APPROXIMATION AND DERIVATIVES OF SURVIVAL
IN STRUCTURAL ANALYSIS AND DESIGN

K. Marti

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Approximation and Derivatives of Probabilities of Survival in Structural Analysis and Design

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Abstract

Yield stresses, allowable stresses, moment capacities (plastic moments), external loadings, manufacturing errors,... are not given fixed quantities in practice, but have to be modelled as random variables with a certain joint probability distribution. Hence, problems from limit (collapse) load analysis or plastic analysis and from plastic and elastic design of structures are treated in the framework of stochastic optimization. Using especially reliability-oriented optimization methods, the behavioral constraints are quantified by means of the corresponding probability $p_s$ of survival. Lower bounds for $p_s$ are obtained by selecting certain redundants in the vector of internal forces/bending-moments; moreover, upper bounds for $p_s$ are constructed by considering a pair of dual linear progrms for the optimization representation of the yield or safety constraints. Whereas the probability $p_s$ can be computed e.g. by sampling methods or by asymptotic expansion techniques based on Laplace integral representations of certain multiple integrals, efficient techniques for the computation of the sensitivities (of various orders) of $p_s$ with respect to input or design variables $X$ and load factors $\lambda$ have yet to be developed. Hence, several new techniques (e.g. Transformation Method, Stochastic Completion Technique) are suggested for the numerical computation of derivatives of $p_s$ with respect to $(X, \lambda)$. 
1. Limit (collapse) load analysis of structures as a linear programming problem.

The collapse load can be defined \( [3],[5],[18] \) "as the load required to generate enough number of local plastic yield points (referred as plastic hinges for bending type members) to cause the structure to become a mechanism and develop excessive deflections". Assuming that the material behaves as an elastic-perfectly plastic material \( [4],[15] \), a conservative estimate of the collapse load factor \( \lambda_T \) is based on the following formulation as a linear program (LP):

\[
\text{maximize } \lambda \\
\text{s.t.} \\
F^L \leq F \leq F^U \\
CF = AR_0.
\]

Here, (1.2) is the equilibrium equation of a statically indeterminate loaded structure involving an \( m \times n \) matrix \( C = (c_{ij}) \), \( m \times n \), of given coefficients \( c_{ij}, \) \( 1 \leq i \leq m, 1 \leq j \leq n \), depending on the undeformed geometry of the structure having \( n_0 \) members (elements); we suppose that \( \text{rank}C=m \). Furthermore, \( R_0 \) is an external load \( m \)-vector, and \( F \) denotes the \( n \)-vector of internal forces and bending-moments in the relevant points (sections, nodes or elements) of the structure. Finally, (1.1) are the yield conditions with the vector of lower and upper bounds \( F^L, F^U \).

For a plane or spatial truss \( [6],[16] \) we have that \( n-n_0 \), the matrix \( C \) contains the direction cosines of the members, and \( F \) involves only the normal (axial) forces, moreover,

\[
F_j^L := \sigma_j^L A_j, \quad F_j^U := \sigma_j^U A_j, \quad j=1, \ldots, n(n-n_0),
\]

where \( A_j \) is the (given) cross-sectional area, and \( \sigma_j^L, \sigma_j^U \) resp., denotes the yield stress in compression (negative values) and tension (positive values) of the \( j \)-th member of the truss. In case of a plane frame, \( F \) is composed of subvectors \( [16] \)

\[
F^{(k)} = \begin{pmatrix} F_1^{(k)} \\ F_2^{(k)} \\ F_3^{(k)} \end{pmatrix} = \begin{pmatrix} t_k^+ \\ m_k^+ \\ m_k^- \end{pmatrix},
\]

(1.4)
where \( F^{(k)}_1 = \tau_k \) denotes the normal (axial) force, and \( F^{(k)}_2 = m^+_k, F^{(k)}_3 = m^-_k \) are the bending-moments at the positive, negative end of the k-th member. In this case \( F^{L}, F^{U} \) contain for each member k - the subvectors

\[
F^{(k)}_L = \begin{pmatrix}
\sigma^L_{yk} A_k
-M_{kpl}^L
-M_{kpl}^L
\end{pmatrix}, \quad F^{(k)}_U = \begin{pmatrix}
\sigma^U_{yk} A_k
M_{kpl}^U
M_{kpl}^U
\end{pmatrix},
\]

resp., where \( M_{kpl}^L \) \( k=1, \ldots, n_o \), denote the plastic moments (moment capacities) given [4],[15] by

\[
M_{kpl}^L = \sigma^U_{yk} W_{kpl}^L,
\]

and \( W_{kpl}^L(A_k) \) is the plastic section modulus of the cross-section of the k-th member (beam) with respect to the local z-axis.

For a spatial frame [6],[16], corresponding to the k-th member (beam), \( F \) contains the subvector

\[
F^{(k)} = (r_k^L m_{kT}^L, m_{kT}^+_k, m_{kT}^-_k, m_{ky}^+_k, m_{ky}^-_k, m_{kz}^+_k, m_{kz}^-_k, m_{ky}^+_k, m_{ky}^-_k, m_{kz}^+_k, m_{kz}^-_k),
\]

where \( r_k \) is the normal (axial) force, \( m_{kT} \) the twisting moment, and \( m_{ky}^+_k, m_{ky}^-_k, m_{kz}^+_k, m_{kz}^-_k \) denote four bending moments with respect to the local \( \hat{y}-, \hat{z}- \) axis at the positive, negative end of the beam, respectively. Finally, corresponding to (1.3),(1.4)-(1.4.2), the bounds \( F^{L}, F^{U} \) for \( F \) are given by

\[
F^{(k)}_L = (\sigma^L_{yk} A_k, M_{kpl}^L/\sigma^L_{yk}, M_{kpl}^L/\sigma^L_{yk}, M_{kpl}^L/\sigma^L_{yk}, M_{kpl}^L/\sigma^L_{yk}),
\]

\[
F^{(k)}_U = (\sigma^U_{yk} A_k, M_{kpl}^U/\sigma^U_{yk}, M_{kpl}^U/\sigma^U_{yk}, M_{kpl}^U/\sigma^U_{yk}, M_{kpl}^U/\sigma^U_{yk}),
\]

where, cf. [4],[15],

\[
M_{kpl}^L = \sigma^L_{yk} W_{kpl}^L, \quad M_{kpl}^U = \sigma^U_{yk} W_{kpl}^U, \quad M_{kpl}^U = \sigma^U_{yk} W_{kpl}^U
\]

are the plastic moments of the cross-section of the k-th element with respect to the local twisting axis, the local \( \hat{y}, \hat{z} \)-axis, respectively. In (1.5.3), \( W_{kpl}^L = W_{kpl}^L(X) \) and \( W_{kpl}^U = W_{kpl}^L(X) \), \( W_{kpl}^L = W_{kpl}^L(X) \), resp., denote the polar, axial modulus of the cross-sectional area of the k-th beam and \( \tau_{yk} \) denotes the yield stress with respect to torsion; we suppose \( \tau_{yk} > \sigma_{yk} \).

**Remark 1.1**

Possible plastic hinges [4],[7],[15] are taken into account by inserting appropriate eccentricities \( e_{kl} > 0, e_{kr} > 0, k=1, \ldots, n_o \), with \( e_{kl}, e_{kr} < L_k \), where \( L_k \) is the length of the k-th beam.

**Remark 1.2**

Working with more general yield polygons [1],[17],[18], the stress condition (1.1) is replaced by the more general system of inequalities

\[
H^{U}_d^{\top} F \leq h.
\]

Here, \((H,h)\) is a given \( \omega \times (n+1) \) matrix, and \( F^{U}_d = (F^{U}_d^{\top}) \) denotes the \( n \times n \)
diagonal matrix of principal axial and bending plastic capacities

\[ F_j^U := \sigma^U y k_j, k_j^U := \sigma^U \tau_j^U, \]

where \( k_j, \tau_j \) are indices as arising in (1.4.1)-(1.5.3). The more general case (1.1a) can be treated by similar methods as the case (1.1) which is considered here.

2. Plastic and elastic design of structures

In the plastic design of trusses, frames [5] having \( n_0 \) members, the n-vectors \( F^L, F^U \) of lower and upper bounds

\[ F^L = F^L(\sigma^L, \sigma^L, X), \quad F^U = F^U(\sigma^U, \sigma^U, X) \]  

(2)

for the n-vector \( F \) of internal member forces and bending-moments \( F_j, j=1, \ldots, n \), are determined [3],[5] by the yield stresses, i.e. compressive limiting stresses (negative values) \( \sigma^L y = (\sigma^L y_1, \ldots, \sigma^L y_n o) \), the tensile yield stresses \( \sigma^U y = (\sigma^U y_1, \ldots, \sigma^U y_n o) \), and the r-vector

\[ X = (x_1, x_2, \ldots, x_r) \]  

(2.1)

of design variables of the structure. In case of trusses we simply have that, cf. (1.3),

\[ F^L = \sigma^L y d A(X) = A(X) d \sigma^L y, \quad F^U = \sigma^U y d A(X) = A(X) d \sigma^U y, \]  

(2.2)

where \( n=n_0 \) and \( \sigma^L y d, \sigma^U y d \) denote the \( n \times n \) diagonal matrices having the diagonal elements \( \sigma^L y_j, \sigma^U y_j \), resp., \( j=1, \ldots, n \); moreover,

\[ A(X) = (A_1(X), \ldots, A_n(X)) \]  

(2.3)

is the n-vector of cross-sectional areas \( A_j = A_j(X), j=1, \ldots, n \), depending on the r-vector \( X \) of design variables \( x_k, k=1, \ldots, r \), and \( A(X) d \) denotes the \( n \times n \) diagonal matrix having the diagonal elements \( A_j = A_j(X), 1 \leq j \leq n \).

Corresponding to (1.2), here we have again the equilibrium equation

\[ CF = R_u \]  

(2.4)

where \( R_u \) describes [5] the ultimate load (representing constant external loads or self-weight expressed in linear terms of \( A(X) \)).

The plastic design of structures can be represented then [1],[2],[5] by the following optimization problem

\[ \min G(X) \]  

(3)

s.t.

\[ F^L(\sigma^L y, \sigma^U y, X) \leq F \leq F^U(\sigma^L y, \sigma^U y, X) \]  

(3.1)

\[ CF = R_u, \]  

(3.2)

where \( G=G(X) \) is a certain objective function, e.g. the volume or weight of the structure.
Remark 2.1

As mentioned in Remark 1.2, working with more general yield polygons, (3.1) is replaced by the condition
\[
H(F^U(\sigma_y^U,X))^{-1}F \leq h. \tag{3.1a}
\]

For the elastic design we have to replace the yield stresses \(\sigma_y^L, \sigma_y^U\) by the allowable stresses \(\sigma_a^L, \sigma_a^U\), and instead of ultimate loads we consider service loads \(R_s\). Hence, instead of (3) we get the related program
\[
\min \ G(X) \tag{4}
\]
\[
\text{s.t.} \quad F_a^L(\sigma_a^L, \sigma_a^U, X) \leq F \leq F_a^U(\sigma_a^L, \sigma_a^U, X) \tag{4.1}
\]
\[
CF = R_s \tag{4.2}
\]
\[
X^L \leq X \leq X^U, \tag{4.3}
\]
where \(X^L, X^U\) still denote lower and upper bounds for \(X\).

3. Analysis and design of structures in case of random data

In practice, yield stresses, allowable stresses, the loads applied to the structure, other material properties and the manufacturing errors are not given fixed quantities, but must be treated as random variables on a certain probability space \((\Omega, \mathcal{F}, P)\). Hence, (1), (3), (4) are stochastic linear/ nonlinear programs (SLP/SNLP) which obviously have the same basic structure represented by a random objective function
\[
Z(\omega) := G(\omega, X), \ \omega \in \Omega, \tag{5}
\]
and by stochastic constraints of the type
\[
\text{CF = } R(\omega) \tag{6}
\]
\[
F^L(\omega) \leq F \leq F^U(\omega), \tag{7}
\]
where \(R=\mathbb{R}(\omega), \ \omega \in \Omega,\) is a random load m-vector given by
\[
\mathbb{R}(\omega) = \lambda \mathbb{R}_0(\omega), \ \mathbb{R}(\omega) = \mathbb{R}_u(\omega), \ \mathbb{R}(\omega) = \mathbb{R}_s(\omega), \tag{6.1}
\]
resp., and for the n-vector \(\mathbb{F}=(\mathbb{F}_j)\) of internal member forces and bending-moments we have the n-vectors of random bounds
\[
\mathbb{F}^L(\omega) = F^L(\omega, X), \ \mathbb{F}^U(\omega) = F^U(\omega, X), \ \omega \in \Omega, \tag{7.1}
\]
depending on an r-vector \(X\) of design variables \(X_k, k=1, \ldots, r\).

Obviously, each realization of the random element \(\omega \in \Omega\) yields new loading conditions, represented by the vector \(R=\mathbb{R}(\omega)\), and therefore, cf. (6), new arrangements \(\mathbb{F}=\mathbb{F}(\omega)\) of internal member forces and bending-moments. Hence, in the present case of "multiple loadings" caused by the random variations of \(R=\mathbb{R}(\omega)\), the survival of the structure, i.e. the existence of certain arrangements \(\mathbb{F}(\omega)\) of internal member forces/bending-moments not overwhelming the strength of the structure, can be evaluated by the probability of survival.
\[ p_s := P(\text{There is } F=F(\omega) \text{ such that } CF(\omega) = R(\omega)) \]
\hspace{1cm} \text{and } F^L(\omega) \leq F(\omega) \leq F^U(\omega), \quad (8)\]

see [2],[14], assuming that, cf. (26),(33),
\[ S(F^L(\cdot),F^U(\cdot),R(\cdot)) := \{ \omega \in \Omega: \text{there is a vector } F=F(\omega) \]
\hspace{1cm} \text{fulfilling (6) and (7)} \}
\hspace{1cm} (8.1)\]
is a measurable set. Denoting by \([F^L,F^U]\) the \(n\)-dimensional interval
\[ [F^L,F^U] := \{ F \in \mathbb{R}^n: F^L \leq F \leq F^U \}, \quad (8.2)\]
we find that
\[ S(F^L(\cdot),F^U(\cdot),R(\cdot)) = \{ \omega \in \Omega: R(\omega) \in C[F^L(\omega),F^U(\omega)] \} \]
\hspace{1cm} \text{with } C[F^L,F^U] = \{ CF: F^L \leq F \leq F^U \}, \text{ and therefore} \]
\[ p_s = \int_{C[F^L,F^U]} P(S(F^L(\cdot),F^U(\cdot),R(\cdot))) \text{ d}F \]
\hspace{1cm} \text{where } \pi \text{ denotes the distribution of the bounds } (F^L(\omega),F^U(\omega)). \text{ Since the}
\hspace{1cm} \text{bounds } F^L,F^U \text{ in (7) depend also on the vector } X \text{ of design variables } X_k,
\hspace{1cm} k=1,\ldots,r, \text{ cf. (7.1), we have } p_s=P(X) \text{ with the probability function} \]
\[ P(X) = P(R(\omega) \in C[F^L(\omega,X),F^U(\omega,X)]); \]
\hspace{1cm} \text{furthermore, if the external load } R=R(\omega) \text{ is given by}
\[ R(\omega) = R(\omega,\lambda) := \sum_{i=1}^{m_R} \lambda_i R_1^{(i)}(\omega), \quad \lambda := (\lambda_1,\ldots,\lambda_{m_R})^T, \]
\hspace{1cm} \text{with random } m \text{-vectors } R^{(i)} = R^{(i)}(\omega), \quad i=1,\ldots,m_R, \text{ and deterministic coe } \]
\hspace{1cm} \text{fficients } \lambda_i, \quad i=1,\ldots,m_R, \text{ then } p_s = P(X,\lambda) \text{ with}
\[ P(\lambda,X) := P(\sum_{i=1}^{m_R} \lambda_i R_1^{(i)}(\omega) \in C[F^L(\omega,X),F^U(\omega,X)]). \]
\hspace{1cm} \text{Especially, in case of trusses, see (1),(3),(4) and (2.2), for dealing}
\hspace{1cm} \text{with the probability of survival } p_s \text{ we have the following probability functions:}
\[ P_A(\lambda,X) := P(\lambda R_0(\omega) \in C[A(X)_{d^L_y(\omega)},A(X)_{d^U_y(\omega)}]), \lambda \in \mathbb{R} \]
\hspace{1cm} \text{and} \]
\[ P_u(X) := P(R_u(\omega) \in C[A(X)_{d^L_y(\omega)},A(X)_{d^U_y(\omega)}]) \]
\hspace{1cm} \text{or}
\[ P_s(X) := P(R_s(\omega) \in C[A(X)_{d^L_y(\omega)},A(X)_{d^U_y(\omega)}]) \]
\hspace{1cm} \text{Since } p_s=P(X) \text{ are very complicated expressions in general, in the fol } \]
\hspace{1cm} \text{lowing we are looking for approximations (lower, upper bounds) of } p_s=P(X) \text{ by}
\hspace{1cm} \text{simpler probability functions.}

\section{Lower and upper bounds for } p_s=P(X)

According to (8.3) we have that
\[ S(F^L(\cdot),F^U(\cdot),R(\cdot)) \subset \bigcap_{i=1}^{m} S_i(F^L(\cdot),F^U(\cdot)), \quad (11)\]

where
\( S_1(F^L(\omega), F^U(\omega), R(\omega)) := (\omega \in \Omega: R_1(\omega) \in C_1[F^L(\omega), F^U(\omega)]) , \quad i=1, \ldots, m, \quad (11.1) \)

and \( C_i \) denotes the \( i \)-th row of \( C \). Hence, we get the inequality

\[
P(X) \leq \min_{1 \leq i \leq m} P^i(X), \quad (12)\]

where

\[
P^i(X) := P(R_1(\omega) \in C_i[F^L(\omega, X), F^U(\omega, X)]). \quad (12.1)\]

Note that related inequalities (with more exact bounds) follow from more general Bonferroni-type inequalities [10]. We find that

\[
C^i_1[F^L(\omega, X), F^U(\omega, X)] = [\gamma^L_i(\omega, X), \gamma^U_i(\omega, X)] \quad (12.2)\]

is an interval in \( \mathbb{N} \) having the bounds

\[
\gamma^L_i(\omega, X) = \min_{1 \leq i \leq J} C_i G^i(\omega, X) \quad (12.3)\]

\[
\gamma^U_i(\omega, X) = \max_{1 \leq i \leq J} C_i G^i(\omega, X),
\]

where \( G^i(\omega, X), i=1, \ldots, J, \) are the extreme points of the interval \([F^L(\omega, X), F^U(\omega, X)]\). Since the components \( G^i(\omega, X), j=1, 2, \ldots, n, \) of \( G^i(\omega, X), i=1, \ldots, J, \) are certain elements of \((F^L(\omega, X), F^U(\omega, X))\), the measurability of the bounds \( F^L, F^U \) with respect to \( \omega \in (\Omega, \mathcal{A}, P) \) yields the measurability of \( \gamma^L_i \) and \( \gamma^U_i \), \( i=1, \ldots, m, \) with respect to \( \omega \). Hence, \( S_1(F^L(\cdot, \cdot), F^U(\cdot, \cdot), R(\cdot)) \), \( i=1, \ldots, m, \) are measurable sets, and we may write

\[
P^i(X) = P(\gamma^L_i(\omega, X) \leq R_1(\omega) \leq \gamma^U_i(\omega, X)) , \quad i=1, \ldots, m. \quad (12.4)\]

4.1. Lower bounds by selection of redundants

For the construction of lower bounds for \( P(X) \), the vector \( F=F(\omega) \) is partitioned

\[
F(\omega) = \begin{bmatrix} F^I \\ N \end{bmatrix} \quad (13)\]

into a certain \((n-m)\)-vector \( N=(F^I_j) \) of redundants \( F^I_j \), \( j=1, \ldots, n-m, \) and an \( m \)-vector \( F^I \) of statically determined member forces/bending-moments. Hence, with a corresponding partition of the \( m \times n \) matrix \( C \) into \( m \times m \), \( m \times (n-m) \) submatrices \( C^I_1, C^I_2, \) resp., where

\[
\text{rank} C^I = \text{rank} C = m, \quad (13.1)\]

the equilibrium equation (6) yields for \( F(\omega) \) the representation

\[
F(\omega) = \begin{bmatrix} F^I \\ N \end{bmatrix} = \begin{bmatrix} C^{-1}_I R(\omega) \\ 0 \end{bmatrix} + \begin{bmatrix} C^{-1}_I C^I_2 \\ I \end{bmatrix} N. \quad (14)\]

Consequently, selecting for each \( \omega \in \Omega \) a vector of redundants

\[
N=N(\omega) = (F^I_j(\omega)) \quad \text{such that } N(.) \text{ is a measurable function on } (\Omega, \mathcal{A}, P),
\]

we get

\[
S(F^L(\cdot, \cdot), F^U(\cdot, \cdot), R(\cdot)) \supseteq \bar{S}(X, N(\cdot), R(\cdot)), \quad (15.1)\]

where \( \bar{S}(X, N(\cdot), R(\cdot)) \) is the measurable set given by

\[
\bar{S}(X, N(\cdot), R(\cdot)) := \{ \omega \in \Omega: F^L(\omega, X) \leq (C^{-1}_I R(\omega) - C^I_2 N(\omega))/N(\omega) \leq F^U(\omega, X) \} \quad (15.1)\]

Thus, for \( P(X) \) we find the lower bound

\[
\]
\[ P(X) \geq \bar{P}(X, N(\cdot)), \quad (16) \]

where

\[ \bar{P}(X, N(\cdot)) := P\left( \begin{array}{c} F^L_{I}(\omega, X) \leq C_{I}^{-1}(R(\omega) - C_{II}N(\omega)) \leq F^U_{I}(\omega, X) \\ F^L_{II}(\omega, X) \leq N(\omega) \leq F^U_{II}(\omega, X) \end{array} \right), \quad (16.1) \]

and \( F^L_{I}, F^U_{I}, F^L_{II}, F^U_{II} \) denotes the partition of \( F^L, F^U \), resp., corresponding to (13). Note that the inequality (16) holds for any choice \( N = (F_{i}) \) of an \((n-m)\)-subvector of redundants such that (13.1) holds and any representation of \( N \) as a random vector \( N = N(\omega) \) on \((\Omega, \mathcal{F}, P)\); especially, \( N \) can be selected as a deterministic vector of redundants:

\[ N(\omega) = z \ a.s. \ (\text{almost sure}), \quad (17) \]

where \( z \in \mathbb{R}^{n-m} \) is a deterministic vector; in this case we set \( \bar{P}(X, N(\cdot)) = \bar{P}(X, z) \).

4.1.1. Special cases. a) In case of trusses, cf. (1.3), (2.2), we have that

\[ \begin{align*}
F^L_{I}(\omega, X) &= A^L_{I}(X) d^L_{I}(\omega), \quad F^U_{I}(\omega, X) = A^U_{I}(X) d^U_{I}(\omega) \\
F^L_{II}(\omega, X) &= A^L_{II}(X) d^L_{II}(\omega), \quad F^U_{II}(\omega, X) = A^U_{II}(X) d^U_{II}(\omega),
\end{align*} \]

where \( A^L_{I}, A^U_{I}, d^L_{I}, d^U_{I}, A^L_{II}, A^U_{II}, d^L_{II}, d^U_{II} \) are the partitions of \( A, d^L, d^U \), resp., corresponding to the partition \( F^L, F^U \) of \( F \), and \( A^L_{I}(X) \) denotes the diagonal matrix which has the components of \( A^L_{I}(X) \) as its diagonal elements. Thus, we get

\[ \bar{P}(X, z) = P\left( \begin{array}{c} A^L_{I}(X) d^L_{I}(\omega) \leq C_{I}^{-1}(R(\omega) - C_{II}z) \leq A^U_{I}(X) d^U_{I}(\omega) \\ A^L_{II}(X) d^L_{II}(\omega) \leq z \leq A^U_{II}(X) d^U_{II}(\omega) \end{array} \right), \quad (18.1) \]

b) Suppose that the partition of \( F^L, F^U \) into \( F^L_{I}, F^U_{I}, F^L_{II}, F^U_{II} \) can be chosen such that \((F^L_{I}(\omega, X), F^U_{I}(\omega, X)), (F^L_{II}(\omega, X), F^U_{II}(\omega, X))\) are stochastically independent. If (17) holds, then

\[ \bar{P}(X, z) := P(F^L_{I}(\omega, X) \leq C_{I}^{-1}(R(\omega) - C_{II}z) \leq F^U_{I}(\omega, X)) \times P(F^L_{II}(\omega, X) \leq z \leq F^U_{II}(\omega, X)). \quad (18.2) \]

5. Failure modes, limit state functions and upper bounds for \( P(X) \)

According to (8), (10) we have that
\begin{equation}
P(X) = P(\text{There is } F=F(\omega) \text{ such that } CF(\omega)=R(\omega) \text{ and})
\begin{align*}
F^U_j(\omega) - F^L_j(\omega, X) &\leq 0, \text{ } j=1, \ldots, n \\
F^L_j(\omega, X) - F^U_j(\omega) &\leq 0, \text{ } j=1, \ldots, n,
\end{align*}
\end{equation}

where we suppose that all bounds \( F^L_j, F^U_j \) are finite, i.e.
\begin{equation}
-\infty < F^L_j(\omega, X) \leq F^U_j(\omega, X) < \infty \text{ a.s., } 1 \leq j \leq n, \text{ for all } X \text{ under consideration.}
\end{equation}

Defining
\begin{equation}
t(\omega, F(\omega), X):= \max_{1 \leq j \leq n} (F^U_j(\omega) - F^U_j(\omega, X), F^L_j(\omega, X) - F^L_j(\omega)),
\end{equation}

we obtain
\begin{equation}
P(X) = P(\text{There is } F=F(\omega) \text{ such that } CF(\omega)=R(\omega) \text{ and } t(\omega, F(\omega), X) \leq 0)
\begin{align*}
&= P(\inf(t(\omega, F(\omega), X): CF(\omega)=R(\omega)) \leq 0) \\
&= P(t^*(\omega, X) \leq 0),
\end{align*}
\end{equation}

where
\begin{equation}
t^*(\omega, X):= \inf\{t(\omega, F(\omega), X): CF(\omega)=R(\omega)\}
\end{equation}
is the minimal value of the program
\begin{equation}
\min t(\omega, F(\omega), X)
\end{equation}
s.t.
\begin{equation}
CF(\omega) = R(\omega)
\end{equation}

being equivalent to the linear program
\begin{equation}
\begin{align*}
\text{minimize } & t \\
s.t. \\
F^U_j - F^L_j(\omega, X) &\leq 0, \text{ } j=1, \ldots, n \\
F^L_j(\omega, X) - F^U_j &\leq 0, \text{ } j=1, \ldots, n \\
CF(\omega) &= R(\omega)
\end{align*}
\end{equation}

with the variables \( F_1, F_2, \ldots, F_n, t, \).

Because of condition (19.1), for each \( (\omega, X) \) we get
\begin{equation}
t(\omega, F(\omega), X) \geq \max_{1 \leq j \leq n} \frac{1}{2} (F^L_j(\omega, X) - F^U_j(\omega, X)) > -\infty
\end{equation}

for arbitrary \( F(\omega) \); hence, the objective function of the linear program (21) is bounded from below for all \( (\omega, X) \). Since the LP (21) always has a feasible solution, for each \( (\omega, X) \) an optimal solution \( (F^*_j)^* \) of (21) is guaranteed, and we have that
\begin{equation}
t^* = t(\omega, F^*_j(\omega), X) = t^*(\omega, X).
\end{equation}

Consequently, by means of duality theory the optimal value \( t^*(\omega, X) \) of the equivalent programs (20.2) and (21) can be represented also by the optimal value of the dual program of (21) given by
\[
\max R(\omega)'u - F^U(\omega,X)'\tilde{u}^+ + F^L(\omega,X)'\tilde{u}^-
\]  
\text{s.t.} 
\[
C'u - \tilde{u}^+ + \tilde{u}^- = 0 \\
1'\tilde{u}^+ + 1'\tilde{u}^- = 1 \\
\tilde{u}^+ \geq 0, \tilde{u}^- \geq 0,
\]  
where \(u \in \mathbb{R}^n\) is not restricted.

\textbf{Remark 5.1}

Obviously, (23.1) is the member-node (or joint) displacement equation.

According to its mechanical meaning, we call (21), (23), resp., the static, \textit{kinematic} linear program (LP), cf. [2], [14].

Having (23), \(t^*(\omega,X)\) reads
\[
t^*(\omega,X) = \max (\begin{pmatrix} R(\omega) \\ -F^U(\omega,X) \end{pmatrix}', \begin{pmatrix} u \\ \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix}; : \delta \in \Delta_0),
\]
where \(\Delta_0\) denotes the convex polyhedron in \(\mathbb{R}^{m+2n}\) represented by the constraints (23.1)-(23.3) of the LP (23). Taking any subset \(\Delta_1 \subset \Delta_0\) of \(\Delta_0\), and defining then \(t^*_1(\omega,X)\) by
\[
t^*_1(\omega,X) := \sup (\begin{pmatrix} R(\omega) \\ -F^U(\omega,X) \end{pmatrix}', \delta; : \delta \in \Delta_1),
\]
we get
\[
t^*(\omega,X) \geq t^*_1(\omega,X),
\]
which yields for \(P(X)\) the following upper bound:
\[
P(X) = P(t^*(\omega,X) \leq 0) \leq P(t^*_1(\omega,X) \leq 0).
\]

Moreover, if
\[
\delta^{(l)} = \begin{pmatrix} u^{(l)} \\ \tilde{u}^+(l) \\ \tilde{u}^-(l) \end{pmatrix}, \ l = 1, \ldots, l_0,
\]
denote the extreme points of the convex polyhedron \(\Delta_0\), then
\[
t^*(\omega,X) = \max_{1 \leq l \leq l_0} R(\omega)'u^{(l)} - F^U(\omega,X)'\tilde{u}^+(l) + F^L(\omega,X)'\tilde{u}^-(l),
\]
which shows that \(t^*(\cdot,X)\) is measurable. Hence, \(S(F^L(\cdot), F^U(\cdot), R(\cdot)) = (\omega \in \Omega: t^*(\omega,X) \leq 0)\) is measurable, cf. (10), (20), and we get
\[
P(X) = P(R(\omega)'u^{(l)} - F^U(\omega,X)'\tilde{u}^+(l) + F^L(\omega,X)'\tilde{u}^-(l) \leq 0, 1 \leq l \leq l_0).
\]

According to (6), (7), the survival, failure, resp., of the underlying structure can be described by the inequality
\[
t^*(\omega,X) \leq 0, t^*(\omega,X) > 0, \text{ respectively.}
\]

Thus, the structure fails if and only if
- 10 -

\[ R(\omega)'u^{(\ell)} + F^U(\omega,X)'\bar{u}^+(\ell) - F^L(\omega,X)'\bar{u}^-(\ell) > 0 \text{ for at least one } 1 \leq \ell \leq \ell_o; \]  

(27.1)

obviously, (27.1) represents the different failure modes of the structure.

Having a certain number \( \ell_1 \leq \ell_o \) of basic solutions \( \delta^{(\ell)}_r \), \( r = 1, \ldots, \ell_1 \), of the LP (23), and defining

\[ \bar{t}_1^*(\omega,X) := \max_{1 \leq r \leq \ell_1} \left( R(\omega)'u^{(\ell)} - F^U(\omega,X)'\bar{u}^+(\ell) + F^L(\omega,X)'\bar{u}^-(\ell) \right), \]  

(28)

corresponding to (25.1), here we get

\[ t^*(\omega,X) \geq \bar{t}_1^*(\omega,X) \]  

(28.1)

and therefore

\[ P(X) = P(t^*(\omega,X) \leq 0) \leq P(\bar{t}_1^*(\omega,X) \leq 0). \]  

(28.2)

6. The probability of failure \( p_f \)

According to (8), (10), (20) and (27), for the probability of failure

\[ p_f := 1 - p_s = 1 - P(X) \] we obtain

\[ p_f = P(t^*(\omega,X) > 0) \]

\[ = P(R(\omega)'u - F^U(\omega,X)'\bar{u}^+ + F^L(\omega,X)'\bar{u}^- > 0 \text{ for at least one } \begin{pmatrix} \omega \\ \bar{u}^+ \\ \bar{u}^- \end{pmatrix} \in \Delta_o) \]

\[ = P(R(\omega)'u^{(\ell)} - F^U(\omega,X)'\bar{u}^+(\ell) + F^L(\omega,X)'\bar{u}^-(\ell) > 0 \text{ for at least one } 1 \leq \ell \leq \ell_o) \]  

(29)

\[ = P(\bigcup_{\ell=1}^{\ell_o} F_\ell(X)), \]

where \( F_\ell(X) \) denotes the \( \ell \)-th failure domain

\[ F_\ell(X) := \{ \omega \in \Omega : R(\omega)'u^{(\ell)} - F^U(\omega,X)'\bar{u}^+(\ell) + F^L(\omega,X)'\bar{u}^-(\ell) > 0 \} \]  

(29.1)

\[ = \{ \omega \in \Omega : M_\ell(\omega,X) < 0 \} \]

with the corresponding limit state function

\[ M_\ell(\omega,X) := F^U(\omega,X)'\bar{u}^+(\ell) - F^L(\omega,X)'\bar{u}^-(\ell) - R(\omega)'u^{(\ell)}, \]  

(29.2)

\( \ell = 1, \ldots, \ell_o \); especially, for trusses, cf. (2.2), (18.1), we find

\[ M_\ell(\omega,X) := \sigma^U(\omega)'A(X)'u^+(\ell) - \sigma^L(\omega)'A(X)'u^-(\ell) - R(\omega)'u^{(\ell)}. \]  

(29.3)

Using known inequalities for probabilities, for \( p_f \) we find the bounds

\[ \max_{1 \leq \ell \leq \ell_o} p_{f,\ell} \leq p_f \leq \sum_{\ell=1}^{\ell_o} p_{f,\ell}, \]  

(30)

where \( p_{f,\ell} \) is given by

\[ p_{f,\ell} := P(F_\ell(X)) - P(M_\ell(\omega,X) < 0) \]

\[ = P(F^U(\omega,X)'u^+(\ell) - F^L(\omega,X)'u^-(\ell) < R(\omega)'u^{(\ell)}) \]

\[ = 1 - P(R(\omega)'u^{(\ell)} \leq F^U(\omega,X)'u^+(\ell) - F^L(\omega,X)'u^-(\ell)). \]  

(30.1)
and sharper bounds can be obtained by using more general Bonferroni-type inequalities for probabilities.

7. Representation of \( p_s \) by using cones

According to (9) we have that

\[ p_s = P(\omega) \in C[F^L(\omega), F^U(\omega))] , \]

where \([F^L, F^U]\) is given by (8.2). Representing therefore the vector \( F \) of internal member forces/bending-moments by

\[ F = F^L + \Delta F^U = F^U - \Delta F^L \]

with \( n \)-vectors \( \Delta F^U, \Delta F^L \geq 0 \), the condition \( R \in C[F^L, F^U] \) can be represented by

\[
\begin{align*}
R &- CF^U & - C \Delta F^U \\
- (F^U \cdot F^L) & - \Delta F^U - \Delta F^L \\
\Delta F^U &\geq 0, \Delta F^L \geq 0.
\end{align*}
\]

(31)

Thus, we consider the cone \( Y_o \subset \mathbb{R}^{m+n} \) defined by

\[
Y_o := \left\{ \begin{pmatrix} C & 0 \\ I & I \end{pmatrix} \begin{pmatrix} \Delta F^U \\ \Delta F^L \end{pmatrix} : \Delta F^U \geq 0, \Delta F^L \geq 0 \right\}
\]

(32)

\[
\sum_{k=1}^{2n} \alpha_k y_k : \alpha_k \geq 0, k=1, \ldots, 2n,
\]

where the cone-generators \( y_k(k) \), \( k=1, \ldots, 2n \), are given by

\[
y_k(k) := \begin{pmatrix} c_k \\ e_k \end{pmatrix}, \quad 1 \leq k \leq n, \quad y_k(k) := \begin{pmatrix} 0 \\ e_k \end{pmatrix}, \quad n < k \leq 2n,
\]

(32.1)

and \( c_k, e_k \) denotes the \( k \)-th column of \( C \), of the \( n \times n \) identity matrix \( I \), respectively. Having \( Y_o \), the set \( S(F^L(\cdot), F^U(\cdot), R(\cdot)) \) defined in (8.1) can be described by

\[
S(F^L(\cdot), F^U(\cdot), R(\cdot)) = \{ \omega \in \Omega : \begin{pmatrix} R(\omega) - CF^U(\omega, X) \\ -F^U(\omega, X) + F^L(\omega, X) \end{pmatrix} \in (-1)Y_o \}, \]

(33)

which shows again the measurability of \( S(F^L(\cdot), F^U(\cdot), R(\cdot)) \).

Moreover, the probability function \( P(\lambda, X) \) representing (10)-(10.3) can be given by

\[
P(\lambda, X) = P\left( \begin{pmatrix} CF^U(\omega, X) - \lambda R_o(\omega) \\ F^U(\omega, X) - F^L(\omega, X) \end{pmatrix} \in Y_o \right).
\]

(33.1)

**Remark 7.1**

The cone \( Y_o \) contains all pairs \((C \Delta F^U, \Delta F^L + \Delta F^U)\) of admissible pairs of external load/strength increments.

According to the representation (32) of \( Y_o \), there are a finite number of boundary hyperplanes in \( \mathbb{R}^{m+n} \), represented by vectors \( \eta(\ell) = \omega(\ell) \)

\[ \ell = 1, \ldots, \ell' \]

such that

\[
Y_o = \{ y = \begin{pmatrix} w \\ v \end{pmatrix} \in \mathbb{R}^{m+n} : y' \eta(\ell) = w' \omega(\ell) + v' v(\ell) \geq 0, 1 \leq \ell \leq \ell' \}.
\]

(34)

Hence, because of (33) and (34), the survival of the structure can be
represented also by the inequalities
\[(R(\omega) - CF^U(\omega, X))'w^{(\ell)} + (-F^U(\omega, X) + F^L(\omega, X))'v^{(\ell)} \leq 0, \ 1 \leq \ell \leq \ell_0',\]
which yields
\[(R(\omega)'w^{(\ell)} - F^U(\omega, X)'(C'w^{(\ell)} + v^{(\ell)}) + F^L(\omega, X)'v^{(\ell)} \leq 0, \ 1 \leq \ell \leq \ell_0'.\] (35)
and therefore
\[ps = P(R(\omega)'w^{(\ell)} - F^U(\omega, X)'(C'w^{(\ell)} + v^{(\ell)}) + F^L(\omega, X)'v^{(\ell)} \leq 0, \ 1 \leq \ell \leq \ell_0'.\] (35.1)

Obviously, the conditions for structural safety given by (27) and (35) coincide.

For arbitrary subset \[Y^{(\ell)}_o, \ \ell=1,2,\] such that
\[Y^{(1)}_o \subseteq Y^{(2)}_o,\] (36)
we get
\[p_{s_{(1)}} \leq p_s \leq p_{s_{(2)}},\] (36.1)
where the bounds \[p_{s\_\ell}, \ \ell=1,2,\] are defined by
\[p_{s\_\ell} := P\left(\frac{CF^U(\omega, X) - R(\omega)}{F^U(\omega, X) - F^L(\omega, X)} \in Y^{(\ell)}_o\right), \ \ell=1,2.\] (36.2)

7.1. Construction of approximating cones

Suppose next to that we have a cone axis (center or middle line)
\[g = (\lambda\hat{y}: \lambda \geq 0)\] (37)
generated by a vector \[\hat{y} \in Y_o, \ \hat{y} \not= 0,\] which is defined later on.

![Diagram](image)

**Fig. 7.1.** Cone \(Y_o\) with cone-generators \(y^{(\ell)}, \ \ell=1,\ldots,6,\) cone-axis \(g,\) hyperplane \(E_o\) and convex polyhedron \(K_o = \text{conv}(\overline{y^{(1)}},\ldots,\overline{y^{(6)}}).\)
Let $E_0$ denote the hyperplane
\[(y \cdot \tilde{y})' \tilde{y} = 0 \iff y' \tilde{y} = ||y||^2\] (37.1)
through $\tilde{y}$ and orthogonal to axis $g$. Consider then the points $y^{(\ell)} \in Y_0$, $\ell=1, \ldots, 2n$, lying on $E_0$ and on the straight lines through 0 and $y^{(\ell)}$, $\ell=1, \ldots, 2n$, hence,
\[y^{(\ell)} := ||\tilde{y}||^2 / y^{(\ell)}' \tilde{y}, \ell=1, \ldots, 2n,\] (38)
see Figure 7.1. Obviously, we get \[y^{(\ell)}' \tilde{y} > 0, \ell=1, \ldots, 2n,\] (38.1)
as a condition for $\tilde{y}$. Since the equation $\Sigma_{k=1}^n \alpha_k y^{(k)} = 0$, $\alpha_k \geq 0$, $k=1, \ldots, 2n$, cf. (32), (32.1), has no nonzero solution $\alpha$, according to Gordan's transposition theorem, system (38.1) has solutions $\tilde{y} \neq 0$. Let
\[K_0 := \text{conv}(\tilde{y}^{(1)}, \tilde{y}^{(2)}, \ldots, \tilde{y}^{(2n)})\]
denote the convex polyhedron on $E_0$ generated by the points $\tilde{y}^{(\ell)}$, $\ell=1, \ldots, 2n$. By (38.1) and (38.2) we get
\[y' \tilde{y} > 0 \text{ for all } y \in K_0.\] (38.1a)
According to (38) and (38.1), the cone $Y_0$ can be represented by
\[Y_0 = \bigcup_{\lambda \geq 0} K_0 = \{\lambda y : \lambda > 0, y \in K_0\}.\] (38.3)

Based on the above representation of $Y_0$, approximations $\tilde{Y}_0$ of $Y_0$ of the type (36) can be obtained by replacing the generating polyhedron $K_0$ in $Y_0$ by suitable approximations, e.g., ellipsoids, spheres, denoted by $\tilde{K}_0$:

\[K_0^{(sph,2)} \quad K_0^{(sph,1)} \quad K_0^{(ell,1)} \quad K_0^{(ell,2)} \]

\[\tilde{K}_0^{(sph,j)} \quad \tilde{K}_0^{(ell,j)}\]

**Fig. 7.2.** Convex polyhedron $K_0$ in hyperplane $E_0$ and (second order) approximations $\tilde{K}_0^{(sph,j)}$, $\tilde{K}_0^{(ell,j)}$, $j=1,2$
a) Ellipsoidal approximations: According to (38.1a), an outer ellipsoidal approximation \( K_{o}^{(e11,2)} \supset K_{o} \) can be defined, for \( j=2 \), by
\[
K_{o}^{(e11,2)} := \{ y \in \mathbb{R}^{n} : y^\top y > 0, \ (y-\tilde{y})' \Gamma^{(1)} \Gamma^{(1)\top} (y-\tilde{y}) \leq 1 \},
\]
where the \((n+m) \times (n+m)\) matrix \( \Gamma^{(2)} \) is chosen such that
\[
(y(\hat{\xi}) - \tilde{y})' \Gamma^{(2)}' (y(\hat{\xi}) - \tilde{y}) \leq 1, \ \ell=1, \ldots, 2n
\]
\[
y^\top y = 0 \implies (\Gamma y)'y = 0 \quad (39.1)
\]
\[
2n \sum_{\ell=1}^{2n} (y(\hat{\xi}) - \tilde{y})' \Gamma^{(2)}' (y(\hat{\xi}) - \tilde{y}) \implies \min
\]
Conversely, an "inner" ellipsoidal approximation \( K_{o}^{(e11,1)} \) of \( K_{o} \) can be determined, cf. (39), by choosing a matrix \( \Gamma^{(1)} \) such that
\[
(y(\hat{\xi}) - \tilde{y})' \Gamma^{(1)}' (y(\hat{\xi}) - \tilde{y}) \geq 1, \ \ell=1, \ldots, 2n
\]
\[
y^\top y = 0 \implies (\Gamma y)'y = 0 \quad (39.2)
\]
\[
2n \sum_{\ell=1}^{2n} (y(\hat{\xi}) - \tilde{y})' \Gamma^{(1)}' (y(\hat{\xi}) - \tilde{y}) \implies \min
\]
Note that \( K_{o}^{(e11,1)} \setminus K_{o} \neq \emptyset \) may happen.

b) Spherical approximations: An outer spherical approximation \( K_{o}^{(sph,2)} \supset K_{o} \) is defined, cf. (38.1a), by
\[
K_{o}^{(sph,2)} := \{ y \in \mathbb{R}^{n} : y^\top y > 0, \ |y-\tilde{y}| < \rho_{2} \},
\]
where the radius \( \rho_{2} \) is given by
\[
\rho_{2} := \max_{1 \leq i \leq 2n} |\tilde{y}(\hat{\xi}) - \tilde{y}|
\]
and \(|.|\) denotes the Euclidean norm. Likewise, an "inner" spherical approximation \( K_{o}^{(sph,1)} \) of \( K_{o} \) can be determined, cf. (40), by choosing the radius
\[
\rho_{1} := \min_{1 \leq i \leq 2n} |\tilde{y}(\hat{\xi}) - \tilde{y}|
\]
where, as for \( K_{o}^{(e11,1)} \), the relation \( K_{o}^{(sph,1)} \setminus K_{o} \neq \emptyset \) is not excluded in general. Of course, an inner spherical approximation \( K_{o}^{(sph,1)} \subset K_{o} \) is obtained by selecting the radius
\[
\rho_{1} := \min_{y \in \partial K_{o}} |y-\tilde{y}|, \quad \rho_{2} := \max_{1 \leq i \leq 2n} |\tilde{y}(\hat{\xi}) - \tilde{y}|
\]
where \( \partial K_{o} \) denotes the boundary of \( K_{o} \).

Having an approximation \( K_{o} \) of \( K_{o} \) as described by (39) and (40), the cone \( Y_{o} \) can be approximated now by the cone
\[
Y_{o} := \{ y \in \mathbb{R}^{n} : y^\top y > 0, \ y \in K_{o} \}.
\]
Fig. 7.3. Approximating cone $\hat{Y}_0$ generated by $\hat{R}_0$

For any point $y \in \mathbb{R}^{m+n}$, $y \neq 0$, the intersection $y_o$ of the hyperplane $E_o$ and the straight line through 0 and $y$ is given by

$$y_o := \frac{||\hat{y}||^2}{y'y} y.$$  \hspace{1cm} (41.1)

Hence, according to the definition (41) of $\hat{V}_o$ we have that $y \in \hat{V}_o$, $y \neq 0$, if and only if the following simple relations for $y$ hold:

$$y'y > 0$$  \hspace{1cm} (42)

$$||\Gamma'(||\hat{y}||^2y - y'y \hat{y})|| \leq y'y$$  \hspace{1cm} (42.1a)

$$||y|| \leq \frac{(\rho^2 + ||\hat{y}||^2)^{1/2}}{||\hat{y}||^2} y'y, \hat{y},$$  \hspace{1cm} (42.1b)

where $\Gamma = \Gamma^{(1)}, \Gamma^{(2)}$ and $\rho = \rho_1, \rho_2$. Obviously, (42)-(42.1b) are convex conditions for $y$.

Having an approximation $\tilde{Y}_o$ of $Y_o$, the probability function $P = P(\lambda, X)$ given by (33.1) can be approximated then, cf. (36.2), by

$$\tilde{P}(\lambda, X) := P\left(\begin{array}{c} CF^{U}(\omega, X) - \lambda X \\ F^{U}(\omega, X) - F^{L}(\omega, X) \end{array}\right)_{\tilde{V}_o}.$$  \hspace{1cm} (43)

Hence, in the above case from (42)-(42.1b) we obtain

$$\tilde{F}(\lambda, X) = P(\omega) P(\Gamma'(||\hat{y}||^2y - y'y(\omega) \hat{y}) || \leq y(\omega)' \hat{y}),$$  \hspace{1cm} (43.1a)

$$\tilde{F}(\lambda, X) = P(\omega) P(\omega) \leq \frac{(\rho^2 + ||\hat{y}||^2)^{1/2}}{||\hat{y}||^2} y(\omega)' \hat{y}),$$  \hspace{1cm} (43.1b)
resp., where
\[
y(\omega) := \begin{pmatrix} c_{1}^{U}(\omega, X) - \lambda R_{o}(\omega) \\ F_{c}^{U}(\omega, X) - F_{c}^{L}(\omega, X) \end{pmatrix}
\] (43.2)

Finally, we have to determine the axis \( g = (\lambda \hat{y}: \lambda \geq 0) \) by selecting a generating vector \( \hat{y} \in Y_{o}, \hat{y} \neq 0 \). Having \( \hat{y} = \sum_{k=1}^{2n} \alpha_{k} y^{(k)} \) with \( \alpha_{k} \geq 0 \), \( k = 1, \ldots, 2n \), condition (38.1) can be fulfilled by choosing the coefficients \( \alpha = (\alpha_{1}, \ldots, \alpha_{2n})' \) such that
\[
Y_{o}^{M} \alpha > 0,
\] (44)

where the \( 2n \times 2n \) matrix \( Y_{o}^{M} \) is given by
\[
Y_{o}^{M} := (y^{(\ell)'}, y^{(k)})_{\ell, k = 1, \ldots, 2n}
\] (44.1)

\[
\begin{pmatrix}
||c_{1}||^{2} + 1 & c_{1}^{c} c_{2} & \ldots & c_{1}^{c} c_{n} \\
c_{2}^{c} c_{1} & ||c_{2}||^{2} + 1 & \ldots & c_{2}^{c} c_{n} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & ||c_{n}||^{2} + 1 \\
c_{n}^{c} c_{1} & c_{n}^{c} c_{2} & \ldots & ||c_{n}||^{2} + 1 \\
\end{pmatrix}
\]

\[-
\begin{pmatrix}
I_{n} \\
I_{n}
\end{pmatrix}
\]

where \( I_{n} \) is the \( n \times n \) identity matrix. Thus, we may select then \( \alpha \) such that
\[
\min_{1 \leq \ell \leq 2n} y^{(\ell)} \hat{y} \text{ is maximized, which yields the linear program}
\] (45)

\[
\max t \quad \begin{pmatrix}
- Y_{o}^{M} \\
I_{2n}
\end{pmatrix} \begin{pmatrix}
\alpha \\
0
\end{pmatrix} \leq \begin{pmatrix}
1 \\
0
\end{pmatrix}
\] (45.1)

\[
0 \leq \alpha \leq a_{0}, \quad t \geq 0,
\] (45.2)

where \( a_{0} = (a_{o1}, a_{o2}, \ldots, a_{o2n})' \) is a given \( 2n \)-vector having positive components, and the constraint \( \alpha \leq a_{0} \) is imposed because the direction of \( \hat{y} \) is needed only.

On the other hand, for any vector \( \hat{y}, \hat{y} \neq 0 \), we consider the hyperplane \( E_{o} \), defined by (37.1) and the points \( y^{(\ell)} \), \( \ell = 1, \ldots, 2n \), on \( E_{o} \) defined by (38):
Fig. 7.4. Construction of a cone axis $g$ of $Y_o$: Approximative generators $\tilde{y}_I$, $\tilde{y}_{II}$.

Since the axis $g=(\lambda \tilde{y}: \lambda \geq 0)$ should pass through the center of the cone $Y_o$, the deviations between $\tilde{y}$ and the points $\tilde{y}^{(k)}$, $k=1, \ldots, 2n$, should be well-balanced. Thus, $\tilde{y}$ is chosen such that the quantity

$$\max_{1 \leq k \leq 2n} \frac{||\tilde{y} - \tilde{y}^{(k)}||}{\max_{0 \leq \lambda \leq 2n} \left(\frac{||\tilde{y}||}{2} - ||\tilde{y}^{(k)}|| \right)^2}$$

(cf. (38), is minimized, which yields, see (38.1), the optimization problem

$$\min \max_{1 \leq k \leq 2n} \frac{||\tilde{y} - \tilde{y}^{(k)}||}{||\tilde{y}||}$$

s.t.

$$y^{(k)}, \tilde{y} > 0, k=1, \ldots, 2n$$

$$\tilde{y} \in Y_o, ||\tilde{y}||=1;$$

since only the direction of $\tilde{y}$ is relevant for our purposes, the norm constraint $||\tilde{y}||=1$ is added. According to (46), problem (47) is equivalent to the program

$$\max_t$$

s.t.

$$t - \frac{y^{(k)} \tilde{y}}{||y^{(k)}||} \leq 0, k=1, \ldots, 2n$$

$$t \geq 0, \tilde{y} \in Y_o, ||\tilde{y}||=1.$$
Note that
\[
\frac{\gamma(\hat{\lambda}) - \gamma(\check{\lambda})}{||\gamma(\hat{\lambda})||} = \cos \hat{\gamma}(\gamma(\hat{\lambda}), \check{\gamma}),
\]
is the cosine of the angle between the vectors \(\gamma(\hat{\lambda})\) and \(\check{\gamma}\).

Representing \(\check{\gamma} \in Y_0\) by \(\check{\gamma} = \sum_{k=1}^{2n} \alpha_k y^{(k)}\), \(\alpha_k \geq 0\), \(k=1, \ldots, 2n\), we observe that (47)' is closely related to (45).

8. Sensitivity analysis of probabilities of survival/failure

In the following we consider the sensitivity of the probability of survival/failure with respect to the variables \((\lambda, X, z)\), i.e., with respect to the \(m\)-vector \(\lambda\) of deterministic load factors, the \(r\)-vector \(X\) of design variables, and the \((n-m)\)-vector \(z\) of deterministic redundants in the \(n\)-vector \(F-F(\omega)\) of member (element) loads. As developed in [8-13], derivatives of the probability functions \(P=P(\lambda, X, \check{F}=\bar{F}(\lambda, X, z)\) can be obtained - under weak mathematical assumptions - by means of the Transformation Method in combination with a stochastic completion technique.

8.1. The probability functions (10.3)-(10.5). In order to show the differentiation of the probability functions given by (10.3)-(10.5) it is sufficient to consider probability functions of the type

\[
P(\lambda, X) = P(\lambda R_0(\omega) \in [A(X) \sigma_{L}(\omega), A(X) \sigma_{U}(\omega)]),
\]
where we suppose that the random vectors \(R_0(\omega)\), \((\sigma_{L}(\omega), \sigma_{U}(\omega))\) have sufficiently smooth probability densities \(\psi(\cdot, \sigma_{L})\), \(\psi(\cdot, \sigma_{U})\) on \(R^m, R^{2n}\), respectively. Hence, using also (33), we have that

\[
P(\lambda, X) = P(\lambda, A=A(X) = (A_1(X), \ldots, A_n(X))),
\]
cf. (2.3), where

\[
P(\lambda, A) = \int_{A_d \sigma_{L} \in \mathcal{C}[A_d \sigma_{L}, A_d \sigma_{U}]} \varphi(\sigma_{L}) \psi(\sigma_{L}, \sigma_{U}) d\sigma_{L} d\sigma_{U}
\]

\[
\lambda R_0 \in \mathcal{C}[A_d \sigma_{L}, A_d \sigma_{U}],
\]

\[
\begin{pmatrix}
C_{A_d \sigma_{U}} - \lambda R_0 \\
A_d \sigma_{U} - A_d \sigma_{L}
\end{pmatrix} \in Y_0
\]

Applying, for given variables \((\lambda, A)\) with \(\lambda \neq 0\), \(A_i \neq 0\), \(i=1, \ldots, n-n_0\), resp., to (48.2) the transformation \(T(\lambda, A) : (R, F^L, F^U) \rightarrow (R_0, \sigma_{L}, \sigma_{U})\) in \(R^m \times R^{2n}\) defined by

\[
R_0 := \frac{1}{\lambda} R, \quad \sigma_{L} := A^{-1}_d \sigma_{L}, \quad \sigma_{U} := A^{-1}_d \sigma_{U},
\]

we obtain
\[ P(\lambda, A) = \int_{\mathbb{R}^m} \varphi^{(1)}(\lambda)^{R} \psi(A_d^{-1}F, A_d^{-1}F) \times \frac{1}{|\lambda|^m} \prod_{j=1}^n \frac{1}{A_j} \ dR dF dF. \]  

(48.4)

Under weak assumptions [10],[13] we may interchange in (48.4) differentiation and integration, hence,

\[ \frac{\partial P}{\partial \lambda}(\lambda, A) = -\frac{1}{\lambda} \int_{\mathbb{R}^m} (\nabla \varphi^{(1)}(\lambda)^{R} + m \varphi^{(1)}(\lambda)^{R}) \psi(A_d^{-1}F, A_d^{-1}F) \times \frac{1}{|\lambda|^m} \prod_{j=1}^n \frac{1}{A_j} \ dR dF dF. \]  

(49)

Using the inverse transformation \( T^{(-1)}(\lambda, A) \), (49) yields

\[ \frac{\partial P}{\partial \lambda}(\lambda, A) = -\frac{1}{\lambda} \int_{\mathbb{R}^m} \text{div}(R_0 \varphi(R_0)) \psi(\sigma^L, \sigma^U) dR_0 d\sigma d\sigma \]  

(49.1)

with the conditional probability function

\[ P(\lambda, A|R_0) = P(\lambda R_0 \in C[A_d^{-1}\sigma^L, A_d^{-1}\sigma^U]). \]  

(49.2)

Considering now the derivative \( \frac{\partial P}{\partial x_k}(\lambda, X) \) with respect to a design variable \( x_k \), by means of (49.1) we find

\[ \frac{\partial P}{\partial x_k}(\lambda, X) = \sum_{j=1}^n \frac{\partial}{\partial A_j} \frac{1}{A_k} \psi^{(1)} \]  

(50)

and, using again (48.4), by interchanging differentiation and integration there we get

\[ \frac{\partial P}{\partial A_j}(\lambda, A) = -\frac{1}{A_j} \int_{\mathbb{R}^m} \psi^{(1)}[2\psi(A_d^{-1}F, A_d^{-1}F) \]  

(51)

Using the inverse \( T^{(-1)}(\lambda, A) \) of (48.3) again, from (50.1) we get

\[ \frac{\partial P}{\partial A_j}(\lambda, A) = -\frac{1}{A_j} \int_{\lambda R_0 \in C[A_d^{-1}\sigma^L, A_d^{-1}\sigma^U]} \psi^{(1)}[2\psi(\sigma^L, \sigma^U) \]  

(51.1)

with the conditional probability function

\[ P(\lambda, A|\sigma^L, \sigma^U) = P(\lambda R_0(\omega) \in C[A_d^{-1}\sigma^L, A_d^{-1}\sigma^U]), \sigma^L, \sigma^U \in \mathbb{R}. \]  

(51.2)

and
\begin{equation}
\langle \sigma^L, \sigma^U \rangle (j) := (0, \ldots, 0, \sigma^L_j, 0, \ldots, 0, \sigma^U_j, 0, \ldots, 0)', \tag{51.3}
\end{equation}

where \( \sigma^L_j, \sigma^U_j \) are placed at the \( j \)-th, \((n+j)\)-th position, respectively.

According to the representation (31) of the event \( \{ R \in C[F^L, F^U] \} \), the conditional probability functions \( P(\lambda, A|R_0) \), \( P(\lambda, A|\sigma^L, \sigma^U) \) can be represented, cf. (31)-(33.1), (48.2), by

\begin{align}
P(\lambda, A|R_0) &= P(\gamma(\omega|R_0) \in Y_0), \tag{52}
\end{align}

\begin{align}
P(\lambda, A|\sigma^L, \sigma^U) &= P(\gamma(\omega|\sigma^L, \sigma^U) \in Y_0),
\end{align}

where

\begin{align}
y(\omega|R_0) := & \left( \begin{array}{c}
CA_d \sigma^U(\omega) - LR_0 \\
A_d \sigma^U(\omega) - A_d \sigma^L(\omega)
\end{array} \right), \tag{52.1}
\end{align}

\begin{align}
y(\omega|\sigma^L, \sigma^U) := & \left( \begin{array}{c}
CA_d \sigma^U(\omega) - LR_0 \\
A_d \sigma^U(\omega) - A_d \sigma^L(\omega)
\end{array} \right).
\end{align}

Hence, corresponding to the approximation \( \widetilde{P}(\lambda, A) \) of \( P(\lambda, A) \), cf. (43)-(43.2), the above conditional probability functions (52) can be approximated by

\begin{align}
\widetilde{P}(\lambda, A|R_0) &:= P(\gamma(\omega|R_0) \in \widetilde{Y}_0), \tag{52.2}
\end{align}

\begin{align}
\widetilde{P}(\lambda, A|\sigma^L, \sigma^U) &:= P(\gamma(\omega|\sigma^L, \sigma^U) \in \widetilde{Y}_0),
\end{align}

where \( \widetilde{Y}_0 \) is the approximation of the convex cone as described in Section 7.

By a slight modification in the equations (49.1), (51.1), resp., the derivatives of \( P^* P(\lambda, A) \) can also be represented by means of expectations:

**Theorem 8.1.** Under the assumptions in Section 8 we get

\begin{align}
a) \frac{\partial P}{\partial \lambda}(\lambda, A) &= - \frac{1}{\lambda} E \frac{\text{div}(R_0(\omega)\varphi(R_0(\omega)))}{\varphi(R_0(\omega))} \frac{1}{C[A_d \sigma^L(\omega), A_d \sigma^U(\omega)]} (LR_0(\omega)),
\text{div}(R_0(\omega)\varphi(R_0(\omega))) \\
&= - \frac{1}{\lambda} E \frac{\text{div}(R_0(\omega)\varphi(R_0(\omega)))}{\varphi(R_0(\omega))} P(\lambda, A|R_0(\omega)), \tag{53}
\end{align}

\begin{align}
b) \frac{\partial P}{\partial A}(\lambda, A) &= - \frac{1}{A_j} E \frac{\text{div}((\sigma^L(\omega), \sigma^U(\omega)) (j) \psi(\sigma^L(\omega), \sigma^U(\omega)))}{\psi(\sigma^L(\omega), \sigma^U(\omega))} \times \frac{1}{C[A_d \sigma^L(\omega), A_d \sigma^U(\omega)]} (LR_0(\omega)) \\
&= - \frac{1}{A_j} E \frac{\text{div}((\sigma^L(\omega), \sigma^U(\omega)) (j) \psi(\sigma^L(\omega), \sigma^U(\omega)))}{\psi(\sigma^L(\omega), \sigma^U(\omega))} P(\lambda, A|\sigma^L(\omega), \sigma^U(\omega)). \tag{53.1}
\end{align}

**Remark 8.1.**

a) Selecting fixed variables \( (\lambda, \tilde{A}) \) such that \( \tilde{A} > 0, \tilde{A}_i > 0, i=1, \ldots, n \), and applying then - instead of \( T^{-1}(\lambda, A) \) - the inverse transformation \( T^{-1}(\lambda, A) \) to \( (\lambda, \tilde{A}) \) (49), we obtain
\[
\frac{\partial P}{\partial \lambda}(\lambda, A) = \frac{1}{\lambda} \int_{(\lambda, A)} \left( \varphi_{\lambda} (\mathcal{R}_0) - \int_{\lambda} \Omega + \varphi_{\lambda} (\mathcal{R}_0) \right) \times \psi(A, \lambda, A, \mathcal{R}_0, \sigma, U) d\mathcal{R}_0 \cdot d\sigma \cdot dU,
\]
and \( \frac{\partial P}{\partial A}(\lambda, A) \) can be represented in the same way. Hence, the derivatives \( \frac{\partial P}{\partial \lambda}, \frac{\partial P}{\partial A} \) may be represented by integrals over the fixed domain \( (\lambda, A) \) in the \( (\mathcal{R}_0, \sigma, U) \)-space.

b) Since the domain of integration in the integral representation (49), (51), resp., of \( \frac{\partial P}{\partial \lambda}, \frac{\partial P}{\partial A} \) is independent of the variables \( \lambda, A=(A_1, \ldots, A_n)' \), the higher order derivatives - of arbitrary order - of \( P=P(\lambda, A) \) can be obtained by further differentiation of the equations (49), (51) with respect to \( \lambda, A_j, 1 \leq j \leq n \).

c) Having the mean value representations (53), (53.1), gradient estimates - as well as estimates of the probability function itself - can be obtained by a suitable sampling procedure.

8.2 The probability function (18.1). Corresponding to (48)-(48.2) we note first that
\[
\bar{P}(X, z) = \bar{P}(A, z), \quad A=X=(A_1(X), \ldots, A_n(X))',
\]
cf. (2.3), where
\[
\bar{P}(A, z) = P\left( \begin{array}{c}
A_1, d\sigma_1^L(\omega) \leq C_1^{-1}(R(\omega) - C_1 z) \leq A_1, d\sigma_1^U(\omega) \\
A_2, d\sigma_2^L(\omega) \leq z \leq A_2, d\sigma_2^U(\omega)
\end{array} \right).
\]

Supposing that the random vectors \( R(\omega), (\sigma_1^L(\omega), \sigma_1^U(\omega)) \) have sufficiently smooth densities \( \varphi, \psi(\sigma_1^L, \sigma_1^U) \) on \( \mathbb{R}^m, \mathbb{R}^{2m} \), resp., and defining here the integral transformation \( T(z, A): (S, F_1, q_{11}, F_1, q_{11}) \rightarrow (R, \sigma_1^L, \sigma_1^U) \) by \( R := C_1 z + S \),
\[
\sigma_1^L := A_1, d^{-1} F_1, \sigma_1^U := A_1, d^{-1}_U (z + q_{11}),
\]
\[
\sigma_1^L := A_1, d^{-1} F_1, \sigma_1^U := A_1, d^{-1}_U (z + q_{11}),
\]
where \( A_j > 0, 1 \leq j \leq n \), we find
\[
\bar{P}(A, z) = \int_{F_1 \subseteq S_1, -S \leq F_1 \leq S} \varphi(C_1 z + S) \psi\left( \begin{array}{c}
A_1, d^{-1} F_1 \\
A_2, d^{-1}_U (z + q_{11})
\end{array} \right) \times \frac{1}{2} \int_{q_{11} \leq 0} q_{11} \quad \times \frac{1}{2} \int_{q_{11} \leq 0} q_{11} \quad \times \frac{1}{2} \int_{q_{11} \leq 0} q_{11}
\quad j=1 A_j
\]
\[
\phi_{\lambda} (\mathcal{R}_0, \sigma, U) d\mathcal{R}_0 \cdot d\sigma \cdot dU.
\]

Having (54.3), the derivatives of \( \bar{P} = \bar{P}(A, z) \) - of various orders - follow again under weak assumptions [10],[13] by interchanging differentiation.
and integration in (54.3). Hence, corresponding to (51)-(51.3), for
\[ j=1,2,...,n \] we find
\[ \frac{\partial \bar{P}}{\partial A_j}(A,z) = \frac{1}{A_j} \int_{A_j} \text{div}(\sigma_L, \sigma_U)(j) \psi(\sigma_L, \sigma_U) \times \bar{P}(A,z|\sigma_L, \sigma_U) d\sigma_L d\sigma_U, \] (55)

where \( \bar{P}(A,z|\sigma_L, \sigma_U) \) is the conditional probability function given by
\[ \bar{P}(A,z|\sigma_L, \sigma_U) := \bar{P}(A, d_{\sigma I}^L \leq z \leq A, d_{\sigma I}^U). \] (55.1)

Moreover, for \( j \in \{ j_1: 1 \leq j \leq n, n \} \) we get
\[ \frac{\partial \bar{P}}{\partial z_j}(A,z) = \int \psi(R) c_j \bar{P}(A,z|R) dR + \frac{1}{A_j} \int_{A_j} \text{div}(\sigma_L, \sigma_U)(j) \psi(\sigma_L, \sigma_U) \times \bar{P}(A,z|\sigma_L, \sigma_U) d\sigma_L d\sigma_U, \] (55.2)

where \( \bar{P}(A,z|R) := \bar{P}(A, d_{\sigma I}^L(\omega) \leq z \leq A, d_{\sigma I}^U(\omega)) \). (55.3)

**Remark 8.2.**

Using the inverse transformation \( T^{-1}_{(z,A)} \) with given fixed variables \( z, \dot{A} \), cf. Remark 8.1, we also obtain integral representations of the derivatives having a fixed domain of integration in the space of the original \( (R, \sigma_L, \sigma_U) \)-variables.

**8.3. The probability function (26.1)**

According to the definition of \( F^U(\omega, X), F^U(\omega, X) \) given in Section 1 for different cases, the probability function (26.1) can be represented by
\[ P(\lambda, X) = P(a(\omega)^T V(\lambda, X) \delta \leq 0 \text{ for all } \delta \in \Delta_0), \] (56)

where
\[ a(\omega) := \begin{pmatrix} R_0(\omega) \\ U(\omega) \\ u(\omega) \\ a_L(\omega) \end{pmatrix}, \quad \delta := \begin{pmatrix} x \\ u \\ \bar{u} \\ \bar{u} \end{pmatrix}, \] (56.1)

\( \Delta_0 \) is the convex polyhedron of elements \( \delta \) represented by the system of linear equalities/inequalities (23.1)-(23.3), and
\[ V = V(\lambda, X) = V(\lambda, A(X), \bar{V}(X), \bar{W}(X), \bar{W}(X)) \] (56.1)
is an \((m+2n)\times(m+2n)\) matrix given analytically; in case of trusses we have that
\[ V(\lambda, X) := \begin{pmatrix} I_2 \\ -A_q(x) \\ A_q(x) \end{pmatrix} \] (56.2)

Having the extreme points \( \delta(\ell), \ell=1,...,\ell_o, \) of \( \Delta_0 \), we also get
\[ P(\lambda, X) = P(a(\omega)^T V(\lambda, X) \delta(\ell) \leq 0, \ell=1,...,\ell_o). \] (56.3)
Since the number \( \ell_0 \) of extreme points of \( \Delta_0 \) may be very large, and the numerical computation of \( \delta^{(\ell)} \), \( \ell=1,\ldots,\ell_0 \), is very time consuming in general, first we are looking for upper and lower bounds of \( P=P(\lambda,X) \):

Considering an arbitrary sequence of elements

\[ \delta_1, \delta_2, \ldots, \delta_j, \ldots \in \Delta_0, \]

we find for any \( \nu \in \mathbb{N} \) the upper bounds

\[
P_{\nu}(\lambda,X;\delta_1,\ldots,\delta_{\nu}) := P\left( e(\omega)^\nu V(\lambda,X)\delta_{j=0}, j=1,\ldots,\nu \right),
\]

where

\[
P_{\nu}(\lambda,X;\delta_1,\ldots,\delta_{\nu}) \geq P_{\nu+1}(\lambda,X;\delta_1,\ldots,\delta_{\nu+1}) \geq P(\lambda,X).
\]

(58)

for each \( \nu=1,2,\ldots \). Obviously, minimal upper bounds \( P_{\nu}(\lambda,X) \) are obtained by minimizing (57) with respect to \( \delta_{j=0}, j=1,\ldots,\nu \), hence, we put

\[
P_{\nu}(\lambda,X) := \min\{ P_{\nu}(\lambda,X;\delta_1,\ldots,\delta_{\nu}) : \delta_{j=0}, j=1,\ldots,\nu \}.
\]

(59)

By means of optimum upper bounds the true probability \( P(\lambda,X) \) can be reached in a finite number of steps:

**Lemma 8.1.** There is an integer \( \nu_0 = \nu(\lambda,X), \nu_0 \leq \ell_0 \), such that

\[
P_{\nu_0}(\lambda,X) = P(\lambda,X).
\]

(60)

The assertion follows from the inequalities

\[
P(\lambda,X) = P_{\ell_0}(\lambda,X;\delta_{(1)},\ldots,\delta_{(\ell_0)}),
\]

\[
\leq P_{\nu}(\lambda,X;\delta_{(1)},\ldots,\delta_{(\nu)}),
\]

for each \( \nu=1,2,\ldots,\ell_0 \).

According to [19], suboptimal upper bounds for \( P(\lambda,X) \) can be obtained iteratively as follows:

**Stage 1.** Define

\[
P_{1}(\lambda,X) := p_{1}(\lambda,X) = \min\{ p_{1}(\lambda,X;\delta_{1}) : \delta_{1} \in \Delta_0 \},
\]

(61a)

and let \( \delta_{1}^*(\lambda,X) \) denote an element of \( \Delta_0 \) such that

\[
P_{1}(\lambda,X) = P_{1}(\lambda,X;\delta_{1}^*(\lambda,X)).
\]

**Stage \( \nu \).** Having \( \delta_{j}^* = \delta_{j}^*(\lambda,X), j=1,\ldots,\nu-1 \), for \( \nu > 1 \) define

\[
P_{\nu}(\lambda,X) := \min\{ p_{\nu}(\lambda,X;\delta_{1}^*(\lambda,X),\ldots,\delta_{\nu}^*(\lambda,X),\delta_{\nu}) : \delta_{\nu} \in \Delta_0 \},
\]

(61b)

and denote by \( \delta_{\nu}^* = \delta_{\nu}^*(\lambda,X;\delta_{1}^*,\ldots,\delta_{\nu-1}^*) \) an optimal solution in (61b). Obviously we have that

\[
P_{\nu}(\lambda,X) \geq P_{\nu}(\lambda,X) \geq P(\lambda,X), \nu=1,2,\ldots.
\]

Clearly, a big advantage in (61b) is that we have only one single decision vector \( \delta_{\nu} \), whereas in (59) we have to deal with \( \nu \) decision vectors \( \delta_{1},\ldots,\delta_{\nu} \). On the other hand, with the suboptimal upper bounds \( P_{\nu}(\lambda,X) \) the exact value \( P(\lambda,X) \) can not be reached in general in a finite number of steps.
According to (56) we find, cf. (29), (29.1),
\[ P(\lambda, X) = 1 - \mathcal{P}(F_1 \cup \ldots \cup F_{\ell_o}) \]
with the failure domains \( F_{\ell} \) given by
\[ F_{\ell} := (\omega \in \Omega: a(\omega)' V(\lambda, X) \delta_j > 0), \quad \ell = 1, \ldots, \ell_o. \]

Hence, lower bounds for \( P(\lambda, X) \) follow by applying Bonferroni-bounds [10]
\[ \ell_o \]
to \( P(\bigcup_{\ell=1}^{\ell_o} F_{\ell}) \). These bounds can be described by means of an estimate
\[ \ell_o \]
\[ Q_{\nu}(\lambda, X; \delta_1, \ldots, \delta_{\nu}) := \mathcal{P}(a(\omega)' V(\lambda, X) \delta_j > 0, \ j = 1, \ldots, \nu) \quad (62) \]
similar to \( P_{\nu}(\lambda, X; \delta_1, \ldots, \delta_{\nu}) \), \( \delta_j \in \Delta_o \), \( j = 1, \ldots, \nu \).

E.g., for \( \nu = 1 \) we get
\[ P(\lambda, X) \geq 1 - \sum_{\ell=1}^{\ell_o} \mathcal{P}(a(\omega)' V(\lambda, X) \delta_j > 0) \]
\[ = 1 - \sum_{\ell=1}^{\ell_o} Q_{1}(\lambda, X; \delta_j) \geq 1 - \ell_o Q_{1}^{*}(\lambda, X), \]
where, for \( \nu = 1, 2, \ldots, \)
\[ Q_{\nu}^{*}(\lambda, X) := \max(Q_{1}(\lambda, X; \delta_1, \ldots, \delta_{\nu}) : \delta_j \in \Delta_o, \ 1 \leq j \leq \nu). \quad (62.1) \]

Consequently, for the approximative computation of the probability
\[ P(\lambda, X) \] we have to solve optimization problem of the type
\[ \min_{\delta_j \in \Delta_o, 1 \leq j \leq \nu} P_{\nu}(\lambda, X; \delta_1, \ldots, \delta_{\nu}). \quad (63) \]
and
\[ \max_{\delta_j \in \Delta_o, 1 \leq j \leq \nu} Q_{\nu}(\lambda, X; \delta_1, \ldots, \delta_{\nu}). \quad (63.1) \]
Moreover, for the approximative solution of reliability-oriented optimization problems of the type
\[ \max_{x \in \mathcal{D}} P(\lambda_o, X) \]
with a given load factor \( \lambda_o \in \mathbb{R} \), we have the maxmin-problem
\[ \max_{x \in \mathcal{D}} \min_{\delta_j \in \Delta_o, 1 \leq j \leq \nu} P_{\nu}(\lambda, X; \delta_1, \ldots, \delta_{\nu}). \quad (64.1) \]

Thus, in each case the derivatives of the probability functions \( P_{\nu}, Q_{\nu} \), resp., are needed.

By the transformation
\[ \theta_j := V(\lambda, X) \delta_j, \quad j = 1, \ldots, \nu, \quad (65) \]
for \( P_{\nu} = P_{\nu}(\lambda, X; \delta_1, \ldots, \delta_{\nu}) \) we find the representation
\[ P_{\nu}(\lambda, X; \delta_1, \ldots, \delta_{\nu}) = W_{\nu}(V(\lambda, X) \delta_1, \ldots, V(\lambda, X) \delta_{\nu}), \quad (66) \]
where
\[ W_{\nu}(\theta_1, \ldots, \theta_{\nu}) := \mathcal{P}(\theta_j a(\omega) \leq 0, \ j = 1, \ldots, \nu), \quad (67) \]
and $Q_\nu$ can be represented in the same way. Since the derivatives of $V-V(\lambda,X)$ can be obtained analytically, the remaining problem is the differentiation of $W_\nu$.

8.3.1. **Exact differentiation formulas in case of $\nu \leq \text{dim} \ a(.)$**

In many practical cases the stage number $\nu$ is small, hence, the assumption

$$\nu \leq d := \text{dim} \ a(.) = m + 2n$$

is not too restrictive. For the computation of the partial derivative $\frac{\partial W_\nu}{\partial \theta_{jk}}$ for a given pair $(j,k)$, $1 \leq j \leq \nu$, $1 \leq k \leq d$, we consider a partition $\theta = (\theta_I, \theta_{II})$ of the $\nu \times d$ matrix

$$\theta := \begin{pmatrix}
\theta_1 \\
\theta_2 \\
\vdots \\
\theta_\nu
\end{pmatrix}$$

such that

i) $\theta_{jk}$ is an element of $\theta_I$ and

ii) rank $\theta_I = \nu$.

Supposing then that the random vector $a = a(\omega)$ has a probability density $f = f(a)$, and partitioning the vector $a \in \mathbb{R}^d$ in the same way $a = (a_I, a_{II})$ as the matrix $\theta$, by the integral transformation

$$a = \begin{pmatrix}
a_I \\
a_{II}
\end{pmatrix} = \begin{pmatrix}
\theta_I^{-1}p_I \\
p_{II}
\end{pmatrix}$$

we find

$$W_\nu(\theta_1', \ldots, \theta_\nu) = \int_{\text{det}\theta_I = 0} f(\theta_I^{-1}p_I, p_{II}) \frac{dp_I}{p_{II}^{\nu+1}}$$

Obviously, by means of this transformation, the domain of integration in (71) is now independent of the element $\theta_{jk}$ and - of course - also independent of all other elements $\theta_{ik}$ contained in $\theta_I$. Thus, if the density $f$ of $a(\omega)$ is sufficiently smooth the derivative follows, cf. [10], [13], by interchanging differentiation and integration:

**Theorem 8.2.** Under appropriate assumptions [10], [13] on the density $f = f(a)$ of $a(\omega)$, the partial derivative $\frac{\partial}{\partial \theta_{jk}} W_\nu$ is given by

$$\frac{\partial W_\nu}{\partial \theta_{jk}}(\theta_1', \ldots, \theta_\nu) = - \int_{\text{det}\theta_I = 0} \nabla f(a(\omega)) \bigg( \theta_I^{-1} \bigg)_{jk} a_k(a(\omega)) \bigg( \theta_I^{-1} \bigg)_{jk} \text{d}a,$$

which can be represented also by the expectation

$$\frac{\partial W_\nu}{\partial \theta_{jk}}(\theta_1', \ldots, \theta_\nu) = - \mathbb{E} \left( \frac{\partial f(a(\omega))}{\partial \theta_{jk}} \bigg( \theta_I^{-1} \bigg)_{jk} a_k(a(\omega)) \right) \bigg|_{\text{det}\theta_I = 0}.$$ 

(72.1)
Corollary 8.1. The derivatives \( \frac{\partial \tilde{W}}{\partial \theta_{ik}} \) of \( W_\nu \) with respect to elements \( \theta_{ik} \) of \( \Theta_\nu \) have the same form.

8.3.2. Approximative derivatives of \( W_\nu \)

If condition (71) can not be fulfilled, e.g. in the case \( \nu > d \), then approximative derivatives of \( W_\nu \) - of an arbitrary high accuracy - can be obtained by the following stochastic completion technique [8],[13]:

Let \( z_j^\nu = z_j^\nu (\omega), j=1,\ldots,\nu \), denote real random variables such that

i) \( z_j^\nu (\omega), j=1,\ldots,\nu \), and \( a(\omega) \) are stochastically independent

ii) \( E z_j^\nu (\omega) = 0 \), and \( z_j^\nu (\omega) \) has a continuous probability density \( \psi_j(t), j=1,\ldots,\nu \). \hfill (73)

By means of the "stochastic completion terms" \( z_j^\nu = z_j^\nu (\omega), j=1,\ldots,\nu \), for \( W_\nu \) we obtain the approximative probability function

\[ \tilde{W}_\nu(\theta_1^\nu,\ldots,\theta_\nu) := P(\theta_j^\nu a(\omega) + z_j^\nu (\omega) \leq 0, j=1,\ldots,\nu). \] \hfill (74)

We find

\[ \tilde{W}_\nu(\theta_1^\nu,\ldots,\theta_\nu) = P(z_j^\nu (\omega) \leq \theta_j^\nu a(\omega), j=1,\ldots,\nu) \]

\[ = E \prod_{j=1}^\nu \psi_j(-\theta_j^\nu a(\omega)), \]

where \( \psi_j \) denotes the distribution function of \( z_j^\nu (\omega) \); if \( z_j^\nu (\omega) \) has the normal distribution \( N(0,\sigma_j^2) \), then

\[ \tilde{W}_\nu(\theta_1^\nu,\ldots,\theta_\nu) = E \prod_{j=1}^\nu \Phi(-\frac{1}{\sigma_j^2} \theta_j^\nu a(\omega)), \]

where \( \Phi \) is the distribution function of \( N(0,1) \).

The partial derivatives of \( \tilde{W}_\nu \) read

\[ \frac{\partial \tilde{W}_\nu}{\partial \theta_{jk}} = - E \prod_{k=1}^\nu \psi_k(-\theta_k^\nu a(\omega))\psi_j(-\theta_j^\nu a(\omega))a_k(\omega), \] \hfill (76)

\( j=1,\ldots,\nu \), \( k=1,\ldots,d \), and higher order derivatives of \( \tilde{W}_\nu \) can be obtained in the same way. If

\[ z(\omega) = (z_1(\omega),\ldots,z_\nu(\omega))' \xrightarrow{\text{w.p.1}} 0 \] \hfill (77)

(with probability one), then under some regularity assumptions [9],[13]

\[ \tilde{W}_\nu(\theta_1^\nu,\ldots,\theta_\nu) \longrightarrow W_\nu(\theta_1^\nu,\ldots,\theta_\nu), \]

\[ \frac{\partial \tilde{W}_\nu}{\partial \theta_{jk}}(\theta_1^\nu,\ldots,\theta_\nu) \longrightarrow \frac{\partial W_\nu}{\partial \theta_{jk}}(\theta_1^\nu,\ldots,\theta_\nu). \] \hfill (78)

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