for several different size parameters (ka), normalized to unit values of $\delta T_g/T_\infty$ and v_g/c_∞. These expressions yield exactly the same dependence of the scattering on θ, k, and a as the expressions in

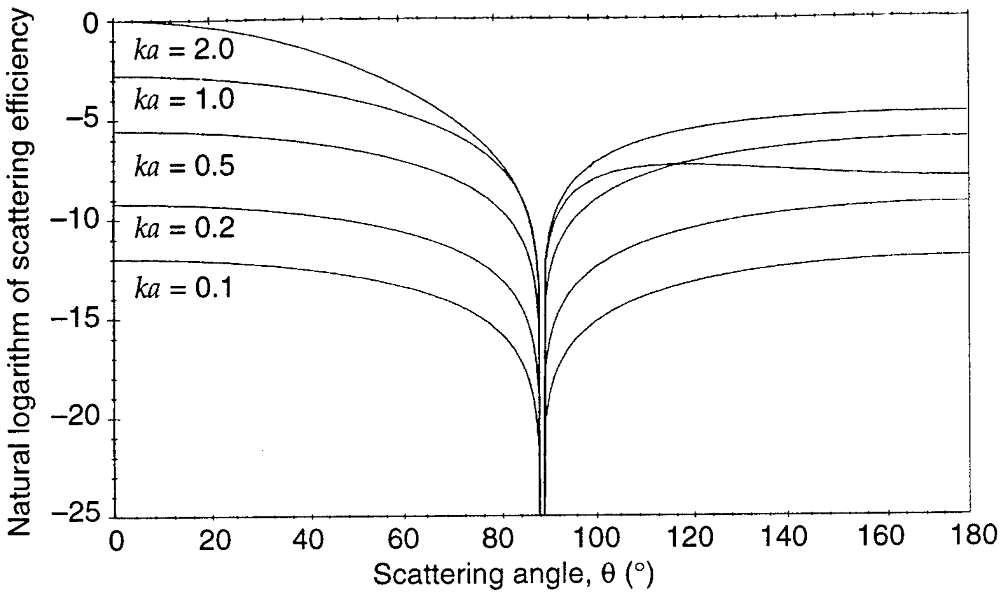

Figure 1. Temperature scattering efficiency $\overline{\sigma}_T/\pi a^2$ versus scattering angle θ, for $\delta T/T_\infty =$ 1.0, for a Gaussian isotropic ensemble of turbules of scale length a, for several size parameters $ka = 2\pi a/\lambda$, $\lambda =$ acoustic wavelength, in Born approximation.

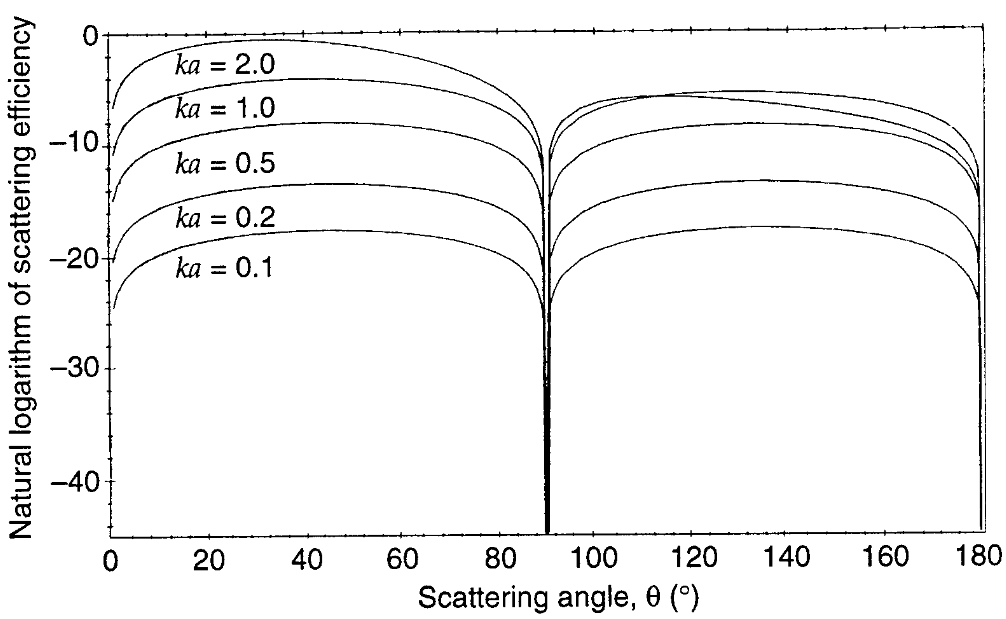

Figure 2. Velocity scattering efficiency $\overline{\sigma}_v(\theta)/\pi a^2$ versus scattering angle θ, for $V/c_\infty = 1.0$, for a Gaussian isotropic ensemble of turbules of scale length a, for several size parameters $ka = 2\pi a/\lambda$, $\lambda =$ acoustic wavelength, in Born approximation.

Mellert [18] for a Gaussian spectrum of temperature and wind velocity fluctuations. That is, a Gaussian envelope function for an isotropic ensemble of turbules of a single scale length is the same as a Gaussian spectrum for isotropic turbulence of a single scale length, as would be expected.

The total efficiencies are given by

$$\begin{aligned}\overline{\sigma}_{Tg}/\pi a^2 &= 2\pi(\delta T_g/4T_\infty)^2(x-2+2/x) - e^{-2x}(x+2+2/x), && (106)\\ \overline{\sigma}_{vg}/\pi a^2 &= (2\pi/3)(v_g/2c_\infty)^2 x\Big[(1+e^{-2x})(2+24/x^2) \\ &\quad - (1-e^{-2x})(10/x+24/x^3)\Big], && (107)\end{aligned}$$

where $x=(ka)^2$.

In figure 3, the numerical integration of the differential scattering efficiencies are plotted versus ka for unit $\delta T/T_\infty$ and v/c_∞. For small ka, the total efficiencies reduce to

$$\begin{aligned}\overline{\sigma}_{Tg}/\pi a^2 &\approx (4\pi/3)(\delta T_g/4T_\infty)^2(ka)^4, && (108)\\ \overline{\sigma}_{vg}/\pi a^2 &\approx \tfrac{1}{3}(8\pi/15)(v_g/2c_\infty)^2(ka)^6. && (109)\end{aligned}$$

For large ka, they reduce to

$$\begin{aligned}\overline{\sigma}_{Tg}/\pi a^2 &\approx 2\pi(\delta T_g/4T_\infty)^2(ka)^2, && (110)\\ \overline{\sigma}_{vg}/\pi a^2 &\approx (4\pi/3)(v_g/2c_\infty)^2(ka)^2. && (111)\end{aligned}$$

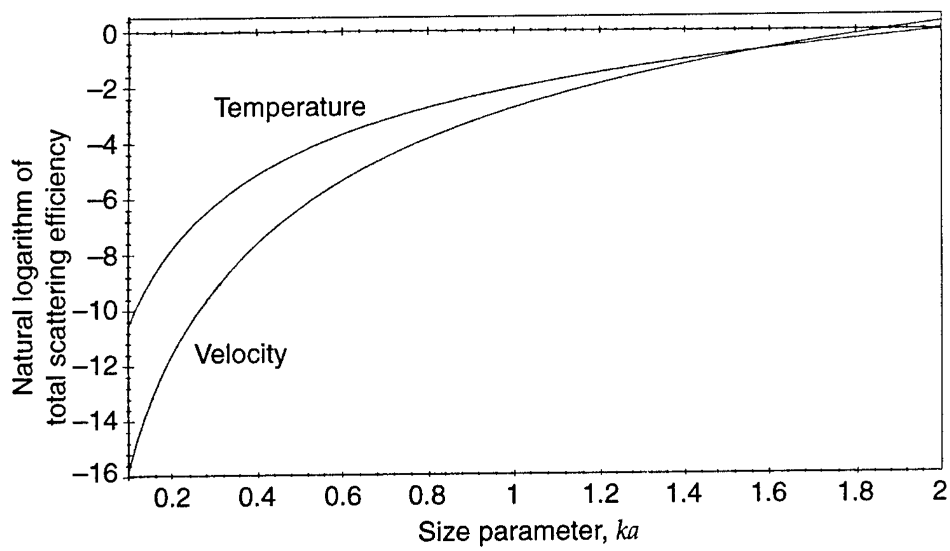

Figure 3. Total Born approximation (temperature, velocity) scattering efficiency $(\overline{\sigma}_T/\pi a^2, \overline{\sigma}_v/\pi a^2)$ versus size parameter $ka = 2\pi a/\lambda$, λ = acoustic wavelength, a = turbule scale length, for a Gaussian isotropic ensemble, for $\delta T/T_\infty = v/c_\infty = 1.0$.

As is well known, the Born approximation fails in the limit of arbitrarily large ka; the actual total scattering efficiencies should be independent of ka in this limit. But we cannot simply apply higher order approximations for the scattering amplitudes for large ka, because the wave equations used are valid only to first order in v_0/c_∞. In order to obtain trustworthy expressions for scattering cross sections for very large size parameters, we must first obtain and solve wave equations valid to all orders of v_0/c_∞ and its derivatives.

Several features of the above results are remarkable. If $(\delta T/T_\infty) \approx (v/c_\infty)$, then $\overline{\sigma}_{vg} \ll \overline{\sigma}_{Tg}$ for small ka, while $\overline{\sigma}_{vg} \approx \overline{\sigma}_{Tg}$ for large ka. Also, $\overline{\sigma}_{vg}(\hat{\mathbf{r}}) = 0$ for $\theta = (0, \pi/2, \pi)$, while $\overline{\sigma}_{Tg}(\hat{\mathbf{r}})$ is a maximum in the forward direction.

(b) *Exponential envelope.* Again for convenience, we let

$$\widetilde{B}_T(y) = \widetilde{B}_v(y) \equiv \widetilde{B}_e(y) = (1 + y^2/\alpha^2)^{-3}, \tag{112}$$

where α is a parameter to be adjusted. This corresponds to the following envelope function in position space:

$$B_e(\xi) \equiv (2\pi)^{-3} \int d^3y e^{i\boldsymbol{\xi}\cdot\mathbf{y}} \widetilde{B}_e(y) \propto (1 + \alpha\xi)e^{-\alpha\xi}. \tag{113}$$

The presence of the factor $(1 + \alpha\xi)$ is necessary in order to ensure that the turbule velocity field is bounded at $\xi = 0$.

Equations (98), (100), and (101) then yield

$$a_{rms}/a = \sqrt{18}/\alpha, \quad \mathcal{E}_{Te} = \left(\frac{7\alpha^3\sqrt{\pi}}{256\sqrt{2}}\right)^{\frac{1}{2}} \delta T_e, \quad \mathcal{E}_{ve} = \left(\frac{\alpha^5\sqrt{\pi}}{256\sqrt{2}}\right) v_e^2. \tag{114}$$

Comparing equation (103) yields

$$\alpha = \sqrt{12}, \quad \delta T_e = (0.84)(\delta T_g), \quad v_e = 0.64 v_g; \tag{115}$$

we make this comparison so that these exponential model turbules of scale length a have the same ensemble average values as the Gaussian model for $(a_{rms}, \mathcal{E}_{Te}, \mathcal{E}_{ve})$, i.e., so that the two ensembles of different average morphology be "comparable."

The ensemble average differential scattering efficiencies then follow from equations (94) and (95):

$$\overline{\sigma}_{Te}(\hat{\mathbf{r}})/\pi a^2 = (0.70)(\delta T_e/4T_\infty)^2 (ka)^4 \cos^2\theta \left[1 + \tfrac{1}{6}(ka)^2(1 - \cos\theta)\right]^{-6}, \tag{116}$$

$$\overline{\sigma}_{ve}(\hat{\mathbf{r}})/\pi a^2 = \tfrac{1}{3}(0.41)(v_e/2c_\infty)^2 (ka)^6 \sin^2\theta \cos^2\theta \left[1 + \tfrac{1}{6}(ka)^2(1 - \cos\theta)\right]^{-6}. \tag{117}$$

These are plotted in figures 4 and 5 as functions of scattering angle θ for several size parameters ka, for unit values of $\delta T_e/T_\infty$ and v_e/c_∞.

The calculations for the total cross sections versus ka are algebraically quite complicated. We give the small ka limits only; the curves of figure 6 were found by direct numerical integration of the differential cross sections, equations (116) and (117). For small size parameters, we have for the total efficiencies

$$\overline{\sigma}_{Te}/\pi a^2 \approx (4\pi/3)(0.70)(\delta T_g/4T_\infty)^2(ka)^4 \approx (0.70)(\overline{\sigma}_{Tg}/\pi a^2), \tag{118}$$

$$\overline{\sigma}_{ve}/\pi a^2 \approx \tfrac{1}{3}(0.41)(8\pi/15)(v_g/2c_\infty)^2(ka)^6 \approx (0.41)(\overline{\sigma}_{vg}/\pi a^2). \tag{119}$$

Thus, for small ka, the total cross sections for the exponential ensemble are smaller, but of the same order as those for the Gaussian ensemble. Figures 3 and 6 allow a comparison of the total efficiencies versus ka.

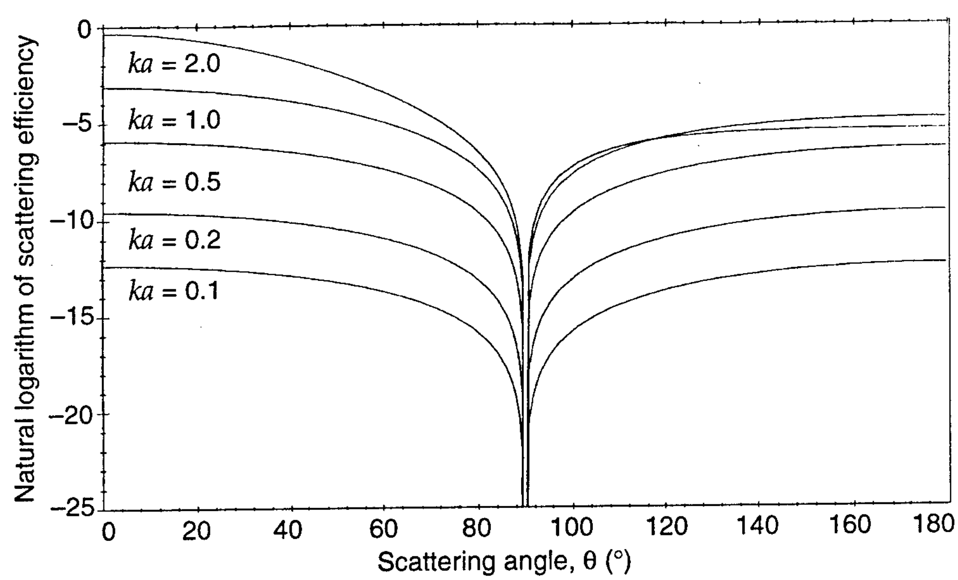

Figure 4. Temperature scattering efficiency $\overline{\sigma}_T\theta/\pi a^2$ versus scattering angle θ, for $\delta T/T_\infty = 1.0$, for an exponential isotropic ensemble of turbules of scale length a, for several size parameters $ka = 2\pi a/\lambda$, $\lambda =$ acoustic wavelength, in Born approximation.

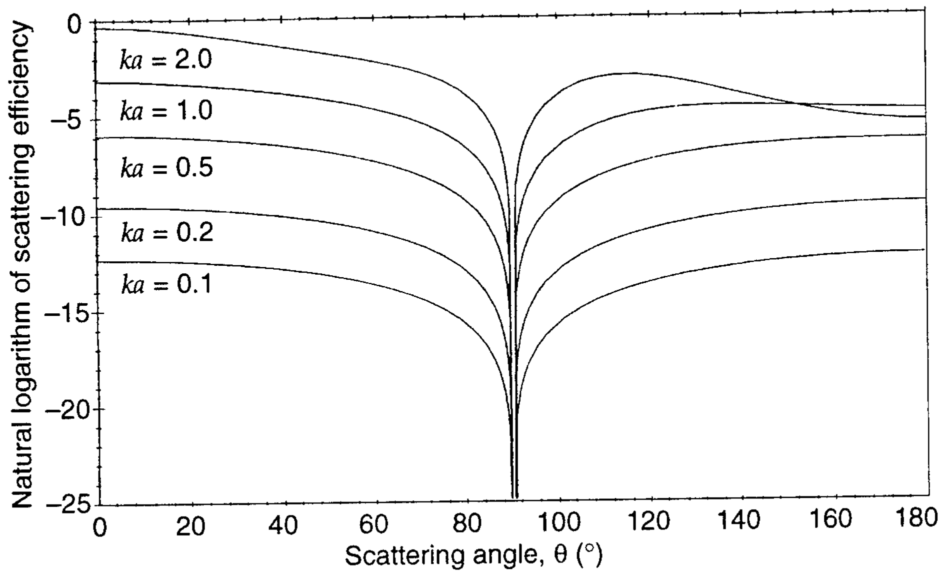

Figure 5. Velocity scattering efficiency $\overline{\sigma}_v\theta/\pi a^2$ versus scattering angle θ, for v/c_∞ = 1.0, for an exponential isotropic ensemble of turbules of scale length a, for several size parameters $ka = 2\pi a/\lambda$, λ = acoustic wavelength, in Born approximation.

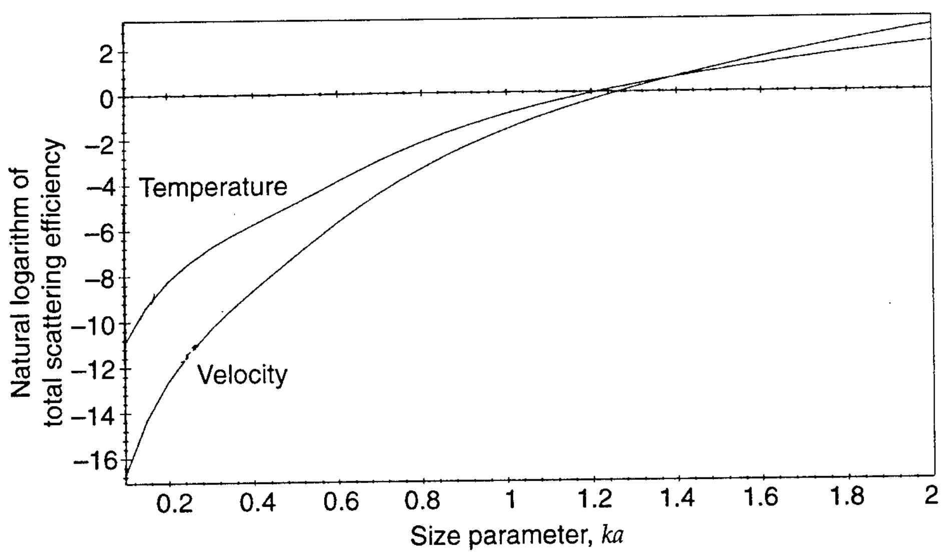

Figure 6. Total Born approximation (temperature, velocity) scattering efficiency $(\overline{\sigma}_T/\pi a^2, \overline{\sigma}_v/\pi a^2)$ versus size parameter $ka = 2\pi a/\lambda$, λ = acoustic wavelength, a = turbule scale length, for an exponential isotropic ensemble, for $\delta T/T_\infty = v/c_\infty = 1.0$.

3.3 Individual Turbule Model

This section illustrates the azimuthal dependence of the velocity scattering from even the simplest model turbule, and shows that all isotropic ensembles may indeed be composed of such simple model turbules. For example, suppose we choose

$$\mathbf{A}(\boldsymbol{\xi}) = \mathbf{u}e^{-\xi^2}, \tag{120}$$

where $\mathbf{u}$ is a constant vector, as the simplest Gaussian turbule model. Then we have, from equation (69),

$$\mathbf{v}_0(\mathbf{r}) = \boldsymbol{\nabla}_\xi \times \mathbf{A}(\boldsymbol{\xi}) = 2\mathbf{u} \times \boldsymbol{\xi} e^{-\xi^2}. \tag{121}$$

If we define

$$\mathbf{u} = \tfrac{1}{2}a\boldsymbol{\Omega} \tag{122}$$

and use the definition of $\boldsymbol{\xi}$ (eq (69)), we get

$$\mathbf{v}_0(\mathbf{r}) = [\boldsymbol{\Omega} \times (\mathbf{r} - \mathbf{b})] \exp\left[-(\mathbf{r} - \mathbf{b})^2/a^2\right]. \tag{123}$$

Thus, this model of a single turbule represents a nonuniform rotation, with angular velocity parameter $\boldsymbol{\Omega}$, of a spherically symmetric Gaussian turbule of scale length a, centered at $\mathbf{r} = \mathbf{b}$.

The Fourier transform of $\mathbf{A}(\boldsymbol{\xi})$ is given by equation (79); we have

$$\widetilde{\mathbf{A}}(\mathbf{y}) = \pi^{3/2}\mathbf{u}e^{-y^2/4}. \tag{124}$$

We write $\hat{\mathbf{u}}$ in terms of its polar and azimuthal angles:

$$\hat{\mathbf{u}} = \mathbf{e}_1 \sin\theta_u \cos\varphi_u + \mathbf{e}_2 \sin\theta_u \sin\varphi_u + \mathbf{e}_3 \cos\theta_u, \tag{125}$$

whereby from equations (80) and (85), the first Born differential scattering efficiency is

$$\sigma_v(\hat{\mathbf{r}})/\pi a^2 = (ka)^6 (v/2c_\infty)^2 \sin^2\theta \cos^2\theta \sin^2\theta_u \sin^2(\varphi - \varphi_u) \exp\left[-(ka)^2(1 - \cos\theta)\right]. \tag{126}$$

This exhibits strong azimuthal dependence. It is zero if $\theta_u = 0$ or $\pi (\hat{\mathbf{u}} = \pm\hat{\mathbf{k}})$, and if $\varphi = (\varphi_u, \varphi_u \pm \pi)$. That is, it goes to zero if $\hat{\mathbf{u}}$ is in the plane of incidence, the plane whose normal is $\hat{\mathbf{k}} \times \hat{\mathbf{r}}$.

If we average over random directions of $\hat{\mathbf{u}}$, using

$$(4\pi)^{-1} \int_0^{2\pi} d\varphi_u \int_0^{\pi} d\theta_u \sin\theta_u \left[\sin^2\theta_u \sin^2(\varphi_u - \varphi)\right] = 1/3, \tag{127}$$

we get

$$\overline{\sigma}_v(\hat{\mathbf{r}})/\pi a^2 = \tfrac{1}{3}(ka)^6(u/2c_\infty)^2 \sin^2\theta \cos^2\theta \exp\left[-(ka)^2(1-\cos\theta)\right], \quad (128)$$

which is identical with equation (105) for the Gaussian ensemble, if $u = v_g$. In fact, from equations (124) and (127), we have

$$\langle \tilde{A}_i(\mathbf{y})\tilde{A}_j^*(\mathbf{y})\rangle = \tfrac{1}{3}\pi^3\delta_{ij}u^2 e^{-y^2/2}, \quad (129)$$

which yields, from equations (91) and (102), for the choice $u = v_g$,

$$\widetilde{B}_g^2(y) = e^{-y^2/2}, \quad (130)$$

the same envelope function as that chosen for the Gaussian ensemble. The same conclusions could be drawn for all individual turbules of the generic form

$$\mathbf{A}(\boldsymbol{\xi}) = \mathbf{u}f(\xi) \rightarrow \widetilde{\mathbf{A}}(\mathbf{y}) = \mathbf{u}\tilde{f}(y), \quad (131)$$

where $\mathbf{u}$ is a constant vector that may have any direction and magnitude, and f is a scalar function of $\xi = |\boldsymbol{\xi}|$. The corresponding velocity field is

$$\mathbf{v}_0(\mathbf{r}) = \boldsymbol{\nabla}_\xi \times \mathbf{A}(\boldsymbol{\xi}) = -(\mathbf{u}\times\boldsymbol{\xi})(\xi^{-1}f'(\xi)). \quad (132)$$

If we define $\boldsymbol{\Omega}$ as in equation (122), and $h(\xi) = -\tfrac{1}{2}\xi^{-1}f'(\xi)$, then we have

$$\mathbf{v}_0(\mathbf{r}) = \left[\boldsymbol{\Omega}\times(\mathbf{r}-\mathbf{b})\right]h(|\mathbf{r}-\mathbf{b}|/a). \quad (133)$$

This shows that the vector $\mathbf{u}$ is along the spin axis $\widehat{\boldsymbol{\Omega}}$ of any simple model turbule. All such "simple" turbules are rotating, in general nonuniformly, with spherically symmetric envelopes.

For random orientations of $\mathbf{u}$, we have $\langle u_i u_j\rangle = \tfrac{1}{3}\delta_{ij}u^2$, and thus

$$\langle \tilde{A}_i(\mathbf{y})\tilde{A}_j^*(\mathbf{y})\rangle = \tfrac{1}{3}u^2|\tilde{f}(y)|^2\delta_{ij}, \quad (134)$$

whereby from equation (91), we can identify

$$\widetilde{B}^2(y) = \pi^{-3}|\tilde{f}(y)|^2(u^2/v^2) \quad (135)$$

as the corresponding isotropic ensemble envelope function.

Similar considerations apply for the temperature. Consider a spherically symmetric turbule with $\Delta T_0(\mathbf{r}) = T(\xi), \xi = |\boldsymbol{\xi}|$. Then,

$$|\widetilde{T}(y)|^2 = \langle|\widetilde{T}(y)|^2\rangle = \pi^3(\delta T)^2\widetilde{B}^2(y), \tag{136}$$

since orientation averaging has no effect on a spherically symmetric function.

Based on these considerations, we can reach the following important conclusion: Every ensemble of randomly oriented static turbules of a given scale length, within which each member has arbitrary morphology, is equivalent to a single turbule with a spherically symmetric temperature variation $\Delta T_0(\mathbf{r}) = T(\xi)$, and to a collection of simple turbules with velocity vector potential $\mathbf{A}(\boldsymbol{\xi})$ (given by eq (131), with $\mathbf{u}$ a randomly oriented constant vector). The specific equivalence is established by equations (135) and (136). Here, "equivalent" means "produces the same (incoherent) acoustic scattering." This theorem is similar to that for "equivalent spheres" that may replace a collection of randomly oriented dielectric particles of arbitrary morphology in incoherent electromagnetic scattering.

As in the electromagnetic case, the theorem is valid only for incoherent scattering from an actual collection of turbules. But this case is of great importance and generality. On the other hand, if coherence is important in the calculation of multiple scattering from an actual collection of randomly oriented turbules of general morphology, then we should not perform the orientation averaging until after we have solved for the scattering cross sections of the whole collection, for each of many realizations of the collection. In such cases, replacement of the actual collection of turbules by a collection of simple turbules will not yield the same result.

4. Summary and Discussion

We describe the following principal results in this report:

i: A complete set of fluid equations involving the pressure, temperature, mass density, and flow velocity is presented for an ideal gas atmosphere for which viscosity and thermal conductivity are neglected. These equations include equations (8) or (9) as the heat flow equation.

ii: From this set, a complete set of coupled linear differential equations is derived for the acoustic flow velocity and the variations of acoustic pressure, temperature, and mass density relative to underlying stationary turbulent fluid fields. Equations relating the turbulent fields are also derived.

iii: We show that the coupled acoustic differential equations yield Monin's wave equation for the acoustic pressure when only first-order terms in the turbulent temperature variation and velocity fields are retained.

iv: We show that the length scale a_s of the scattering volume is just as important as the acoustic wavelength λ and the length scales a of the turbulence in predicting the general behavior of the far-field acoustic scattering by turbulence in a first Born approximation. In particular, if $a \ll a_s$, then the standard results (eq (48) and (49)) for the scattering amplitudes, proportional to $\cos\theta$, result from Monin's equation for all a/λ. If both $a > a_s$ and $a/\lambda \gg 1$, then the scattering amplitudes (eq (50) and (51)) result, which are those predicted by the short wavelength limit of Monin's equation, the same as equations (48) and (49) but without the $\cos\theta$ factor. If $\lambda \geq a \geq a_s$, other results occur.

v: For solenoidal turbulent velocity fields, we show that both forward and backward velocity scattering are essentially zero for all a/λ if $a \ll a_s$, but nonzero if $a > a_s$.

The first-order results of this report for $a \ll a_s$ have been applied to scattering by individual turbules. In a followup report [2], we will develop similar results for ensembles of turbules, and a connection will be made to the Kolmogorov spectrum, a Gaussian spectrum, and the standard structure function development for isotropic homogeneous turbulence.

In the work reported here, we have also found expressions for the mean temperature and velocity differential, as well as total scattering cross sections in first Born approximation for a randomly oriented stationary turbule of a given scale length, but arbitrary morphology. We have obtained and compared expressions for these mean cross sections for both Gaussian and exponential envelope functions of the turbule temperature variation and velocity. We have given a somewhat arbitrary but general recipe for adjusting the parameters for ensembles of turbules with different envelope functions so that they have essentially the same size and energy content. We have shown that the simplest possible model for an individual turbule (a spherically symmetric structure spinning nonuniformly about a fixed axis) still yields strong azimuthal dependence in the velocity scattering. Also, we have shown that we can always find an ensemble of these simplest model turbules, with random orientations of the spin axes, that produces the same ensemble average incoherent scattering as an ensemble of randomly oriented turbules of arbitrary morphology. We have also given formulas that yield the equivalent simple turbule parameters in terms of any given actual ensemble averages $\langle|\tilde{T}(y)|^2\rangle$, $\langle\tilde{A}_i(y)\tilde{A}_j^*(y)\rangle$.

In a future report [2], we will make use of these results to construct an isotropic homogeneous ensemble of turbules having different scale lengths and locations, and to derive the conditions for the existence and the bounds of the Kolmogorov (von Kármán) spectrum and a Gaussian spectrum for the incoherent scattering from such an ensemble. In later work, we hope to be able to use the results of this report to treat the problem of incoherent scattering produced by anisotropic inhomogeneous ensembles of turbules.

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13. ABSTRACT *(Maximum 200 words)*

A new turbulence model is used to describe the acoustical scattering from atmospheric turbulence. A complete set of fluid equations, including the heat flow equation with zero conductivity, is presented for an ideal gas atmosphere. From this set, a complete set of coupled linear differential equations is derived for the acoustic pressure, temperature, mass density, and velocity in the presence of stationary turbulence. From these acoustic wave equations, expressions for acoustic scattering cross sections are derived for individual localized stationary scalable turbules of arbitrary morphology and orientation. Averages over random turbule orientations are also derived. Criteria for comparability of orientationally averaged turbules with different envelope functions are presented and applied, and cross sections for Gaussian and exponential envelopes are compared. The azimuthal dependence of the velocity scattering cross section for a spherically symmetric nonuniformly rotating turbule is illustrated. It is shown that, for incoherent scattering, a collection of randomly oriented turbules of arbitrary morphology may be replaced by an "equivalent" collection of spherically symmetric, nonuniformly rotating turbules with randomly directed rotation axes.

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