Poisson's Ratio for Cubic Crystals

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# Title and Subtitle
POISSON'S RATIO FOR CUBIC CRYSTALS

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# Abstract (Maximum 200 words)
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POISSON’S RATIO FOR CUBIC CRYSTALS

Abstract

General expressions for Poisson's ratio are derived for cubic crystals; simplified forms are given for cases involving symmetry directions.

Introduction

Poisson's ratio, ν, is defined for isotropic media as the quotient of lateral contraction to longitudinal extension arising from application of a simple tensile stress; in most materials, this dimensionless number is positive. In crystals, ν takes on different values, depending on the directions of stress and strain chosen. The ratio finds application in a variety of areas of applied elasticity and solid mechanics, for example, as indication of the mechanical coupling between various vibrational modes of motion.

The maximum value of ν = +1/2 is obtained in the incompressible medium limit, where volume is preserved; for ordinary materials, values of +1/4 to +1/3 are typical, but in crystals ν may vanish, or take on negative values. Analytical formulas for Poisson's ratio are expressed in terms of elastic constants. For the case of crystals of general anisotropy, these expressions are quite unwieldy, but for cubic crystals the symmetry elements reduce the complexity considerably.

Crystals of cubic symmetry include many of the binary semiconductor systems with the piezoelectric zincblende structure, such as GaAs and related alloys. These are extremely important for high technology applications such as cellular radio and microwave collision avoidance radar. All cubic classes have the same elastic matrix scheme, so for our purposes it is not necessary to distinguish between the cubic point groups. The cubic elastic matrix scheme is identical in form to that of isotropic substances; the difference is that for isotropic materials the shear coefficient (s44 or c44) is related to the two other independent coefficients, whereas in cubic crystals it is a third independent quantity.
Expressions Relating Cubic Stiffnesses and Compliances

Relations for Poisson's ratio are most simply expressed in terms of the elastic compliances \([s_{ij}]\). It is often the case, however, that the most accurate determinations of the elastic constants (resonator and transit-time methods) yield values for the stiffnesses \([c_{ij}]\) directly; the conversion relations are given below. For the cubic system, the elastic stiffness and compliance matrices have identical form. Referred to the \([100], [010],\) and \([001]\) directions, the matrices are:

\[
\begin{bmatrix}
  c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\
  c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\
  c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\
  0 & 0 & 0 & c_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & c_{44} & 0 \\
  0 & 0 & 0 & 0 & 0 & c_{44}
\end{bmatrix}
\begin{bmatrix}
  s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\
  s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\
  s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\
  0 & 0 & 0 & s_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & s_{44} & 0 \\
  0 & 0 & 0 & 0 & 0 & s_{44}
\end{bmatrix}
\]

Stiffness and compliance are matrix reciprocals; the three independent components of each are related by:

\[
c_{11} = \frac{(s_{11} + s_{12})}{[(s_{11} - s_{12})(s_{11} + 2s_{12})]}
\]

\[
c_{12} = \frac{(-1s_{12})}{[(s_{11} - s_{12})(s_{11} + 2s_{12})]}
\]

\[
c_{44} = \frac{1}{s_{44}}
\]

These are inverted simply by an interchange of symbols \(c_{ij}\) and \(s_{ij}\). For the case of isotropy, one has the further relations \(s_{44} = 2(s_{11} - s_{12})\) and \(c_{44} = (c_{11} - c_{12})/2\). For cubic crystals, anisotropy factors \(s = (s_{11} - s_{12} - s_{44}/2)\) and \(c = (c_{11} - c_{12} - 2c_{44})\) are defined, in terms of which the departure from isotropy may be quantized. The Poisson's ratios are simply expressed in terms of \(s_{11}, s_{12},\) and \(s\).

Definition of Poisson's Ratio for Crystals

Poisson's ratio for crystals is defined in general as \(\nu_{ij} = s_{ij}'' / s_{ij}'\), where \(x_i\) is the direction of the longitudinal extension, \(x_i\) is the direction of the accompanying lateral contraction, and the \(s_{ij}'\) and \(s_{ij}''\) are the appropriate elastic compliances referred to this right-handed axial set. It suffices to take \(x_1\) as the direction of the longitudinal extension; then two Poisson's ratios are defined by the orientations of the lateral axes \(x_2\) and \(x_3\): \(\nu_{21} = s_{12}'' / s_{11}'\) and \(\nu_{31} = s_{13}'' / s_{11}'\). Application of the definition requires
specification of the orientation of the $x_k$ coordinate set with respect to the crystallographic directions, and transformation of the compliances accordingly.

Relations for Rotated Cubic Compliances - General

The unprimed compliances $s_{11}$, $s_{44}$, and $s_{12}$ are referred to a set of right-handed cubic crystallographic axes aligned with the [100], [010], and [001] directions. Direction cosines $a_{mn}$ relate the transformation from these axes to the set specifying the directions of the longitudinal extension ($x_1$), and the lateral contractions ($x_2$ and $x_3$). General expressions for the transformed compliances that enter the formulas for $v_{21}$ and $v_{31}$ are:

$$s'_{11} = s_{11} \left[ a_{11}^2 \alpha_{11}^4 + a_{12}^4 + a_{13}^4 \right] + (s_{44} + 2 s_{12}) \left[ a_{12}^2 a_{12}^2 + a_{13}^2 a_{13}^2 + a_{11}^2 a_{11}^2 \right]$$

$$s'_{12} = s_{11} \left[ a_{11}^2 a_{21}^2 + a_{12}^2 a_{22}^2 + a_{13}^2 a_{23}^2 \right] +$$

$$s_{44} \left[ a_{12} a_{13} a_{22} a_{23} + a_{13} a_{11} a_{23} a_{21} + a_{11} a_{12} a_{21} a_{22} \right] +$$

$$s_{12} \left[ a_{11}^2 a_{22}^2 + a_{12}^2 a_{21}^2 + a_{11}^2 a_{23}^2 + a_{12}^2 a_{23}^2 + a_{13}^2 a_{22}^2 \right]$$

$$s'_{13} = s_{11} \left[ a_{11}^2 a_{31}^2 + a_{12}^2 a_{32}^2 + a_{13}^2 a_{33}^2 \right] +$$

$$s_{44} \left[ a_{12} a_{13} a_{32} a_{33} + a_{13} a_{11} a_{33} a_{31} + a_{11} a_{12} a_{31} a_{32} \right] +$$

$$s_{12} \left[ a_{11}^2 a_{32}^2 + a_{12}^2 a_{31}^2 + a_{11}^2 a_{33}^2 + a_{12}^2 a_{33}^2 + a_{13}^2 a_{32}^2 \right]$$

Single-Axis Rotations

The general rotation relations given above for $s'_{11}$, $s'_{12}$, and $s'_{13}$ simplify considerably for single-axis rotations, and use of the anisotropy factor $s = (s_{11} - s_{12} - s_{44}/2)$. Longitudinal extension is along the $x_1$ axis, and abbreviations $c(\phi)$ and $s(\phi)$ stand for $\cos(\phi)$ and $\sin(\phi)$, etc.:

(A) Rotation about $x_1$: $s'_{11} = s_{11}$; $s'_{12} = s'_{13} = s_{12}$; $v_{21} = v_{31} = s_{12} / s_{11}$

(B) Rotation about $x_2$: $s'_{11} = s_{11} - 2s [c^2(\psi) s^2(\psi)]$; $s'_{12} = s_{12}$

$$s_{13}' = s_{12} + 2s [c^2(\psi) s^2(\psi)]; v_{21} = s_{12} / s_{11}'; v_{31} = s_{13}' / s_{11}'$$

(C) Rotation about $x_3$: $s'_{11} = s_{11} - 2s [c^2(\phi) s^2(\phi)]$

$$s_{12}' = s_{12} + 2s [c^2(\phi) s^2(\phi)]; s_{13}' = s_{12}; v_{21} = s_{12}' / s_{11}'; v_{31} = s_{12} / s_{11}'$$
Transformation Matrix for General Rotations

In order to derive the Poisson's ratio for the most general case, we consider the transformation matrix for a combination of three coordinate rotations: a first rotation about \( x_3 \) by angle \( \phi \), a second rotation about the new \( x_1 \) by angle \( \theta \), and a third rotation about the resulting \( x_2 \) by angle \( \psi \). When these angles are set to zero, the \( x_1, x_2, x_3 \) axes coincide respectively with the [100], [010], and [001] crystallographic directions. For nonzero angles, the direction cosines \( a_{mn} \) are as follows:

\[
[a_{11} \quad a_{12} \quad a_{13}]
= \begin{bmatrix}
[c(\phi)c(\psi) - s(\phi)s(\theta)s(\psi)] & [s(\phi)c(\psi) + c(\phi)s(\theta)s(\psi)] & [-c(\phi)s(\psi)] \\
[-s(\phi)c(\theta)] & [c(\phi)c(\theta)] & [s(\theta)] \\
[s(\phi)s(\psi) + c(\phi)s(\theta)c(\psi)] & [s(\phi)s(\psi) - c(\phi)s(\theta)c(\psi)] & [c(\phi)c(\psi)]
\end{bmatrix}
\]

Substitution of these \( a_{mn} \) into the expressions for \( s_{11}', s_{12}', \) and \( s_{13}' \), and thence into the formulas \( \nu_{21} = s_{12}' / s_{11}' \) and \( \nu_{31} = s_{13}' / s_{11}' \) formally solves the problem for specified values of \( \phi, \theta, \) and \( \psi \).

Poisson's Ratios for Specific Orientations

1) Longitudinal extension along [100]: \( \phi = \psi = 0 ; \theta \) arbitrary. This is the same as case (A) above. \( \nu_{21} = \nu_{31} = s_{12} / s_{11} \), independent of angle \( \theta \).

2) Longitudinal extension along an axis normal to [001]: \( \psi = 0 ; \phi \) and \( \theta \) arbitrary. Direction cosines are:

\[
[a_{11} \quad a_{12} \quad a_{13}]
= \begin{bmatrix}
[c(\phi)] & [s(\phi)] & [0] \\
[-s(\phi)c(\theta)] & [c(\phi)c(\theta)] & [s(\theta)] \\
[s(\phi)s(\theta)] & [-c(\phi)s(\theta)] & [c(\phi)]
\end{bmatrix}
\]

Rotated compliances are:

\[
s_{11}' = s_{11} - 2s [c^2(\phi) s^2(\phi)]
\]

\[
s_{12}' = s_{12} + 2s [c^2(\phi) s^2(\phi) c^2(\theta)]
\]

\[
s_{13}' = s_{12} + 2s [c^2(\phi) s^2(\phi) s^2(\theta)]
\]

\[
\nu_{21} = s_{12}' / s_{11}'; \nu_{31} = s_{13}' / s_{11}'
\]
2a) When both $\theta$ and $\psi = 0$, but $\varphi$ is arbitrary, this reduces to case (C) above:

$$s_{11}' = s_{11} - 2s[c^2(\varphi)s^2(\varphi)]$$

$$s_{12}' = s_{12} + 2s[c^2(\varphi)s^2(\varphi)]$$

$$s_{13}' = s_{12} ; \nu_{21} = s_{12}' / s_{11}' ; \nu_{31} = s_{12} / s_{11}'$$

2b) When $\varphi = \pi/4$, $\theta$ is arbitrary, and $\psi = 0$, the $x_1$ axis (direction of longitudinal extension) coincides with the [110] direction; the rotated compliances become:

$$s_{11}' = s_{11} - s(1/2) ; s_{12}' = s_{12} + s(1/2)[c^2(\vartheta)] ; s_{13}' = s_{12} + s(1/2)s^2(\vartheta)]$$

$$\nu_{21} = s_{12}' / s_{11}' = (2s_{12} + s[c^2(\vartheta)]) / (2s_{11} - s)$$

$$\nu_{31} = s_{13}' / s_{11}' = (2s_{12} + s[s^2(\vartheta)]) / (2s_{11} - s)$$

2c) When $\varphi = \pi/4$, $\theta = 0$, and $\psi = 0$, the $x_1$ axis (direction of longitudinal extension) coincides with the [110] direction, and the $x_2$ and $x_3$ axes coincide respectively with the [-110] and [001] directions. The rotated compliances then become:

$$s_{11}' = s_{11} - s(1/2) ; s_{12}' = s_{12} + s(1/2) ; s_{13}' = s_{12}$$

The Poisson's ratios are thus:

$$\nu_{21} = s_{12}' / s_{11}' = (2s_{12} + s) / (2s_{11} - s) \text{ and}$$

$$\nu_{31} = s_{13}' / s_{11}' = (2s_{12}) / (2s_{11} - s)$$

When $\theta = \pi/2$ instead of 0, $\nu_{21}$ and $\nu_{31}$ are simply interchanged.

3) Longitudinal extension in the plane containing [110] and [111]:

$\varphi = \pi/4$, $\theta = 0$, $\psi$ arbitrary. The $x_2$ axis coincides with the [-110] direction. Direction cosines are:

$$c(\psi)/\sqrt{2} \quad c(\psi)/\sqrt{2} \quad -s(\psi)$$

$$-1/\sqrt{2} \quad 1/\sqrt{2} \quad 0$$

$$s(\psi)/\sqrt{2} \quad s(\psi)/\sqrt{2} \quad c(\psi)$$
Rotated compliances are:

\[ s_{11}' = s_{11} - 2s \left[ c^2(\psi) \right] \left[ 1 - (3/4)c^2(\psi) \right] \]
\[ s_{12}' = s_{12} + (1/2)s \left[ c^2(\psi) \right] \]
\[ s_{13}' = s_{12} + (3/2)s \left[ c^2(\psi) s^2(\psi) \right] \]

The Poisson's ratios are: \( \nu_{21} = \frac{s_{12}'}{s_{11}'} \); \( \nu_{31} = \frac{s_{13}'}{s_{11}'} \). When \( \psi = 0 \), this reduces to case 2c) above; when \( \psi = \pi/2 \), it is equivalent to case (A) with extension along [001].

4) Longitudinal extension along [-11-1]: \( \phi = \pi/4 \), \( \theta = 0 \), and \( \psi = \text{arc sin} \left(1/\sqrt{3}\right) \). The \( x_2 \) and \( x_3 \) axes coincide, respectively, with the [-110] and [112] directions. Direction cosines \( a_{mn} \) are:

\[
\begin{array}{ccc}
1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\
-1/\sqrt{2} & 1/\sqrt{2} & 0 \\
1/\sqrt{6} & 1/\sqrt{6} & \sqrt{(2/3)}
\end{array}
\]

Rotated compliances are:

\[ s_{11}' = s_{11} - (2/3)s \]
\[ s_{12}' = s_{13}' = s_{12} + (1/3)s \]

The Poisson's ratios are: \( \nu_{21} = \nu_{31} = \frac{s_{12}'}{s_{11}'} = \frac{3s_{12} + s}{3s_{11} - 2s} \)

5) Longitudinal extension along [-11-1]: \( \phi = \pi/4 \) and \( \psi = \text{arc sin} \left(1/\sqrt{3}\right) \). Provision is made for rotating the lateral axes by adding a third rotation, about the resulting \( x_1 \) (i.e., about [-11-1]), by angle \( \theta \), subsequent to the \( \phi \) and \( \psi \) rotations.

Direction cosines \( a_{mn} \) are now:

\[
\begin{array}{ccc}
1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\
1/\sqrt{2} \left[ c(\theta) + s(\theta)/\sqrt{3} \right] & 1/\sqrt{2} \left[ c(\theta) + s(\theta)/\sqrt{3} \right] & \sqrt{(2/3)} s(\theta) \\
1/\sqrt{2} \left[ s(\theta) + c(\theta)/\sqrt{3} \right] & 1/\sqrt{2} \left[ -s(\theta) + c(\theta)/\sqrt{3} \right] & \sqrt{(2/3)} c(\theta)
\end{array}
\]

6
Rotated compliances computed from these direction cosines turn out to be independent of angle \( \theta \), and are identical to those of case 4) above:

\[
\begin{align*}
{s}'_{11} & = s_{11} - \frac{2}{3} s \\
{s}'_{12} & = s_{13}' = s_{12} + \frac{1}{3} s
\end{align*}
\]

The Poisson's ratios are: \( \nu_{21} = \nu_{31} = s_{12}' / s_{11}' = \left( 3 s_{12} + s \right) / \left( 3 s_{11} - 2 s \right) \)

**Conclusions**

Poisson's ratio, with respect to rotated coordinate axes for cubic materials, has been obtained. Three cases are of particular interest:

- For longitudinal extension along [100] (along the cube axis),
  \( \nu_{21} = \nu_{31} = s_{12} / s_{11} \), independent of lateral directions. Case (A).

- For longitudinal extension along [110] (along the face diagonal; normal to the dodecahedral planes (110)), with \( s = (s_{11} - s_{12} - s_{44}/2) \),
  \[
  \begin{align*}
  \nu_{21} & = \left( 2 s_{12} + s \left[ c^2(0) \right] \right) / \left( 2 s_{11} - s \right) \\
  \nu_{31} & = \left( 2 s_{12} + s \left[ s^2(0) \right] \right) / \left( 2 s_{11} - s \right)
  \end{align*}
  \]

When \( \theta = 0 \), the \( x_2 \) and \( x_3 \) axes coincide, respectively, with the [-110] and [001] directions. Case 2c). Poisson's ratios are:

\[
\begin{align*}
\nu_{21} & = (2 s_{12} + s) / (2 s_{11} - s) \\
\nu_{31} & = (2 s_{12}) / (2 s_{11} - s)
\end{align*}
\]

- For longitudinal extension along [111] (along the body diagonal; normal to the octahedral planes (111)). Case 5). Poisson's ratios are independent of the lateral directions:

\[
\begin{align*}
\nu_{21} = \nu_{31} = \left[ 3 s_{12} + s \right] / \left[ 3 s_{11} - 2 s \right]
\end{align*}
\]
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