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|--------------------|----------------------|-----------------------|--------------------------------------|----------------|-----------------------------------------------|
CONTROL OF SLOWLY VARYING LPV SYSTEMS: AN APPLICATION TO FLIGHT CONTROL *

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Abstract

Recent results in parameter-dependent control of linear parameter-varying (LPV) systems are applied to the problem of designing gain-scheduled pitch rate controllers for the F-16 VISTA (Variable-Stability In-Flight Simulator Test Aircraft). These methods, based on parameter-dependent quadratic Lyapunov functions, take advantage of known a priori bounds on the parameters’ rates of variation (the bounds may themselves be parameter-varying). The controller achieves an induced-$L_2$-norm performance objective; Level 1 flying qualities are predicted. Suboptimal solutions are obtained by solving a convex optimization problem described by linear matrix inequalities (LMIs). Incorporation of $D$-$K$ iteration with “constant $D$-scales” provides robustness to time-varying uncertainty. Parameter-varying performance weights are used to shape the desired performance at different points in the design envelope.

1 Introduction

The area of analysis and control of linear parameter-varying (LPV) systems has received much recent attention, primarily in order to develop systematic techniques for gain-scheduling. These systems resemble linear systems that depend on one or more time-varying parameters; nonlinear systems are often modelled in this form via a parameterized family of linearizations. The analysis of LPV systems differ from that of linear time-varying (LTV) systems in that it considers whole families of parameter trajectories; moreover, the parameter values are available only in real time, not in advance.

The classical approach to gain-scheduled $H_\infty$ control involves designing an (LTI) $H_\infty$ controller for each of a parameterized family of linearizations and then interpolating controller gains by operating condition. This heuristic approach yields satisfactory results if the parameters are sufficiently “slowly-varying.” Early results in so-called “LPV synthesis” explicitly account for this time-variation using scaled small-gain arguments or single quadratic Lyapunov functions (SQLF); those designs tend to be conservative, though, partly because they allow the parameters to vary arbitrarily quickly. More recent results use parameter-dependent Lyapunov functions (PDLF) to factor in a priori bounds on the parameters’ rates of variation, reducing this conservatism.

In this paper the PDLF technique is incorporated into the $D$-$K$ iteration framework in order to design robust, parameter-varying pitch rate controllers for the F-16 VISTA (Variable Stability In-Flight Simulator Test Aircraft). Optimal solutions are obtained by solving a convex optimization problem described by a system of linear matrix inequalities (LMIs), for which efficient algorithms are available. Parameter-varying performance weights are used to smoothly vary the desired performance within the design envelope.

The remainder of this paper is organized as follows. Section 2 reviews some results on the control of LPV systems. Section 3 provides a robustness test for LPV systems. Section 4 applies these techniques to the F-16 VISTA. Section 5 summarizes the paper and discusses future prospects.

This paper uses standard notation. In addition, $S^{n\times n}$ denotes the set of real, symmetric, $n \times n$ matrices. For any matrix $X \in S^{n\times n}$, $X > 0$ and $X < 0$ respectively denote positive-definiteness (all of its eigenvalues are positive) and negative-definiteness.
2 Control of LPV Systems

This section reviews some results on the control of continuous-time, linear parameter-varying systems. The reader may consult the applicable references\(^7,^{16,17}\) (and the papers cited therein) for details.

An LPV system \(G(s, \rho)\) is a finite-dimensional linear system
\[
\begin{bmatrix}
    \dot{x}(t) \\
    e(t)
\end{bmatrix} =
\begin{bmatrix}
    A(\rho(t)) & B(\rho(t)) \\
    C(\rho(t)) & D(\rho(t))
\end{bmatrix}
\begin{bmatrix}
    x(t) \\
    d(t)
\end{bmatrix}
\tag{2.1}
\]
whose state-space data are known continuous functions of time-varying parameters denoted by \(\rho \in \mathbb{R}^s\). The \(s\) parameter values are not known in advance; rather, they are measured in real-time. Assume that the bounded vector-valued parameter signal \(\rho \in L_\infty\) is a piecewise-\(C^1\) function of time and that there exists a compact set \(\mathcal{P} \subset \mathbb{R}^s\) for which \(\rho(t) \in \mathcal{P}\) for all \(t \geq 0\).

Also assume that the rates of variation of the first \(\bar{s}\) parameters \(\rho_1, \ldots, \rho_\bar{s}\) are each bounded in magnitude by known positive scalars \(\nu_1, \ldots, \nu_\bar{s}\), i.e.,
\[
|\rho_i(t)| \leq \nu_i \quad \text{for all } t > 0 \quad (i = 1, \ldots, \bar{s}).
\]

Denote these \(\bar{s}\) rate-limited parameters and rate bounds by \(\hat{\rho} := (\rho_1, \ldots, \rho_\bar{s}) \in \mathbb{R}^\bar{s}\) and \(\nu := (\nu_1, \ldots, \nu_\bar{s}) \in \mathbb{R}^{\bar{s}}\), respectively. Now \(\nu\) may itself be a continuous function of the parameters \(\rho\), so that parameter trajectories must obey the differential inclusion
\[
|\dot{\rho}_i(t)| \leq \nu_i(\rho(t)) \quad \text{for all } t > 0 \quad (i = 1, \ldots, \bar{s}).
\]

Parameter trajectories satisfying the above conditions for given \(\mathcal{P}\) and \(\nu\) will be called allowable. Note that, at the cost of added notation, one can expand the results in this section to separate upper and lower bounds of the rates of variation.

2.1 Induced \(L_2\)-norm Analysis

Given a family of allowable parameter trajectories defined by a parameter set \(\mathcal{P}\) and a rate-of-variation bound \(\nu\), one can bound the induced \(L_2\) norm of an LPV system using a parameter-dependent quadratic Lyapunov function.

Lemma 2.2 Given the LPV system in (2.1) and a performance level \(\gamma > 0\), suppose there exists a matrix function \(W \in C^1(\mathbb{R}^s, \mathbb{R}^{m \times n})\) such that \(W(\hat{\rho}) > 0\) and (omitting dependence on \(\rho\) and \(\hat{\rho}\))
\[
\begin{bmatrix}
    A^TW + W A + \sum_{i=1}^{\bar{s}} \beta_i \frac{\partial W}{\partial \rho_i} & W B & C^T \\
    B^T W & -\gamma I & D^T \\
    C & D & -\gamma I
\end{bmatrix} < 0
\]
for all \(\beta_i \in [-\nu_i(\rho), \nu_i(\rho)]\) at each parameter value \(\rho \in \mathcal{P}\). Then for any allowable parameter trajectory the LTV system governed by (2.1) is exponentially stable. Furthermore, there exists \(\gamma_1 \in [0, \gamma)\) for which \(\|e\|_2 \leq \gamma_1 \|d\|_2\) for all \(d \in L_2\) if \(x(0) = 0\).

The constraints \(W(\hat{\rho}) > 0\) and (2.3) represent convex linear matrix inequality (LMI) constraints on the variables \(W\) and \(\gamma\). Although these LMIs are clearly infinite dimensional, one can compute solutions using the approximate method described in the sequel.

2.2 Output-feedback synthesis

Consider an LPV plant in the standard form
\[
\begin{bmatrix}
    \dot{x} \\
    e \\
    y
\end{bmatrix} =
\begin{bmatrix}
    A(\rho) & B_1(\rho) & B_2(\rho) \\
    C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\
    C_2(\rho) & D_{21}(\rho) & D_{22}(\rho)
\end{bmatrix}
\begin{bmatrix}
    x \\
    d \\
    u
\end{bmatrix}
\tag{2.4}
\]

where \(x \in \mathbb{R}^n, e \in \mathbb{R}^{n_e}, d \in \mathbb{R}^{n_d}, u \in \mathbb{R}^{n_u}\), and \(y \in \mathbb{R}^{n_y}\), and other quantities are dimensioned appropriately. By assuming regularity (i.e., \(D_{12}\) and \(D_{21}\) are full rank) and \(D_{11} = 0\), (2.4) can be transformed into
\[
\begin{bmatrix}
    \dot{x} \\
    e_1 \\
    e_2 \\
    y
\end{bmatrix} =
\begin{bmatrix}
    A_1(\rho) & B_{11}(\rho) & B_{12}(\rho) & B_2(\rho) \\
    C_{11}(\rho) & 0 & 0 & 0 \\
    C_{12}(\rho) & 0 & 0 & I \\
    C_2(\rho) & 0 & I & 0
\end{bmatrix}
\begin{bmatrix}
    x \\
    d_1 \\
    d_2 \\
    u
\end{bmatrix}
\tag{2.5}
\]

The LPV \(\gamma\)-Performance/\(\nu\)-Variation problem consists of finding a parameter-varying controller
\[
\begin{bmatrix}
    \dot{x}_c \\
    u
\end{bmatrix} =
\begin{bmatrix}
    A_k(\rho, \hat{\rho}) & B_k(\rho, \hat{\rho}) \\
    C_k(\rho, \hat{\rho}) & D_k(\rho, \hat{\rho})
\end{bmatrix}
\begin{bmatrix}
    x_c \\
    y
\end{bmatrix}
\]
(possibly dependent on \(\hat{\rho}\)) for which the closed-loop system (omitting dependence on \(\rho\))
\[
\begin{bmatrix}
    \dot{\hat{x}} \\
    \dot{\hat{e}} \\
    e_1 \\
    e_2
\end{bmatrix} =
\begin{bmatrix}
    A_{clp} & B_{clp} \\
    C_{clp} & D_{clp}
\end{bmatrix}
\begin{bmatrix}
    x \\
    x_c \\
    \dot{d}_1 \\
    \dot{d}_2
\end{bmatrix}
\]
satisfies the conditions of Lemma 2.2 for a desired closed-loop norm \(\gamma > 0\). The following theorem gives necessary and sufficient conditions for solvability.
Theorem 2.6 Given the compact set \( \mathcal{P} \), the vector-valued function \( \nu(\rho) \), the scalar \( \gamma > 0 \), and the open-loop system (2.5), the LPV \( \gamma \)-Performance/\( \nu \)-Variation problem is solvable if and only if there exist matrix functions \( X \in \mathcal{C}^1(\mathbb{R}^f, \mathbb{S}^{n \times n}) \) and \( Y \in \mathcal{C}^1(\mathbb{R}^f, \mathbb{S}^{n \times n}) \) such that (omitting dependence on \( \rho \) and \( \bar{\rho} \))

\[
\begin{bmatrix}
A^T X + X A - \gamma C_1^T C_2 + \sum_{i=1}^s \frac{\nu_i}{\delta_i} B_i^T X B_i + \gamma I & C_1^T \\
B_i^T X & 0 \\
C_1 & -\gamma I
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
\dot{Y} + Y A - \gamma B_2 B_2^T + \sum_{i=1}^s \frac{\nu_i}{\delta_i} Y C_i & Y C_1^T \\
0 & 0
\end{bmatrix} < 0
\]

(2.7)

\[
\begin{bmatrix}
X & I \\
I & Y
\end{bmatrix} > 0
\]

(2.8)

Similarly, a parameter-varying \( X(\bar{\rho}) \) and constant \( Y \) admit the controller defined by choosing

\[
M = Y^{-1} - X, \quad N = Y
\]

Holding both \( X \) and \( Y \) constant recovers the conservative SQFL controller, which allows for arbitrarily fast parameter variation.

### 2.3 Computing Solutions

The infinite-dimensionality of the inequalities in Theorem 2.6 demands approximate methods of solution for the sake of practical computation. One such method follows:\(^{1,6,17}\)

Pick scalar basis functions \( \{f_i \in \mathcal{C}^1(\mathbb{R}^f, \mathbb{R})\}_{i=1}^{N_X} \) and \( \{g_j \in \mathcal{C}^1(\mathbb{R}^f, \mathbb{R})\}_{j=1}^{N_Y} \), and search over those \( X(\bar{\rho}) \)'s and \( Y(\bar{\rho}) \)'s that are linear combinations

\[
X(\bar{\rho}) = \sum_{i=1}^{N_X} f_i(\bar{\rho}) X_i, \quad Y(\bar{\rho}) = \sum_{j=1}^{N_Y} g_j(\bar{\rho}) Y_j
\]

using constant matrices \( \{X_i \in \mathbb{S}^{n \times n}\}_{i=1}^{N_X} \) and \( \{Y_j \in \mathbb{S}^{n \times n}\}_{i=1}^{N_Y} \). Then (2.7)-(2.9) can be rewritten as (omitting dependence on \( \rho \) and \( \bar{\rho} \))

\[
\begin{bmatrix}
\sum_{i=1}^{N_X} (A^T X_i + X_i \bar{A}) f_i - \gamma C_1^T C_2 \\
\sum_{i=1}^{N_X} \frac{\nu_i}{\delta_i} B_i^T X_i & -\gamma I \\
C_1 & -\gamma I
\end{bmatrix} < 0
\]

(2.10)

\[
\begin{bmatrix}
\sum_{j=1}^{N_Y} (A Y_j + Y_j \bar{A}) g_j - \gamma B_2 B_2^T \\
\sum_{j=1}^{N_Y} \frac{\nu_i}{\delta_i} Y_j C_i & -\gamma I \\
0 & -\gamma I
\end{bmatrix} < 0
\]

(2.11)

\[
\begin{bmatrix}
\sum_{i=1}^{N_X} f_i X_i & I \\
I & \sum_{j=1}^{N_Y} g_j Y_j
\end{bmatrix} > 0
\]

(2.12)

where the shorthand \((*)\) denotes the implied transposes. The problem thus consists of finding real, symmetric matrices \( \{X_i\}_{i=1}^{N_X} \) and \( \{Y_j\}_{j=1}^{N_Y} \) that satisfy the above inequalities for all \( \rho \in \mathcal{P} \), still an infinite-dimensional problem.

Now approximate the parameter set \( \mathcal{P} \) by a grid of \( L \) points \( \{\rho_k \in \mathbb{R}^f\}_{k=1}^L \), defining \( \{\bar{\rho}_k \in \mathbb{R}^f\}_{k=1}^L \) accordingly, and solve the inequalities (2.10)-(2.12) at these grid points. The conditions represent up to \( L(2^s+1) \) LMIs in the \( N_X + N_Y \) matrix variables \( \{X_i\}_{i=1}^{N_X}, \{Y_j\}_{j=1}^{N_Y} \). Note that minimizing \( \gamma \) is a convex optimization problem, since the inequalities are affine in \( \gamma \) as well.
3 Robust LPV Systems

This section derives a simple robustness test and proposes an ad hoc method for making the preceding control synthesis robust. Consider the analysis of an LPV system

\[
\begin{bmatrix}
\dot{z} \\
\dot{e}
\end{bmatrix} = G(s, \rho) \begin{bmatrix}
w \\
d
\end{bmatrix}
\]

put in feedback with a block-diagonal, memoryless linear time-varying (uncertainty) operator

\[ w(t) = \Delta(t)z(t) \]

where \( w, z \in \mathbb{R}^{n_z} \), \( d \in \mathbb{R}^{n_d} \), and \( e \in \mathbb{R}^{n_e} \). Assume that the block structure of \( \Delta \) is consistent with a set \( \Delta \subseteq \mathbb{S}^{n_z \times n_z} \). Define a corresponding set of scalings

\[ S_{\Delta} := \{ S \in \mathbb{S}^{n_z \times n_z} : \Delta S = S\Delta \text{ for all } \Delta \in \Delta \} \]

An elementary small-gain argument establishes the following sufficient conditions for robustness to (arbitrarily quickly) time-varying uncertainty.

**Proposition 3.1** Suppose there exists a continuous matrix function \( S : \mathbb{R}^q \rightarrow S_{\Delta} \) such that for any allowable parameter trajectory \( \rho : \mathbb{R} \rightarrow \mathbb{R}^q \) the resulting LTV system

\[
G_S(s, \rho) = \begin{bmatrix}
S(\rho) & 0 \\
0 & I_{n_e}
\end{bmatrix} G(s, \rho) \begin{bmatrix}
S^{-1}(\rho) & 0 \\
0 & I_{n_e}
\end{bmatrix}
\]

is exponentially stable and \( ||G_S(s, \rho)||_{\infty} \leq \gamma \) (given zero initial conditions). Then for any allowable parameter trajectory \( \rho \) and any \( \Delta \) satisfying

\[
\hat{\sigma}(\Delta(t)) \leq 1/\gamma
\]

the LTV system \( F_u(G(s, \rho), \Delta) \) also is exponentially stable and \( ||F_u(G(s, \rho), \Delta)||_{\infty} \leq \gamma \) for zero initial conditions.

This suggests that one can design robust LPV controllers by alternating (in obvious fashion) between controller synthesis and computation of parameter-varying scaling matrices \( S(\rho) \), as in D-K iteration. The design example in this paper uses an ad hoc choice of \( S(\rho) \): “zeroth-order” fits to frequency-dependent D-scales that are obtained at each “frozen” grid point by applying the appropriate \( \mu \)-Tools commands for closed-loop analysis. There is also a more rigorous iterative approach that will not be addressed in this paper.

4 A Design Example

This section presents an application of LPV synthesis to the pitch rate control of the F-16 VISTA over a specified flight envelope; a similar problem has been addressed using a small-gain method. The physical plant and certain design weights are parameter-varying, and a \( D-K \) iteration-like process is employed to enhance robust performance.

4.1 Plant Modelling

This design uses the standard short-period equations of motion:

\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{q}
\end{bmatrix} = \begin{bmatrix}
Z_\alpha(\rho) & 1 \\
M_\alpha(\rho) & M_q(\rho)
\end{bmatrix} \begin{bmatrix}
\alpha \\
q
\end{bmatrix} + \begin{bmatrix}
Z_{\delta_e}(\rho) \\
M_{\delta_e}(\rho)
\end{bmatrix} \delta_e \tag{4.1}
\]

\[ \rho = \bar{\rho} = (M, h) \]

ignoring the aerodynamic effects of the trailing edge flaps. Only the longitudinal dynamics of the aircraft are considered; the roll, yaw, and sideslip angles are assumed to be zero. In (4.1) the states \( (\alpha, q) \), input \( \delta_e \), and parameters \( (M, h) \) respectively denote angle-of-attack & pitch rate, elevator deflection, and Mach number & altitude.

The Flight Dynamics Directorate of the Wright Laboratory uses a high-fidelity, six degree-of-freedom, nonlinear model to simulate the F-16 VISTA. This simulation model includes accurate descriptions of the propulsion system, actuators, sensors, disturbances, payload, atmosphere, rigid-body equations of motion, and aerodynamics for a wide range of Mach numbers, altitudes, and angles of attack.

The LPV short-period model’s state space data (the dimensional coefficients \( Z_\alpha, M_\alpha, M_q, Z_{\delta_e}, \text{ and } M_{\delta_e} \)) were obtained by trimming and linearizing the nonlinear model at level flight for the flight conditions marked in Table 1. The design region \( \mathcal{P} \) was chosen accordingly, and these 21 grid points were used for the controller synthesis.

<table>
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<th>h/M</th>
<th>0.35</th>
<th>0.45</th>
<th>0.55</th>
<th>0.65</th>
<th>0.75</th>
<th>0.85</th>
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<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
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</tr>
<tr>
<td>15000 ft</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>5000 ft</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>10000 ft</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Grid points used for modeling & synthesis

Data on the excess thrust and rate-of-climb of the
F-16 VISTA suggest the bounds

\[

\nu_1(\rho) = 2 \left( \frac{\left| V_T \right| a(h)}{a(h)} + M^2 \left| \frac{da(h)}{dh} \right| \right) \geq \left| \dot{M}(t) \right| \\

\nu_2(\rho) = a(h) M \geq \left| \dot{h}(t) \right|
\]

on the parameters’ rates of variation, where \( a(h) \) denotes the speed of sound as a function of altitude and \( V_T = a(h) M \) denotes the aircraft’s true velocity. Note that these bounds are conservative; achieving them would require vertical flight, for example.

### 4.2 Problem Setup

The objective here is to design for the F-16 VISTA a pitch-rate controller that provides robust command tracking with predicted Level 1 handling qualities.\textsuperscript{11} Time-domain specifications for pitch-rate response are illustrated in Fig. 1 and listed in Table 2. Note that the “rise-time” parameter \( \Delta t \) varies with the true velocity \( V_T \) (in ft/sec).

![Figure 1: Pitch rate handling qualities specifications](image)

<table>
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<tr>
<th>Parameter</th>
<th>Level 1</th>
<th>Level 2</th>
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<tbody>
<tr>
<td>max ( t_1 ) (sec)</td>
<td>0.12</td>
<td>0.17</td>
</tr>
<tr>
<td>max ( \Delta q_1 )</td>
<td>0.30</td>
<td>0.60</td>
</tr>
<tr>
<td>max ( 500/V_T )</td>
<td>0.30</td>
<td>0.60</td>
</tr>
<tr>
<td>min ( 9.0/V_T )</td>
<td>0.30</td>
<td>0.60</td>
</tr>
</tbody>
</table>

Table 2: Pitch rate handling qualities specifications

This design uses the model-matching control structure shown in Fig. 2. The second-order reference model

\[

G_{ref}(s) = \frac{\omega_n^2 (Ts + 1)}{s^2 + 2\zeta \omega_n s + \omega_n^2},
\]

\( \omega = 4 \text{ rad/s}, \zeta = 0.6, 1/T = 10 \text{ rad/s} \)

meets Level 1 flying qualities over the entire design envelope. The first-order, parameter-varying command and performance weights

\[

W_r(s, \rho) = \frac{s + 100}{100 \ s + 10} q_{max}(\rho) \\

W_p(s, \rho) = \frac{s + 80}{80 \ s + 4 \ 0.05 q_{max}(\rho)}
\]

reflect a uniform 10 rad/s command bandwidth, a maximum pitch-rate command \( q_{max} \) that varies across the design envelope as shown in Table 3, and steady-state tracking within 5%.

<table>
<thead>
<tr>
<th>( h ) ( M )</th>
<th>0.35</th>
<th>0.45</th>
<th>0.55</th>
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<td>14</td>
<td>17</td>
<td>20</td>
<td>23</td>
<td>25</td>
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</table>

Table 3: Max. pitch-rate command \( q_{max} \) (deg/s)

The design plant also includes actuator dynamics, additive sensor noise, multiplicative input uncertainty, and penalties on the elevator deflection angle and rate. The first-order actuator model

\[

G_a(s) = \frac{20.2}{s + 20.2}
\]

reflects a bandwidth of 20.2 rad/s. The control weight

\[

W_a = \begin{bmatrix} 21 & 0 \\ 0 & 70 \end{bmatrix}
\]

reflects elevator deflection angle and rate limits of 21 deg and 70 deg/s, respectively. The parameter-dependent noise weight

\[

W_n(\rho) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.03 q_{max}(\rho) \end{bmatrix}
\]

anticipates 0.5 deg of measurement error for \( \alpha \) and 0.03 \( q_{max} \) deg/s (about 3%) for \( q \). The uncertainty weight

\[

W_u = 0.1
\]

represents 10% parametric and/or dynamic modeling error.

### 4.3 Design Results

Several \( D-K \) iterations (using LPV synthesis) were performed in MATLAB on the 7th-order design plant. LMILab\textsuperscript{10} was used to solve the controller
synthesis LMIs, and μ-Tools\textsuperscript{5} was used for closed-loop robustness analysis. The elementary basis functions

\[ [f_1(\bar{\rho}) f_2(\bar{\rho}) f_3(\bar{\rho})] = [g_1(\bar{\rho}) g_2(\bar{\rho}) g_3(\bar{\rho})] = [1 \ M \ h] \]

selected according to previous experience,\textsuperscript{2,13,16,17} were used to vary the matrices \( X(\bar{\rho}) \) and \( Y(\bar{\rho}) \) with various degrees of complexity: constant \( X \) & \( Y \), varying \( X(\bar{\rho}) \) & constant \( Y \), constant \( X \) & varying \( Y(\bar{\rho}) \), and varying \( X(\bar{\rho}) \) & \( Y(\bar{\rho}) \).

<table>
<thead>
<tr>
<th>( D-K )</th>
<th>( X, Y )</th>
<th>( X, Y(\bar{\rho}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iter. #</td>
<td>( \gamma )</td>
<td>( \bar{\gamma} )</td>
</tr>
<tr>
<td>1</td>
<td>2.60</td>
<td>1.90</td>
</tr>
<tr>
<td>2</td>
<td>2.43</td>
<td>1.75</td>
</tr>
<tr>
<td>3</td>
<td>2.42</td>
<td>1.74</td>
</tr>
<tr>
<td>4</td>
<td>2.42</td>
<td>1.74</td>
</tr>
</tbody>
</table>

Table 4: Closed-loop performance levels

The closed-loop time-varying performance level \( \gamma \) achieved using Theorem 2.6 is shown in Table 4; constraining the rate of variation of \( X(\bar{\rho}) \) clearly offers a significant improvement in performance. Also included are the maximum "frozen-point" (i.e. gainscheduled) \( H_\infty \) norms

\[ \bar{\gamma}_\infty = \max_{\rho \in \mathcal{P}} \| G_{clp}(s, \rho) \|_\infty \]

and structured singular values (cf. Figure 3)

\[ \bar{\mu} = \max_{\rho \in \mathcal{P}} \sup_{\omega \in \mathbb{R}} \mu[G_{clp}(j\omega, \rho)] \]

obtained via pointwise \( H_\infty \) and \( \mu \)-analysis of the closed-loop systems. These indicate the controller’s robustness at constant flight conditions. The two designs with varying \( X(\bar{\rho}) \) offer no appreciable improvement over the corresponding constant-\( X \) designs, so they have been omitted.

4.4 Nonlinear Simulation

Figure 4 shows the results of a high-fidelity, parameter-varying, nonlinear simulation of the closed-loop step response, which demonstrates predicted Level 1 flying qualities. The aircraft is initially perturbed from trimmed, level flight at \( M = 0.75 \) and \( h = 5000 \) ft. The response of the constant-\( X, Y \) controller is also shown, for comparison.

5 Conclusions

Recent results in parameter-dependent control of linear parameter-varying systems are applied to the problem of robust, gain-scheduled pitch rate control for the F-16 VISTA. Using parameter-dependent Lyapunov functions, \( a \ priori \) bounds on the parameters' rates of variation, LMI-based convex optimization, and parameter-varying design weights, this method achieves an induced-\( L_2 \)-norm performance objective while predicting Level 1 flying qual-
ities throughout the design envelope. Straightforward D-K iteration with "constant D-scales" provides robustness to time-varying uncertainty. Ongoing research includes expanding the parameter set (to encompass the full flight envelope) and including a parameter-varying reference model (while maintaining Level 1 flying qualities), and incorporating the robustness scales into the synthesis LMIs.

References


